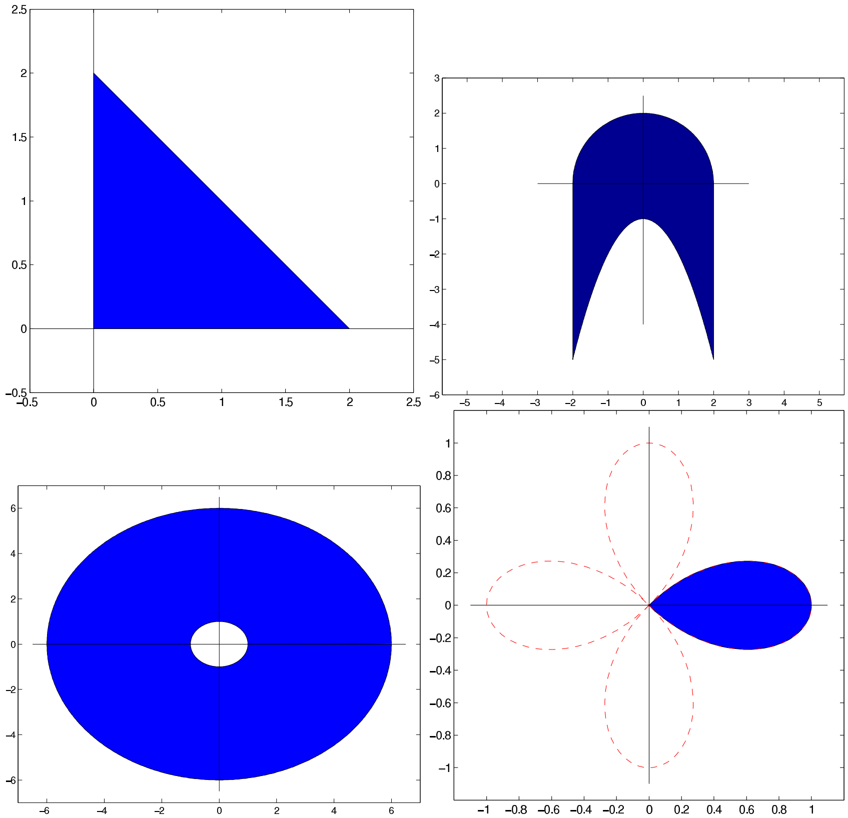
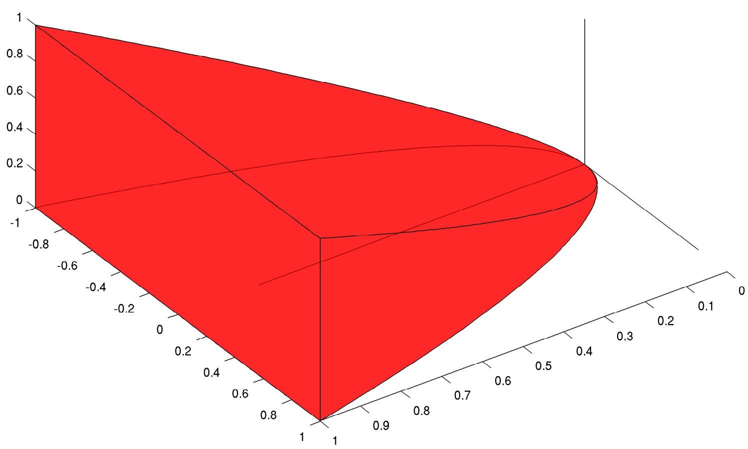
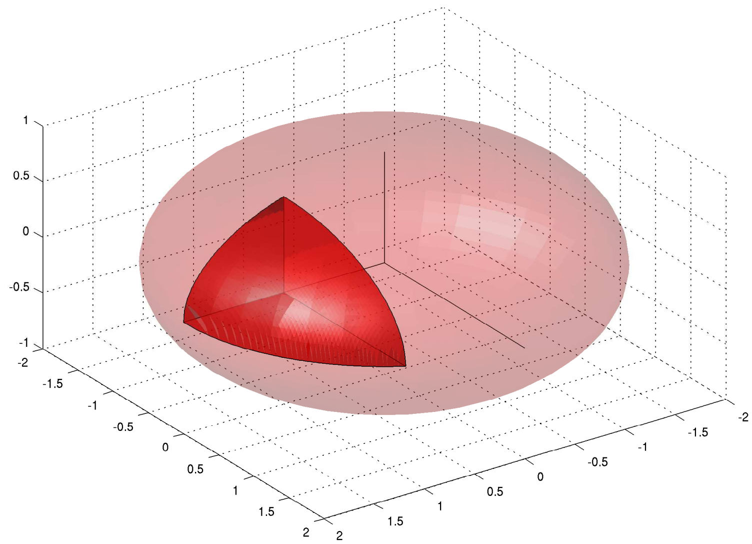
Math 2163, Practice Exam III, Solution

1. 

2. 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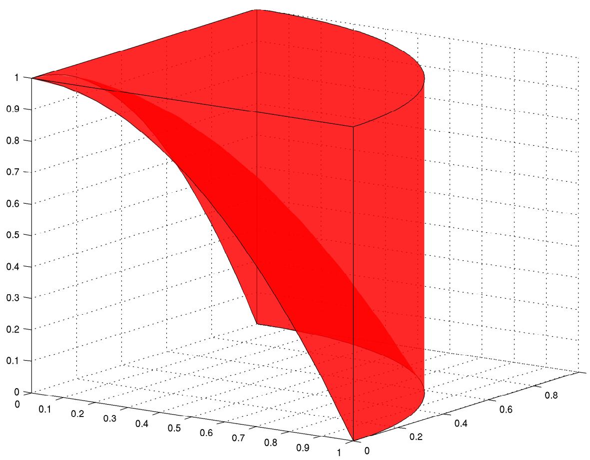
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0.3
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0.2
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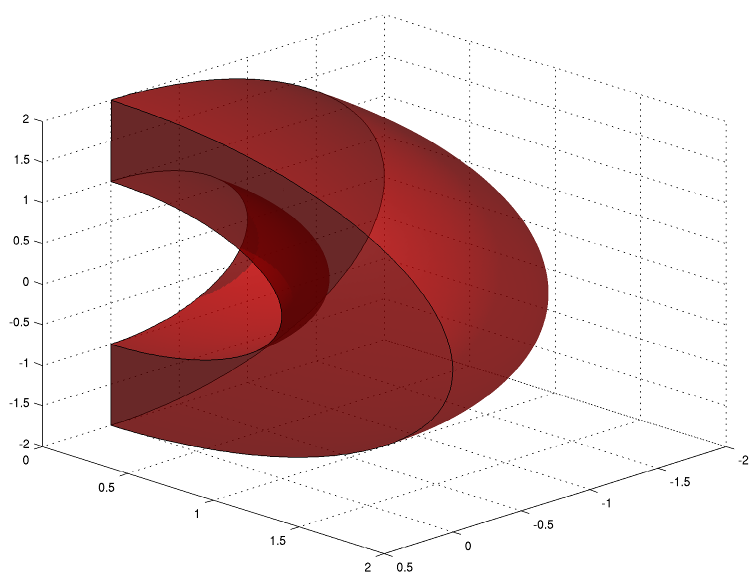
0.6
0.8
1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0
-0.5
-1
-1.5
-2

3. (a)

$$\int_0^1 \int_0^1 ye^{xy}\,dx\,dy = \int_0^1 e^{xy}|_{x=0}^{x=1}\,dy = \int_0^1 (e^y - 1)\,dy = e - 2$$

(b)

$$\begin{aligned}\int_0^1 \int_0^y \int_x^1 6xyz\,dz\,dx\,dy &= \int_0^1 \int_0^y (3xy - 3x^3y)\,dx\,dy \\ &= \int_0^1 (\frac{3}{2}y^3 - \frac{3}{4}y^5)\,dy = \frac{1}{4}\end{aligned}$$

(c)

$$\begin{aligned}\int_0^{\pi/2} \int_0^{\sin 2\theta} r\,dr\,d\theta &= \int_0^{\pi/2} \frac{1}{2}\sin^2(2\theta)\,d\theta \\ &= \int_0^{\pi/2} \frac{1}{4}(1 - \cos 4\theta)\,d\theta = \pi/8\end{aligned}$$

(d)

$$\begin{aligned}\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta &= \int_0^{\pi/2} \int_0^{\pi/2} \frac{7}{3}\sin\phi\,d\phi\,d\theta \\ &= \int_0^{\pi/2} \frac{7}{3}\,d\theta = \frac{7\pi}{6}\end{aligned}$$

4. $D = \{(r, \theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 4 + 3\cos\theta\}$, so

$$\begin{aligned}A(D) &= \iint_D dA = \int_0^{2\pi} \int_0^{4+3\cos\theta} r\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{2}(4 + 3\cos\theta)^2\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (16 + 24\cos\theta + 9\cos^2\theta)\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} (16 + 24\cos\theta + 9(1 + \cos 2\theta)/2)\,d\theta \\ &= \frac{41}{2}\pi\end{aligned}$$

5. Use polar coordinates, $D = \{(r, \theta) \mid 0 \le \theta \le 2\pi,\ 0 \le r \le 3\}$, and

$$f_x = 2x, \qquad f_y = 2y.$$

So

$$A = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\,dA = \int_0^{2\pi} \int_0^3 (\sqrt{4r^2 + 1})r\,dr\,d\theta = \frac{\pi}{6}(37\sqrt{37} - 1)$$

6. **Solution 1:** Use a double integral, D is the intersection of

$$\begin{cases} z = 3x^2 + 3y^2 \\ z = 4 - x^2 - y^2 \end{cases} \quad \Rightarrow \quad x^2 + y^2 = 1.$$

So $D = \{(r, \theta) \,|\, 0 \le \theta \le 2\pi,\ 0 \le r \le 1\}$, and

$$\begin{aligned} V &= \iint_D (\text{top surface} - \text{bottom surface})\, dA \\ &= \int_0^{2\pi} \int_0^1 ((4 - r^2) - 3r^2) r\, dr\, d\theta = 2\pi. \end{aligned}$$

Solution 1: Use triple integral in cylindrical coordinates, then

$$E = \{(r, \theta, z) \,|\, 0 \le \theta \le 2\pi,\ 0 \le r \le 1,\ 3r^2 \le z \le 4 - r^2\}$$

and

$$\begin{aligned} V = \iiint_E dV = \int_0^{2\pi} \int_0^1 \int_{3r^2}^{4-r^2} r\, dz\, dr\, d\theta \\ = 2\pi \end{aligned}$$

7. The tetrahedron is

$$E = \{(x, y, z) \,|\, 0 \le x \le 1,\ 0 \le y \le 2 - 2x,\ 0 \le z \le 3(1 - x - y/2)\}$$

So the mass is

$$mass = \iiint_E \rho\, dV = \int_0^1 \int_0^{2-2x} \int_0^{3(1-x-y/2)} (x^2 + y^2 + z^2)\, dz\, dy\, dx = 7/5$$

and the center of the mass is

$$\begin{aligned} \bar{x} &= \frac{1}{mass} \iiint_E x\rho\, dV = 4/21, \\ \bar{y} &= \frac{1}{mass} \iiint_E y\rho\, dV = 11/21, \\ \bar{z} &= \frac{1}{mass} \iiint_E z\rho\, dV = 8/7. \end{aligned}$$

8. (a)

$$\iiint_E xy\, dV = \int_0^3 \int_0^x \int_0^{x+y} xy\, dz\, dy\, dx = 81/2$$

(b) Use cylindrical coordinates:

$$\begin{aligned}
\iiint_E yz\,dV &= \int_0^{\pi}\int_0^{2}\int_0^{r\sin\theta}(r\sin\theta)zr\,dz\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^{2}\frac{1}{2}r^4\sin^3\theta\,dr\,d\theta \\
&= \frac{16}{5}\int_0^{\pi}\sin^3\theta\,d\theta \\
&= \frac{16}{5}\int_0^{\pi}(1-\cos^2\theta)\sin\theta\,d\theta \\
&= \frac{16}{5}(-\cos\theta+\frac{1}{3}\cos^3\theta)|_0^{\pi} = 64/15
\end{aligned}$$

(c) Use spherical coordinates:

$$\begin{aligned}
\iiint_E z^3\sqrt{x^2+y^2+z^2}\,dV &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^{1}(\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/2}\frac{1}{7}\cos^3\phi\sin\phi\,d\phi\,d\theta \\
&= \pi/14
\end{aligned}$$

9. The Jacobian is

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial s} & \frac{\partial y}{\partial t} \end{vmatrix} \\
&= \begin{vmatrix} e^{s+t} & e^{s+t} \\ e^{s-t} & -e^{s-t} \end{vmatrix} \\
&= e^{s+t}(-e^{s-t}) - e^{s+t}e^{s-t} \\
&= -2e^{s+t}e^{s-t} = -2e^{2s}
\end{aligned}$$

10. First, compute the Jacobian:

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \\
&= \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \\
&= 6
\end{aligned}$$

Second, notice the ellipse becomes

$$9x^2 + 4y^2 = 36 \qquad \Rightarrow \qquad 9(2u)^2 + 4(3v)^2 = 36$$

under the transformation $x = 2u$, $y = 3v$. Simplify it, we end up with

$$36u^2 + 36v^2 = 36 \qquad \Rightarrow \qquad u^2 + v^2 = 1$$

Therefore, the original integral can be written into

$$\iint_R x^2\,dA = \iint_{u^2+v^2\le 1} (2u)^2(6)\,dudv = \iint_{u^2+v^2\le 1} 24u^2\,dudv$$

Using the polar coordinates, we have

$$\iint_{u^2+v^2\le 1} (2u)^2(6)\,dudv = \int_0^{2\pi}\int_0^1 24(r\cos\theta)^2 r\,drd\theta = \cdots = 6\pi$$

11.

$$\begin{aligned}\nabla f(x,y) &= f_x\mathbf{i} + f_y\mathbf{j}\\ &= (xe^{xy}y + e^{xy})\mathbf{i} + (xe^{xy}x)\mathbf{j}\\ &= (xy+1)e^{xy}\mathbf{i} + x^2e^{xy}\mathbf{j}\end{aligned}$$

The sketch is omitted.

12.

$$\begin{aligned}\nabla f(x,y,z) &= f_x\mathbf{i} + f_y\mathbf{j} + f_x\mathbf{k}\\ &= \frac{x}{\sqrt{x^2+y^2+z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2+y^2+z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2+y^2+z^2}}\mathbf{k}\end{aligned}$$