Math 2163, Practice Final Exam

Please also read through previous practice exams. Problem types that have appeared in previous practice exams will not be repeated here!

Part I: Multiple choices. Each problem is worth 5 points. **Please enter your solution in the parentheses in front of each problem.**

1. (D) Suppose you start from the origin, move along the $x-$axis a distance of 8 units in the positive direction, and then move downward a distance of 1 units. What are the coordinates of your position?

   (A) (8, 0, 1)

   (B) (0, 8, 1)

   (C) (8, -1, 0)

   (D) (8, 0, -1)

   (E) (8, 1, 0)

2. (E) Find an equation of the set of all points equidistant from the points (7, -8, -9) and (-4, 2, -10).

   (A) $11x - 10y + z = -37$;

   (B) $11x + 10y + z = 37$;

   (C) $-11x - 10y + z = 37$;

   (D) $11x - 10y - z = 37$;

   (E) $11x - 10y + z = 37$.

3. (C) Find $5\mathbf{a} + 3\mathbf{b}$ where $\mathbf{a} =< -7,\ -6 >$ and $\mathbf{b} =< -4,\ 7 >$.

   (A) $< 9, 47 >$;

   (B) $< -9, -47 >$;

   (C) $< -47, -9 >$;

   (D) $< 47, 9 >$;

   (E) $< 9, 9 >$.

4. (D) Find the angle between vectors $\mathbf{a} =< 6,\ 0 >$ and $\mathbf{b} =< 6,\ 6 >$.

   (A) $\pi/6$;

   (B) $\pi/3$;

   (C) $\pi/2$;

   (D) $\pi/4$;

   (E) $5\pi/6$.

5. (B) Find a unit vector that is orthogonal to both $< 9,\ 9,\ 0 >$ and $< 9,\ 0,\ 9 >$.

(A) $< 1/3,\ -1/3,\ -1/3 >$;

(B) $< 1/\sqrt{3},\ -1/\sqrt{3},\ -1/\sqrt{3} >$;

(C) $< 1/3,\ 1/3,\ 1/3 >$;

(D) $< 1/9,\ 1/9,\ 1/9 >$;

(E) $< 1/9,\ -1/9,\ -1/9 >$.

6. (B) Find the cross product $\mathbf{a} \times \mathbf{b}$ where $\mathbf{a} =< 3,\ 5,\ 1 >$ and $\mathbf{b} =< -5,\ 2,\ -2 >$.

(A) $< -10, -5, 6 >$;

(B) $< -12, 1, 31 >$;

(C) $< 2, -6, -25 >$;

(D) $< -8, -11, -19 >$;

(E) $< -16, -23, 25 >$.

7. (A) Change from rectangular to cylindrical coordinates.

$$(9,\ -9,\ 2)$$

(A) $(9\sqrt{2},\ 7\pi/4,\ 2)$;

(B) $(9\sqrt{2},\ \pi/4,\ 2)$;

(C) $(9\sqrt{2},\ 0,\ 2)$;

(D) $(9,\ 7\pi/4,\ 2)$;

(E) $(-9\sqrt{2},\ \pi/4,\ 2)$.

8. (D) Evaluate the integral $\int (e^{7t}\mathbf{i} + 4t\mathbf{j} + \ln t\mathbf{k})\, dt$

(A) $\dfrac{e^{7t}}{7}\mathbf{i} + 2t^2\mathbf{j} + (\ln t - 1)\mathbf{k} + \mathbf{C}$;

(B) $\dfrac{e^{7t}}{7}\mathbf{i} + 4t^2\mathbf{j} + (\ln t - 1)\mathbf{k} + \mathbf{C}$;

(C) $e^{7t}\mathbf{i} + 2t^2\mathbf{j} + (\ln t - 1)\mathbf{k} + \mathbf{C}$;

(D) $\dfrac{e^{7t}}{7}\mathbf{i} + 2t^2\mathbf{j} + t(\ln t - 1)\mathbf{k} + \mathbf{C}$;

(E) $e^{7t}\mathbf{i} + 4t^2\mathbf{j} + (\ln t - 1)\mathbf{k} + \mathbf{C}$.

9. (A) Find $\mathbf{r}(t)$ if $\mathbf{r}'(t) = \sin t\mathbf{i} - \cos t\mathbf{j} + 6t\mathbf{k}$ and $\mathbf{r}(\mathbf{0}) = \mathbf{i} + \mathbf{j} + 5\mathbf{k}$.

(A) $(-\cos t + 2)\mathbf{i} - (\sin t - 1)\mathbf{j} + (3t^2 + 5)\mathbf{k}$;

(B) $(-\cos t + 2)\mathbf{i} + (\sin t + 1)\mathbf{j} + (3t^2 + 5)\mathbf{k}$;

(C) $\cos t\mathbf{i} - (\sin t - 1)\mathbf{j} + (3t^2 + 5)\mathbf{k}$;

(D) $\cos t\mathbf{i} + (\sin t + 1)\mathbf{j} + (3t^2 + 5)\mathbf{k}$;

(E) $(-\cos t+1)\mathbf{i}-(\sin t-1)\mathbf{j}+(3t^2+5)\mathbf{k}$.

10. (B) Determine the largest set on which the function is continuous.

$$F(x,y)=\arctan(10x+8\sqrt{y-3})$$

(A) $\{(x,y), y>3$ and $|10x+8\sqrt{y}|\leq 1\}$;

(B) $\{(x,y), y\geq 3\}$;

(C) $\{(x,y), y\leq 0\}$;

(D) $\{(x,y), x\geq 0\}$;

(E) $\{(x,y), x\geq 1\}$.

11. (B) Find $f_x(x,y)$ if $f(x,y)=4x^2-9xy+2y^2$.

(A) $8x-9xy$;

(B) $8x-9y$;

(C) $8x-9$;

(D) $4y-9x$;

(E) $4x-9y$.

12. (D) Find and indicate the partial derivative f_{xxx} where $f(x,y)=x^2y^4-3x^4y$.

(A) $3xy$;

(B) $-3xy$;

(C) $72x^2y$;

(D) $-72xy$;

(E) $-45xy$.

13. (A) Use the chain rule to find $\frac{\partial z}{\partial s}$ where

$$z=e^r\cos(\theta), \qquad r=10st, \qquad \theta=\sqrt{s^2+t^2}$$

(A) $e^r\left(10t\cos(\theta)-\dfrac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right)$;

(B) $e^r\left(t\cos(\theta)-\dfrac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right)$;

(C) $e^r\left(10t\cos(\theta)+\dfrac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right)$;

(D) $e^t\left(\cos(\theta)+\dfrac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right)$;

(E) $\left(10t\cos(\theta)+\dfrac{se^r\sin(\theta)}{\sqrt{s^2+t^2}}\right)$.

14. (B) Find the direction in which the maximum rate of change of f at the given point occurs.

$$f(x, y) = \sin(xy), \qquad (1, 0)$$

(A) $< 1/\sqrt{2},\, 1/\sqrt{2} >$;

(B) $< 0,\, 1 >$;

(C) $< 1,\, 0 >$;

(D) $< \sqrt{2},\, 0 >$;

(E) $< 1,\, -\sqrt{2} >$.

Part II: Partial credit problems.

15. Find the three angles of the triangle with given vertices $A(1,0)$, $B(3,6)$, $C(-1,4)$. Decide whether the triangle is right-angled.

Solution The angle formed by edges AB and AC is given as the angles between vectors:

$$AB: \ (3,\,6)-(1,\,0) =< 2,\,6 >$$
$$AC: \ (-1,\,4)-(1,\,0) =< -2,\,4 >$$

Therefore the angle is

$$\angle A = \arccos \frac{< 2,\,6 > \cdot < -2,\,4 >}{|< 2,\,6 >|\,|< -2,\,4 >|} = \arccos \frac{20}{\sqrt{40}\sqrt{20}} = \arccos \frac{1}{\sqrt{2}} = \pi/4$$

Similarly, we can find the angle between BA and BC is

$$\angle B = \arccos \frac{< -2,\,-6 > \cdot < -4,\,-2 >}{|< -2,\,-6 >|\,|< -4,\,-2 >|} = \arccos \frac{20}{\sqrt{40}\sqrt{20}} = \pi/4$$

and the angle between CA and CB is

$$\angle C = \arccos \frac{< 2,\,-4 > \cdot < 4,\,2 >}{|< 2,\,-4 >|\,|< 4,\,2 >|} = \arccos \frac{0}{\sqrt{20}\sqrt{20}} = \arccos 0 = \pi/2$$

Therefore, the triangle is right-angled.

16. Find the parametric equations for the line segment from point $A(10,3,1)$ to $B(5,6,-3)$.

Solution The line segment between two points is given by

$$\begin{aligned}\mathbf{r}(t) &= A + t(B-A) = (1-t)A + tB \\ &= (1-t) < 10,3,1 > + t < 5,6,-3 > \\ &=< 10-10t, 3-3t, 1-t > + < 5t, 6t, -3t > \\ &=< 10-5t, 3+3t, 1-4t >\end{aligned}$$

for $0 \le t \le 1$. So the parametric equation is

$$\begin{cases} x = 10-5t \\ y = 3+3t \qquad\qquad 0 \le t \le 1 \\ z = 1-4t \end{cases}$$

17. Find the plane pass through the point $(-1,2,-1)$ and contains the line $x = y/2 = z/3$.

Solution We will randomly take two different point from the given line. Notice both $(1,2,3)$, $(-1,-2,-3)$ are on the line $x = y/2 = z/3$. Now we have three points on the plane, this gives us two vectors

$$(1,2,3)-(-1,2,-1) =< 2,0,4 >$$
$$(-1,-2,-3)-(-1,2,-1) =< 0,-4,-2 >$$

Then the normal vector of the plane is

$$\mathbf{n} =< 2, 0, 4 > \times < 0, -4, -2 >=< 16, 4, -8 >$$

So the plane is

$$\begin{aligned} & \mathbf{n} \cdot (< x, y, z > - < -1, 2, -1 >) = 0 \\ \Longrightarrow \quad & 16(x+1) + 4(y-2) - 8(z+1) = 0 \end{aligned}$$

18. Find the tangent plane of $5x^2 + 3y^2 + 8z^2 = 353$ at point $(3, 6, 5)$.

    Solution Given a surface $z = f(x, y)$, then the the tangent plane at point (x_0, y_0, z_0) is

    $$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0$$

    Now the given point is $(x_0, y_0, z_0) = (3, 6, 5)$, all we need to do is to calculate $f_x = \frac{\partial z}{\partial x}$ and $f_y = \frac{\partial z}{\partial y}$ at point (x_0, y_0, z_0). Define $F(x, y, z) = 5x^2 + 3y^2 + 8z^2 - 353 = 0$, by the Implicit Function Theorem,

    $$\begin{aligned} \frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{10x}{16z} \\ \frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{6y}{16z} \end{aligned}$$

    Hence at point $(x_0, y_0, z_0) = (3, 6, 5)$,

    $$\begin{aligned} f_x(3, 6) &= -\frac{10(3)}{16(5)} = -\frac{3}{8} \\ f_y(3, 6) &= -\frac{6(6)}{16(5)} = -\frac{9}{20} \end{aligned}$$

    And the tangent plane is

    $$-\frac{3}{8}(x-3) - \frac{9}{20}(y-6) - (z-5) = 0$$

19. Find the local maximum and minimum values of $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$.

    Solution First we calculate the critical points.

    $$\begin{cases} f_x = 6xy - 6x = 0 \\ f_y = 3x^2 + 3y^2 - 6y = 0 \end{cases}$$

    By solving the first equation, we have either $x = 0$ or $y = 1$. First, if $x = 0$, substitute it into the second equation gives $3y^2 - 6y = 0$ which implies $y = 0$ or $y = 2$. So we have two critical points $(0, 0)$ and $(0, 2)$. Second, if $y = 1$, substitute it into the second equation gives $3x^2 - 3 = 0$ which implies $x = 1$ or $x = -1$. This gives another two critical points $(1, 1)$ and $(-1, 1)$. Combine all the above, we have four critical points $(0, 0)$, $(0, 2)$, $(1, 1)$ and $(-1, 1)$.

Now we classify these critical points. Clearly

$$f_{xx} = 6y - 6, \quad f_{xy} = 6x, \quad f_{yy} = 6y - 6$$

By the formula

$$D(x_0, y_0) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (6y_0 - 6)^2 - (6x_0)^2$$

We have

$D(0,0) = 36 > 0, \qquad f_{xx}(0,0) = -6 < 0 \qquad \Rightarrow f(0,0)$ is local maximum
$D(0,2) = 36 > 0, \qquad f_{xx}(0,2) = 12 > 0 \qquad \Rightarrow f(0,2)$ is local minimum
$D(1,1) = -36 < 0, \qquad (1,1)$ is a saddle point
$D(-1,1) = -36 < 0, \qquad (-1,1)$ is a saddle point

Finally, calculate the local maximum $f(0,0) = 2$, and local minimum $f(0,2) = -2$.

20. Find the region for which the triple integral $\iiint (1 - x^2 - y^2 - z^2)\, dV$ is a maximum.

Solution Think of $(1 - x^2 - y^2 - z^2)$ as the "weight" function. When the "weight" is positive, it will add to the total integral. And when the "weight" is negative, it will lower the total integral. To achieve maximum value of the integral, we only want to integrate on regions with positive "weight", which is

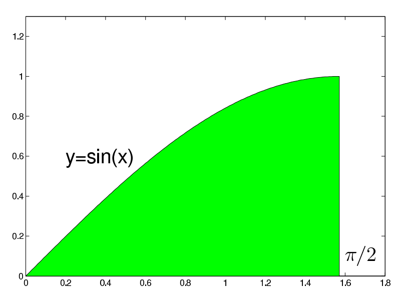
$$\{(x,y,z) |\, 1 - x^2 - y^2 - z^2 > 0\} = \{(x,y,z) |\, x^2 + y^2 + z^2 < 1\}$$

In other words, the region is bounded inside the unit ball.

21. Evaluate the integral by first reversing the order of integration:

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{3 + \cos^2 x}\, dx dy$$

Solution The region of the integral is shown in the graph.



Then

$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{3+\cos^2 x}\, dxdy$$

$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x\sqrt{3+\cos^2 x}\, dydx$$

$$= \int_0^{\pi/2} \cos x\sqrt{3+\cos^2 x}\, y|_0^{\sin x}\, dx$$

$$= \int_0^{\pi/2} \cos x\sqrt{3+\cos^2 x}\ \sin x\, dx$$

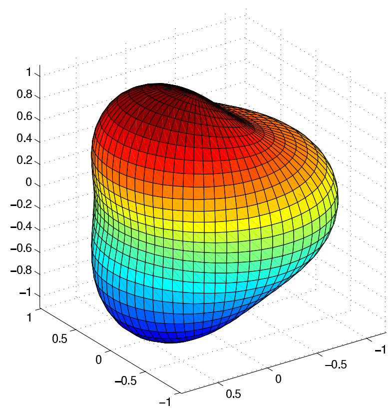
(Let $u = \cos x,$ then $du = -\sin x\, dx$)

$$= \int_1^0 u\sqrt{3+u^2}(-du)$$

$$= -\frac{1}{3}(3+u^2)^{3/2}|_1^0 = \frac{8}{3} - \sqrt{3}$$

22. Find the volume of a "bumpy sphere", the surface $\rho = 1 + \frac{1}{5}\sin\theta\sin 3\phi$.

Solution Using the spherical coordinates, the volume is



$$\iiint_E dV = \int_0^\pi \int_0^{2\pi} \int_0^{1+\frac{1}{5}\sin\theta\sin 3\phi} \rho^2 \sin\phi\, d\rho d\theta d\phi$$

$$= \int_0^\pi \int_0^{2\pi} \frac{1}{3}(1+\frac{1}{5}\sin\theta\sin 3\phi)^3 \sin\phi\, d\theta d\phi$$

$$= \int_0^\pi \frac{\sin\phi}{3}(\theta - \frac{3}{5}\cos\theta\sin 3\phi + \frac{3}{50}(\theta - \frac{\sin 2\theta}{2})\sin^2 3\phi$$

$$-\frac{1}{125}\sin^3 3\phi\sin\phi(\cos\theta - \frac{\cos^3\theta}{3}))|_0^{2\pi}\, d\phi$$

$$= \cdots = \frac{3608}{2625}\pi$$

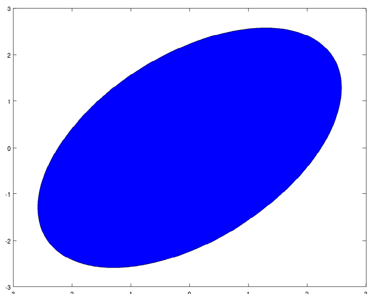
23. Find the Jacobian of the transformation $x = 5\alpha\sin\beta$, $y = 4\alpha\cos\beta$.

Solution The Jacobian is

$$\begin{aligned}\frac{\partial(x,y)}{\partial(\alpha,\beta)} &= \begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} \\ &= \begin{vmatrix} 5\sin\beta & 5\alpha\cos\beta \\ 4\cos\beta & -4\alpha\sin\beta \end{vmatrix} = -20\alpha\sin^2\beta - 20\alpha\cos^2\beta = -20\alpha\end{aligned}$$

24. Use the transformation $x = \sqrt{5}u - \sqrt{\frac{5}{3}}v$, $y = \sqrt{5}u + \sqrt{\frac{5}{3}}v$ to evaluate the integral $\iint_D (x^2 - xy + y^2)\, dA$ where D is the region bounded by the ellipse $x^2 - xy + y^2 = 5$.

Solution Substitute $x = \sqrt{5}u - \sqrt{\frac{5}{3}}v$, $y = \sqrt{5}u + \sqrt{\frac{5}{3}}v$ into the ellipse $x^2 - xy + y^2 = 5$, and then simplify



$$\begin{aligned}&(\sqrt{5}u - \sqrt{\frac{5}{3}}v)^2 - (\sqrt{5}u - \sqrt{\frac{5}{3}}v)(\sqrt{5}u + \sqrt{\frac{5}{3}}v) + (\sqrt{5}u + \sqrt{\frac{5}{3}}v)^2 = 5 \\ \Rightarrow\ & 5u^2 + 5v^2 = 5 \\ \Rightarrow\ & u^2 + v^2 = 1\end{aligned}$$

The Jacobian is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{5} & -\sqrt{\frac{5}{3}} \\ \sqrt{5} & \sqrt{\frac{5}{3}} \end{vmatrix} = \frac{10}{\sqrt{3}}$$

Hence

$$\begin{aligned}\iint_D (x^2 - xy + y^2)\, dA &= \iint_{u^2+v^2\le 1} (5u^2 + 5v^2) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \\ &= \frac{10}{\sqrt{3}} \iint_{u^2+v^2\le 1} (5u^2 + 5v^2)\, dudv \\ &\text{(using polar coordinates)} \\ &= \frac{10}{\sqrt{3}} \int_0^{2\pi} \int_0^1 (5r^2) r\, dr d\theta \\ &= \frac{25}{\sqrt{3}}\pi\end{aligned}$$

25. Find the gradient vector field of $f(x, y) = \ln(x + 8y)$.

Solution $\nabla f =< \frac{1}{x+8y}, \frac{8}{x+8y} >$

26. Evaluate $\int_C xy^4\, ds$ where C is the right half of the circle $x^2 + y^2 = 1$ in counter-clockwise direction.

Solution

$$\begin{aligned}\int_C xy^4\, ds &= \int_{-\pi/2}^{\pi/2} \cos t(\sin t)^4 \sqrt{(-\sin t)^2 + (\cos t)^2}\, dt \\ &= \int_{-\pi/2}^{\pi/2} \cos t(\sin t)^4\, dt = \frac{1}{5}(\sin t)^5|_{-\pi/2}^{\pi/2} = \frac{2}{5}\end{aligned}$$

27. Evaluate $\int_C x^2\, dx + y^2\, dy$, where C consists of line segments from $(0, 0)$ to $(1, 2)$ and then from $(1, 2)$ to $(3, 2)$.

Solution 1 The parametric equation for the line segment C_1 from $(0, 0)$ to $(1, 2)$ is $x = t$, $y = 2t$, $0 \leq t \leq 1$. Therefore

$$\int_{C_1} x^2\, dx + y^2\, dy = \int_0^1 t^2\, dt + \int_0^1 (2t)^2\, 2dt = 3$$

The parametric equation for the line segment C_2 from $(1, 2)$ to $(3, 2)$ is $x = 1 + 2t$, $y = 2$, $0 \leq t \leq 1$. Therefore

$$\int_{C_2} x^2\, dx + y^2\, dy = \int_0^1 (1 + 2t)^2\, 2dt + \int_0^1 (2)^2\, 0\, dt = 9 - \frac{1}{3}$$

Hence

$$\int_C x^2\, dx + y^2\, dy = \int_{C_1} x^2\, dx + y^2\, dy + \int_{C_2} x^2\, dx + y^2\, dy = 9 + \frac{8}{3}$$

Solution 2: Notice that $< x^2, y^2 >$ is a conservative vector field, and the potential function is $f(x, y) = (x^3 + y^3)/3$. Applying the Fundamental Theorem on C_1 and C_2 (both are smooth curves) separately, then

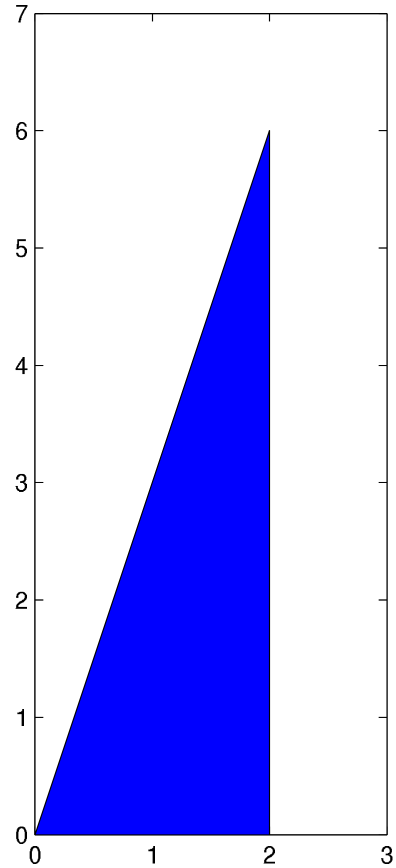
$$\int_C x^2\, dx + y^2\, dy = \int_{C_1} x^2\, dx + y^2\, dy + \int_{C_2} x^2\, dx + y^2\, dy$$
$$=[f(1, 2) - f(0, 0)] + [f(3, 2) - f(1, 2)] = f(3, 2) - f(0, 0) = 9 + \frac{8}{3}$$

28. Evaluate $\int_C yz\, dy + xy\, dz$, where C is given by $x = 4\sqrt{t}$, $y = 5t$, $z = 2t^2$, $0 \leq t \leq 1$.
Solution

$$\begin{aligned}\int_C yz\, dy + xy\, dz &= \int_0^1 (5t)(2t^2)\, 5dt + \int_0^1 (4\sqrt{t})(5t)\, 4tdt \\ &= \frac{25}{2} + \frac{160}{7}\end{aligned}$$

29. Use Green's Theorem to evaluate $\int_C y^2\,dx + 2x\,dy$ where C is the triangle from $(0,0)$ to $(2,6)$ to $(2,0)$ and then back to $(0,0)$.

Solution Notice that the line segments in C are arranged in the negative orientation. According to the Green's Theorem



$$
\begin{aligned}
\int_C y^2\,dx + 2x\,dy &= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\
&= -\iint_D (2-2y)\,dA = -\int_0^2 \int_0^{3x} (2-2y)\,dydx \\
&= -\int_0^2 (6x - 9x^2)\,dx = 12
\end{aligned}
$$