Math 2163, Exam III, Nov. 23, 2010

Name: ____________________

Score:

1. (6 points) Sketch the region on which the given double integral is defined. Then evaluate the integral:

$$\int_1^4 \int_y^4 \sqrt{x+y}\, dx\, dy$$

Solution The graph is omitted.

$$\begin{aligned}
\int_1^4 \int_y^4 \sqrt{x+y}\, dx\, dy &= \int_1^4 \left(\frac{2}{3}(x+y)^{3/2} \Big|_{x=y}^{x=4} \right) dy \\
&= \frac{2}{3}\int_1^4 \left((y+4)^{3/2} - (2y)^{3/2} \right) dy \\
&= \frac{2}{3}\left(\frac{2}{5}(y+4)^{5/2} - 2^{3/2}\frac{2}{5}y^{5/2} \right)\Big|_{y=1}^{y=4} \\
&= \cdots = \frac{4}{15}(66\sqrt{2} - 25\sqrt{5})
\end{aligned}$$

2. (6 points) The boundary of a thin plate consists of the semicircles $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ together with the portions of x-axis that join them. Find the center of mass of the thin plate if the density function is $\rho(x, y) = 5\sqrt{x^2+y^2}$.

Solution The domain can be written using the polar coordinates

$$D = \{(r,\, \theta) |\, 0 \le \theta \le \pi,\, 1 \le r \le 2\}$$

The density function can be written into

$$\rho = 5\sqrt{x^2+y^2} = 5r.$$

Hence we have the mass m

$$\begin{aligned} m &= \iint_D \rho\, dA = \int_0^\pi \int_1^2 5r \cdot r\, dr d\theta \\ &= \int_0^\pi \int_1^2 5r^2\, dr d\theta = \frac{35}{3}\pi \end{aligned}$$

Then the center of mass is $(\bar{x},\, \bar{y})$ where

$$\begin{aligned} \bar{x} &= \frac{1}{m} \iint_D x\rho\, dA = \frac{1}{\frac{35}{3}\pi} \int_0^\pi \int_1^2 5r^3 \cos\theta\, dr d\theta \\ &= \cdots = 0 \end{aligned}$$

$$\begin{aligned} \bar{y} &= \frac{1}{m} \iint_D y\rho\, dA = \frac{1}{\frac{35}{3}\pi} \int_0^\pi \int_1^2 5r^3 \sin\theta\, dr d\theta \\ &= \cdots = \frac{45}{14\pi} \end{aligned}$$

The center of mass is $(0,\, \frac{45}{14\pi})$.

3. (6 points) Compute

$$\iiint_E x^2\,dV$$

where E is the solid tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$.

Solution The region can be written as

$$E = \{(x,y,z)|\, 0 \le x \le 1,\ 0 \le y \le 1-x,\ 0 \le x \le 1-x-y\}.$$

Hence

$$\begin{aligned}
\iiint_E x^2\,dV &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^2\,dzdydx \\
&= \int_0^1 \int_0^{1-x} x^2 z|_{z=0}^{z=1-x-y}\,dydx \\
&= \int_0^1 \int_0^{1-x} x^2(1-x-y)\,dydx \\
&= \int_0^1 \int_0^{1-x} \big(x^2(1-x) - x^2 y\big)\,dydx \\
&= \int_0^1 \left(x^2(1-x)y - x^2\frac{y^2}{2}\right)\Bigg|_{y=0}^{y=1-x}\,dx \\
&= \int_0^1 \left(x^2(1-x)^2 - x^2\frac{(1-x)^2}{2}\right)\,dx \\
&= \int_0^1 \frac{1}{2}x^2(1-x)^2\,dx \\
&= \cdots = \frac{1}{60}
\end{aligned}$$

4. (6 points) Evaluate the triple integral $\iiint_E 3x dV$ where E is the region lies above the xy-plane, under the plane $z = 5 + y$, bounded by the cylinder $x^2 + y^2 = 4$ and is inside the first octant.

Solution The region can be written in cylindrical coordinates as

$$E = \{(r,\, \theta,\, z) |\, 0 \le \theta \le \pi/2,\, 0 \le r \le 2,\, 0 \le z \le 5 + r\sin\theta\}$$

Hence

$$\begin{aligned}
\iiint_E 3x\, dV &= \int_0^{\pi/2}\int_0^2\int_0^{5+r\sin\theta} 3r\cos\theta \cdot r\, dzdrd\theta \\
&= \int_0^{\pi/2}\int_0^2\int_0^{5+r\sin\theta} 3r^2\cos\theta\, dzdrd\theta \\
&= \int_0^{\pi/2}\int_0^2 3r^2\cos\theta(5 + r\sin\theta)\, drd\theta \\
&= \int_0^{\pi/2}\int_0^2 \left(15r^2\cos\theta + 3r^3\cos\theta\sin\theta\right)\, drd\theta \\
&= \int_0^{\pi/2} \left(5r^3\cos\theta + 3\frac{r^4}{4}\cos\theta\sin\theta\right)\Bigg|_{r=0}^{r=2} d\theta \\
&= \int_0^{\pi/2} (40\cos\theta + 12\sin\theta\cos\theta)\, d\theta \\
&= \left(40\sin\theta + 6\sin^2\theta\right)\Big|_{\theta=0}^{\theta=\pi/2} \\
&= 46
\end{aligned}$$

5. (6 points) Use spherical coordinates to find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 9$, above the xy-plane and below the cone $z = \sqrt{x^2 + y^2}$.

Solution The region can be written in spherical coordinates as

$$E = \{(\rho, \theta, \phi) |\, 0 \leq \theta \leq 2\pi,\ \pi/4 \leq \phi \leq \pi/2,\ 0 \leq \rho \leq 3\}$$

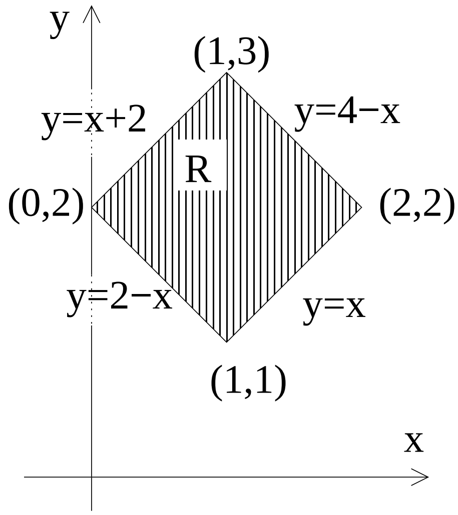
Then

$$\begin{aligned} V &= \iiint_E dV = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^3 \rho^2 \sin\phi \, d\rho d\phi d\theta \\ &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \left(\frac{\rho^3}{3} \sin\phi \right) \Bigg|_{\rho=0}^{\rho=3} d\phi d\theta \\ &= \int_0^{2\pi} \int_{\pi/4}^{\pi/2} 9 \sin\phi \, d\phi d\theta \\ &= 9 \int_0^{2\pi} (-\cos\phi)|_{\phi=\pi/4}^{\phi=\pi/2} \, d\theta \\ &= 9 \int_0^{2\pi} \frac{\sqrt{2}}{2} \, d\theta \\ &= 9\sqrt{2}\pi \end{aligned}$$

6. (8 points) Use the transformation $u = x - y$, $v = x + y$ to evaluate

$$\iint_R (x-y)\,dA$$

where R is the square with vertices $(0,2)$, $(1,1)$, $(2,2)$ and $(1,3)$. (The graph of R and the equation of its four edges are given in the figure below)



Solution Notice that

$$\begin{cases} u = x - y \\ v = x + y \end{cases} \qquad \Rightarrow \qquad \begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$$

Hence the Jacobian is

$$J = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = 1/2$$

The boundaries of the rectangle becomes

$$\begin{aligned} y = x &\qquad \Rightarrow \qquad u = 0 \\ y = 4 - x &\qquad \Rightarrow \qquad v = 4 \\ y = x + 2 &\qquad \Rightarrow \qquad u = -2 \\ y = 2 - x &\qquad \Rightarrow \qquad v = 2 \end{aligned}$$

Hence

$$\iint_R (x-y)\,dA = \int_{-2}^{0}\int_{2}^{4} u|J|\,dvdu = \frac{1}{2}\int_{-2}^{0}\int_{2}^{4} u\,dvdu = -2$$

7. (6 points) Find the Jacobian of the transformation

$$x = v + w^2, \quad y = w + u^2, \quad z = u + v^2$$

Solution

$$J = \begin{vmatrix} 0 & 1 & 2w \\ 2u & 0 & 1 \\ 1 & 2v & 0 \end{vmatrix} = 1 + 8uvw$$

8. (6 points) Find the gradient vector field of $f(x, y) = xy$ and sketch this vector field.

Solution $\nabla f = < y,\, x >$ and the vector field is