**Math 2163, Exam III,** Apr. 16, 2013

Name: ____________________

Score:

**Please read the instructions on each problem carefully, and indicate answers as directed. Show details in your work. If you simply give a solution without steps of how you derive this solution, you may not get credit for it.**

**The total is 50 points**

1. (6 points) Evaluate the double integral $\iint_D x\cos y\,dA$ where $D$ is bounded by $y=0$, $y=x^2$, and $x=2$.

   **Solution** Note that $y=0$ and $y=x^2$ intersects at point $(0,0)$, hence

$$D=\{(x,y)|\ 0\le y\le x^2,\ 0\le x\le 2\}$$

   Therefore, we have

$$\begin{aligned}\iint_D x\cos y\,dA &= \int_0^2\int_0^{x^2} x\cos y\,dy\,dx\\ &= \int_0^2 x\sin x^2\,dx\\ &\text{(use substitution } u=x^2\text{ )}\\ &= \int_0^4 \frac{1}{2}\sin u\,du\\ &= -\frac{1}{2}(\cos 4-\cos 0)\\ &= \frac{1}{2}(1-\cos 4)\end{aligned}$$

2. (8 points) Evaluate the integral by changing to polar coordinates:

$$\iint_D e^{-x^2-y^2}\,dA$$

where $D$ is the region bounded by the semicircle $x = \sqrt{16-y^2}$ and the $y$-axis.

**Solution** Note $D$ is the right half-disk with radius equal to $4$, therefore

$$D = \{(r,\theta) \mid -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 4\}$$

and

$$x^2 + y^2 = r^2.$$

Hence

$$\begin{aligned}
\iint_D e^{-x^2-y^2}\,dA &= \int_{-\pi/2}^{\pi/2}\int_0^4 e^{-r^2} r\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(-\frac{1}{2}e^{-r^2}\right)\Big|_{r=0}^{r=4}\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\left(-\frac{1}{2}e^{-16}+\frac{1}{2}e^{0}\right)\,d\theta \\
&= \pi\left(-\frac{1}{2}e^{-16}+\frac{1}{2}e^{0}\right) \\
&= \frac{\pi}{2}(1-e^{-16})
\end{aligned}$$

3. (8 points) Find the mass of the lamina that occupies the region $D$ and has the given density function $\rho$:

$$D = \{(x, y) | \, 0 \leq x \leq \pi/2, \, 0 \leq y \leq \cos x\}, \qquad \rho(x, y) = 10y$$

Formula: $\sin 2x = 2 \sin x \cos x$, $\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$.

**Solution** The mass is

$$\iint_D \rho \, dA = \int_0^{\pi/2} \int_0^{\cos x} 10y \, dy \, dx$$
$$= \int_0^{\pi/2} 5\cos^2 x \, dx$$

(use the formula $\cos 2x = 2\cos^2 x - 1$ we have $\cos^2 x = \frac{1}{2}(\cos 2x + 1)$ )

$$= \int_0^{\pi/2} \frac{5}{2}(\cos 2x + 1) \, dx$$
$$= \frac{5}{2}\left(\frac{1}{2}\sin 2x + x\right) \Big|_0^{\pi/2}$$
$$= \frac{5}{2}\left(\frac{1}{2}\sin \pi + \frac{\pi}{2}\right)$$
$$= \frac{5}{4}\pi$$

4. (8 points) Use triple integral to find the volume of the solid bounded by the cylinder $x = y^2$ and the planes $z = 0$ and $x + z = 4$.

**Solution** The triple integral is

$$
\begin{aligned}
&\int_{-2}^{2}\int_{y^2}^{4}\int_{0}^{4-x} dz\,dx\,dy \\
=&\int_{-2}^{2}\int_{y^2}^{4}(4-x)\,dx\,dy \\
=&\int_{-2}^{2}(8-4y^2+\frac{1}{2}y^4)\,dy \\
=&(8y-\frac{4}{3}y^3+\frac{1}{10}y^5)|_{-2}^{2} \\
=&\frac{256}{15}
\end{aligned}
$$

**Alternatively** one can use

$$\int_{0}^{4}\int_{-\sqrt{x}}^{\sqrt{x}}\int_{0}^{4-x} dz\,dy\,dx$$

or

$$2\int_{0}^{2}\int_{y^2}^{4}\int_{0}^{4-x} dz\,dx\,dy$$

or

$$2\int_{0}^{4}\int_{0}^{\sqrt{x}}\int_{0}^{4-x} dz\,dy\,dx$$

5. (6 points) Rewrite the following integral using cylindrical coordinates. You do NOT need to evaluate the integral.

$$\int_{-3}^{3} \int_{0}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz dy dx$$

**Solution**

$$\int_{0}^{\pi} \int_{0}^{3} \int_{0}^{9-r^2} r^2 \, dz \, dr \, d\theta$$

6. (6 points) Rewrite the following integral using spherical coordinates. You do NOT need to evaluate the integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} xyz\, dzdxdy$$

**Solution**

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1}(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\theta\,d\phi$$

7. (8 points) Use spherical coordinates to evaluate the triple integral

$$\iiint_V (x^2 + y^2 + z^2)^2 \, dV$$

where $V$ is the upper half of a ball centered at the origin with radius $5$.

**Solution** Use the spherical coordinates, we have

$$\begin{aligned}
&\int_0^{\pi/2} \int_0^{2\pi} \int_0^5 \rho^4 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \\
=&\int_0^{\pi/2} \int_0^{2\pi} \int_0^5 \rho^6 \sin\phi \, d\rho \, d\theta \, d\phi \\
=&\int_0^{\pi/2} \int_0^{2\pi} \frac{5^7}{7} \sin\phi \, d\theta \, d\phi \\
=&\int_0^{\pi/2} 2\pi \frac{5^7}{7} \sin\phi \, d\phi \\
=&-2\pi \frac{5^7}{7} (\cos\frac{\pi}{2} - \cos 0) \\
=&2\pi \frac{5^7}{7}
\end{aligned}$$