**Math 2163, Exam II,** Mar. 12, 2013

Name: ____________________

Score:

**Please read the instructions on each problem carefully, and indicate answers as directed. Show details in your work. If you simply give a solution without steps of how you derive this solution, you may not get credit for it.**

**The total is 50 points**

1. (6 points) Use chain rule to compute

   (a) $z = x^2 + y^2 + xy$, $x = \sin t$, $y = e^t$, compute $\frac{dz}{dt}$

   (b) $z = e^x + 2y$, $x = s/t$, $y = t/s$, compute $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$

   **Solution**

   (a) $\frac{dz}{dt} = (2x + y)\cos t + (2y + x)e^t = (2\sin t + e^t)\cos t + (2e^t + \sin t)e^t$

   (b) $\frac{\partial t}{\partial s} = e^x \frac{1}{t} + 2(-\frac{t}{s^2}) = e^{s/t}\frac{1}{t} + 2(-\frac{t}{s^2})$

   $\frac{\partial t}{\partial t} = e^x(-\frac{s}{t^2}) + 2\frac{1}{s} = e^{s/t}(-\frac{s}{t^2}) + 2\frac{1}{s}$

2. (5 points) Find $\frac{dy}{dx}$ where $x$, $y$ satisfy

$$\cos(x-y) = xe^y$$

**Solution** Define implicit function $F(x,y) = \cos(x-y) - xe^y = 0$, then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(x-y) - e^y}{\sin(x-y) - xe^y}$$

3. (6 points) Find the directional derivative of $f(x,y) = \sqrt{6x - 5y}$ at point $(5,1)$ in the direction indicated by the angle $\theta = -\pi/6$.

**Solution** The unit vector indicated by angle $\theta$ is

$$\mathbf{v} = <\cos\theta, \sin\theta> = <\cos(-\pi/6), \sin(-\pi/6)> = <\frac{\sqrt{3}}{2}, -\frac{1}{2}>$$

The directional derivative is

$$D_{\mathbf{v}}f = \nabla f \cdot \mathbf{v} = <\frac{6}{2\sqrt{6x-5y}}, \frac{-5}{2\sqrt{6x-5y}}> \cdot <\frac{\sqrt{3}}{2}, -\frac{1}{2}>$$

Then, at point $(5,1)$,

$$D_{\mathbf{v}}f(5,1) = <\frac{6}{2\sqrt{25}}, \frac{-5}{2\sqrt{25}}> \cdot <\frac{\sqrt{3}}{2}, -\frac{1}{2}> = \frac{3\sqrt{3}}{10} + \frac{1}{4}$$

4. (5 points) Find the maximum rate of change of $f(x, y) = \sin(xy)$ at the point $(0, 1)$, and also find the direction in which it occurs.

**Solution** $\nabla f =< y\cos(xy),\ x\cos(xy) >$. At point $(0, 1)$, we have

$$\nabla f(0, 1) =< \cos 0,\ 0 >=< 1, 0 >$$

Then, the maximum rate of change is $|\nabla f(0, 1)| = |< 1, 0 >| = 1$, and it occurs in the same direction as $\nabla f(0, 1) =< 1, 0 >$.

5. (6 points) Find all critical points of (you do NOT need to classify them)

$$f(x,y) = 5 + 12xy + 6x^2 + 32y + \frac{y^3}{3}$$

**Solution**

$$\begin{cases} f_x = 12y + 12x = 0 \\ f_y = 12x + 32 + y^2 = 0 \end{cases}$$

From $12y + 12x = 0$, we have $x = -y$. Substitute this into the second equation gives:

$$-12y + 32 + y^2 = 0 \quad \Rightarrow \quad y^2 - 12y + 32 = 0 \quad \Rightarrow \quad (y-4)(y-8) = 0$$

Therefore, we have $y = 4$ or $y = 8$. Use the fact that $x = -y$, this gives two critical point:

$$(-4, 4) \qquad \text{and} \qquad (-8, 8)$$

6. (5 points) We known that $(0,0)$ and $(2,1)$ are two critical points of $f(x,y) = x^3 - 12xy + 8y^3$. Determine whether they are local maximums, local minimums, saddle points, or others.

   **Solution** Note that

$$f_x = 3x^2 - 12y, \qquad f_y = 24y^2 - 12x$$

   and

$$f_{xx} = 6x, \quad f_{xy} = -12, \qquad f_{yy} = 48y$$

   We have

$$D = f_{xx}f_{yy} - f_{xy}^2 = 288xy - 144$$

   - At point $(0,0)$, we have $D = -144 < 0$. Therefore, $(0,0)$ is a saddle point
   - At point $(2,1)$, we have $D = 288 \times 2 - 144 > 0$ and $f_{xx} = 6 \times 2 > 0$. Therefore, $(2,1)$ is a local minimum.

7. (6 points) Use Lagrange multipliers to find the points whose coordinates are all positive at which the function $f(x, y, z) = 5xyz$ has its maximum value, subject to the constraint $5x^2 + 15y^2 + 25z^2 = 6$.

   **Solution** Use the Lagrange multipliers, we have

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 6 \end{cases} \qquad \Rightarrow \qquad \begin{cases} 5yz = 10\lambda x \\ 5xz = 30\lambda y \\ 5xy = 50\lambda z \\ 5x^2 + 15y^2 + 25z^2 = 6 \end{cases}$$

   Multiply the 1st, 2nd and 3rd equations by $x$, $y$, and $z$, respectively, we have

$$\begin{cases} 5xyz = 10\lambda x^2 \\ 5xyz = 30\lambda y^2 \\ 5xyz = 50\lambda z^2 \end{cases}$$

   Hence $10\lambda x^2 = 30\lambda y^2 = 50\lambda z^2$, which implies either $\lambda = 0$ or $10x^2 = 30y^2 = 50z^2$.

   If $\lambda = 0$, from the equation $5yz = 10\lambda x$, either $y$ or $z$ must be $0$. This is not a solution we are seeking, since we want all coordinates to be positive. Now the only choice is $10x^2 = 30y^2 = 50z^2$, which is equivalent to $5x^2 = 15y^2 = 25z^2$. Combine this with $5x^2 + 15y^2 + 25z^2 = 6$, clearly

$$5x^2 = 15y^2 = 25z^2 = 2.$$

   Note all coordinates should be positive, hence the solution is

$$x = \sqrt{2/5}, \quad y = \sqrt{2/15}, \quad z = \sqrt{2/25}.$$

8. (6 points) Find an approximation for the double integral

$$\iint_R (4x - 5y^2)\, dA$$

using a double Riemann sum with $m = n = 2$ and the sample point in the upper right corner. Here $R = \{(x, y) |\, 0 \le x \le 8,\ 0 \le y \le 4\}$.

**Solution** Using the Riemann sum, we have $\Delta A = 8$ and

$$\begin{aligned}\iint_R (4x - 5y^2)\, dA &\approx 8f(4,2) + 8f(4,4) + 8f(8,2) + 8f(8,4) \\ &= 8(-4 - 64 + 12 - 48) = -832\end{aligned}$$

9. (5 points) Calculate the iterated integral

$$\int_1^4 \int_0^3 (1+2xy)\, dxdy$$

**Solution**

$$\begin{aligned}
\int_1^4 \int_0^3 (1+2xy)\, dxdy &= \int_1^4 (x+x^2y)|_{x=0}^{x=3} dy \\
&= \int_1^4 (3+9y) dy \\
&= (3y+\frac{9}{2}y^2)|_{y=1}^{y=4} \\
&= (12+72)-(3+\frac{9}{2}) \\
&= 76.5
\end{aligned}$$