Math 2163, Exam I, Feb. 7, 2013

Name: ____________________

Score:

Please read the instructions on each problem carefully, and indicate answers as directed. Show details in your work. If you simply give a solution without steps of how you derive this solution, you may not get credit for it.

The total is 50 points

1. (6 points) Calculate the following:

   (a) $\cos\theta$ where θ is the angle between $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and $4\mathbf{i}-3\mathbf{k}$.

   (b) $<6,3,-1> \times <0,1,2>$

   Solution to (a)

$$\cos\theta = \frac{(\mathbf{i}+2\mathbf{j}-2\mathbf{k})\cdot(4\mathbf{i}-3\mathbf{k})}{|\mathbf{i}+2\mathbf{j}-2\mathbf{k}|\,|4\mathbf{i}-3\mathbf{k}|} = \frac{10}{\sqrt{9}\sqrt{25}} = \frac{2}{3}$$

   Solution to (b)

$$<6,3,-1> \times <0,1,2> = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 3 & -1 \\ 0 & 1 & 2 \end{vmatrix} = 7\mathbf{i}-12\mathbf{j}+6\mathbf{k}$$

2. (5 points) Find an equation for the line that passes through the point $(0,\ 5,\ -3)$ and is perpendicular to the plane $3x + 4y - 5z = 7$.

Solution Note the directional vector of the line is the same as the normal vector of the plane $3x + 4y - 5z = 7$, which is $< 3, 4, -5 >$. Therefore the line is

$$\frac{x}{3} = \frac{y-5}{4} = \frac{z+2}{-5}$$

3. (6 points) Find an equation for the plane that passes through the point $(1, -1, 1)$ and contains the line $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$.

Solution Note the line goes through point $(0, 0, 0)$, because $(0, 0, 0)$ satisfies the equation $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$. The normal vector of the plane must be perpendicular to

$$(1, -1, 1) - (0, 0, 0) =< 1, -1, 1 >$$

The normal vector should also be perpendicular to the directional vector of the line $\frac{x}{6} = \frac{y}{3} = \frac{z}{2}$, which is $< 6, 3, 2 >$. Combine both, the normal vector can be written as

$$< 6, 3, 2 > \times < 1, -1, 1 >= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 3 & 2 \\ 1 & -1 & 1 \end{vmatrix} =< 5, -4, -9 >$$

Therefore, the plane has equation

$$< 5, -4, -9 > \cdot (< x, y, z > - < 1, -1, 1 >) = 0$$

or you can simplify it into

$$5x - 4y - 9z = 0$$

4. (5 points) Calculate the unit tangent vector $\mathbf{T}$ of the curve $\mathbf{r}(t) =< e^t \cos t,\ e^t \sin t,\ t >$ at the point $(1, 0, 0)$.

Solution First, we have

$$\mathbf{r}'(t) =< e^t(\cos t - \sin t),\ e^t(\sin t + \cos t),\ 1 >$$

Note that at point $(1, 0, 0)$, we clearly have $t = 0$. Use this to get

$$\mathbf{r}'(0) =< e^0(\cos 0 - \sin 0),\ e^0(\sin 0 + \cos 0),\ 1 >=< 1, 1, 1 >$$

Therefore

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{< 1, 1, 1 >}{\sqrt{3}} =< \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} >$$

5. (5 points) Find the domain of $f(x,y) = \sqrt{1-x^2} - \sqrt{1-y^2}$. Sketch the graph of the domain in the xy-plane.

Solution The domain is

$$D = \{(x,y)|\, 1-x^2 \geq 0,\ 1-y^2 \geq 0\} = \{(x,y)|\, -1 \leq x \leq 1,\ -1 \leq y \leq 1\}$$

The graph is a square and is omitted.

6. (6 points) Examine the following limit along the paths $y = 0$, $x = 0$, $x = y$ and $x = y^4$. Does the limit exist?

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$$

Solution

(a) Set $y = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{x\to 0} \frac{0}{x^2 + 0} = \lim_{x\to 0} 0 = 0$$

(b) Set $x = 0$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{y\to 0} \frac{0}{0 + y^8} = \lim_{y\to 0} 0 = 0$$

(c) Set $y = x$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{x\to 0} \frac{x^5}{x^2 + x^8} = \lim_{x\to 0} \frac{x^3}{1 + x^6} = \frac{0}{1 + 0} = 0$$

(d) Set $x = y^4$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{y\to 0} \frac{y^8}{y^8 + x^8} = \lim_{y\to 0} \frac{y^8}{2y^8} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

The overall limit does not exist since different paths give different limits.

7. (5 points) Find all first order derivative of $f(x, y) = xe^{-x^2-y^2}$.

Solution

$$\begin{aligned} f_x(x, y) &= e^{-x^2-y^2} + xe^{-x^2-y^2}(-2x) \\ f_y(x, y) &= xe^{-x^2-y^2}(-2y) \end{aligned}$$

8. (6 points) Use implicit differentiation to calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where z is defined by

$$yz = \ln(x+z)$$

Solution First, we compute $\frac{\partial z}{\partial x}$. Take partial derivative of both side of the equation with respect to x gives

$$\begin{aligned}
& \frac{\partial(yz)}{\partial x} = \frac{\partial \ln(x+z)}{\partial x} \\
\Rightarrow \quad & y\frac{\partial z}{\partial x} = \frac{1}{x+z}\left(1+\frac{\partial z}{\partial x}\right) \\
\Rightarrow \quad & y\frac{\partial z}{\partial x} = \frac{1}{x+z} + \frac{1}{x+z}\frac{\partial z}{\partial x} \\
\Rightarrow \quad & \left(y - \frac{1}{x+z}\right)\frac{\partial z}{\partial x} = \frac{1}{x+z} \\
\Rightarrow \quad & \frac{\partial z}{\partial x} = \frac{\frac{1}{x+z}}{y-\frac{1}{x+z}} = \frac{1}{(x+z)y-1}
\end{aligned}$$

Next we compute $\frac{\partial z}{\partial y}$. Take partial derivative of both side of the equation with respect to y gives

$$\begin{aligned}
& \frac{\partial(yz)}{\partial y} = \frac{\partial \ln(x+z)}{\partial y} \\
\Rightarrow \quad & z + y\frac{\partial z}{\partial y} = \frac{1}{x+z}\frac{\partial z}{\partial y} \\
\Rightarrow \quad & \left(\frac{1}{x+z} - y\right)\frac{\partial z}{\partial y} = z \\
\Rightarrow \quad & \frac{\partial z}{\partial y} = \frac{z}{\frac{1}{x+z}-y} = \frac{(x+z)z}{1-(x+z)y}
\end{aligned}$$

9. (6 points) Find linear approximation of the function $z = \sqrt{20 - x^2 - 7y^2}$ at $(x, y) = (2, 1)$, and use it to approximate $f(1.95, 1.08)$.

Solution First, note that

$$f_x(x,y) = \frac{-2x}{2\sqrt{20 - x^2 - 7y^2}}, \qquad f_y(x,y) = \frac{-14y}{2\sqrt{20 - x^2 - 7y^2}}$$

The linear approximation is given by

$$\begin{aligned} & f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1) \\ =& 3 + \frac{-4}{6}(x-2) + \frac{-14}{6}(y-1) \\ =& 3 - \frac{2}{3}(x-2) - \frac{7}{3}(y-1) \end{aligned}$$

Now, set $x = 1.95$ and $y = 1.08$, the linear approximation is

$$3 - \frac{2}{3}(1.95 - 2) - \frac{7}{3}(1.08 - 1) = 3 - \frac{0.46}{3} \approx 2.84667$$