Math 2163, Exam I, Sept. 23, 2010

Name: ____________________

Score:

Please read the instructions on each problem carefully, and indicate answers as directed. Show details in your work. If you simply give a solution without steps of how you derive this solution, you may not get credit for it.

The total is 50 points

1. (6 points) Calculate the following:

   (a) $(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{k})$

   (b) $< 1,\ -2,\ 4 > \times < -3,\ 2,\ -1 >$

   (c) $|2 < 1,\ 4,\ -1 > |$

   Solution

   (a) $(\mathbf{i} - \mathbf{j} + 3\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{k}) = (1)(2) + (-1)(0) + (3)(-3) = 2 - 9 = -7$

   (b)

$$
\begin{aligned}
& < 1,\ -2,\ 4 > \times < -3,\ 2,\ -1 > \\
= & \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 4 \\ -3 & 2 & -1 \end{vmatrix} \\
= & \mathbf{i}(-2 \times (-1) - 2 \times 4) - \mathbf{j}(1 \times (-1) - 4 \times (-3)) + \mathbf{k}(1 \times 2 - (-2) \times (-3)) \\
= & -6\mathbf{i} - 11\mathbf{j} - 4\mathbf{k}
\end{aligned}
$$

   (c) $|2 < 1,\ 4,\ -1 > | = | < 2,\ 8,\ -2 > | = \sqrt{4 + 64 + 4} = \sqrt{72} = 6\sqrt{2}$

2. (5 points) Find an equation for the line that passes through the point $(2, -1, 3)$ and is parallel to the line $\frac{2x-3}{4} = \frac{4-y}{3} = \frac{z+2}{5}$.

Solution Notice that

$$\begin{aligned} & \frac{2x-3}{4} = \frac{4-y}{3} = \frac{z+2}{5} \\ \Rightarrow \quad & \frac{x-3/2}{2} = \frac{y-4}{-3} = \frac{z+2}{5} \end{aligned}$$

The given line has direction vector $< 2, -3, 5 >$. Hence the line parallel to it and passing through $(2, -1, 3)$ is

$$\frac{x-2}{2} = \frac{y+1}{-3} = \frac{z-3}{5}$$

It can also be written in the parametric form

$$\begin{cases} x = 2 + 2t \\ y = -1 - 3t \\ z = 3 + 5t \end{cases}$$

3. (5 points) Find an equation for the plane that passes through the point $(2,\ -1,\ 3)$ and is parallel to the plane $z = 3x + 4y - 5$.

Solution The give plane is

$$\begin{aligned} & z = 3x + 4y - 5 \\ \Rightarrow\quad & 3x + 4y - z = 5 \end{aligned}$$

It has normal vector $< 3,\ 4,\ -1 >$. The plane parallel to it should also have normal vector $< 3,\ 4,\ -1 >$. Therefore, we have

$$3(x-2) + 4(y+1) - (z-3) = 0$$

which can also be written into

$$3x + 4y - z = -1$$

4. (5 points) Find the length of the curve $x = 1+\sqrt{8}t$, $y = t^2$, $z = \frac{t^3}{3}+t$, for $0 \le t \le 1$.

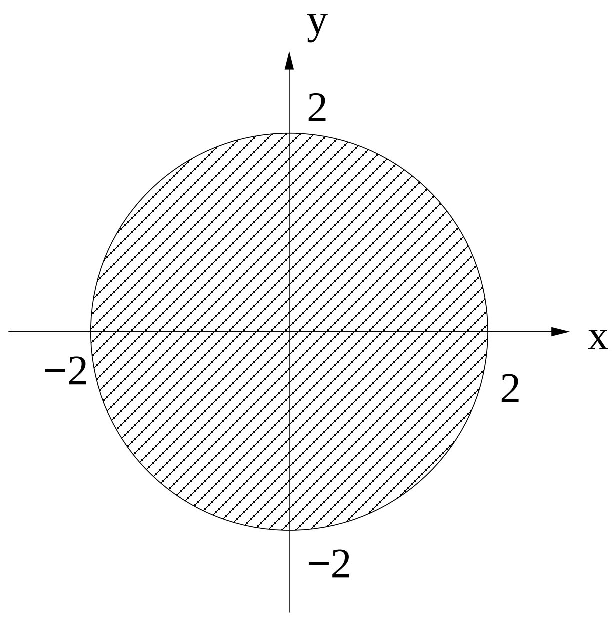
Solution

$$
\begin{aligned}
L &= \int_0^1 |\mathbf{r}'(t)|\,dt \\
&= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt \\
&= \int_0^1 \sqrt{(\sqrt{8})^2 + (2t)^2 + (t^2+1)^2}\,dt \\
&= \int_0^1 \sqrt{8 + 4t^2 + t^4 + 2t^2 + 1}\,dt \\
&= \int_0^1 \sqrt{t^4 + 6t^2 + 9}\,dt \\
&= \int_0^1 \sqrt{(t^2+3)^2}\,dt \\
&= \int_0^1 (t^2+3)\,dt \\
&= (\frac{t^3}{3} + 3t)|_0^1 \\
&= \frac{1}{3} + 3
\end{aligned}
$$

5. (4 points) Find the domain of $f(x,y) = \sqrt{16 - 4x^2 - 4y^2}$. Sketch the graph of the domain in the xy-plane.

Solution The domain is

$$\begin{aligned} D &= \{(x,y)|\, 16 - 4x^2 - 4y^2 \geq 0\} \\ &= \{(x,y)|\, 4x^2 + 4y^2 \leq 16\} \\ &= \{(x,y)|\, x^2 + y^2 \leq 4\} \end{aligned}$$



6. (5 points) Use polar coordinates to compute

$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2}$$

Solution By using the polar coordinate, we have $r^2 = x^2 + y^2$. Therefore

$$\begin{aligned}
& \lim_{(x,y)\to(0,0)} \frac{e^{-x^2-y^2}-1}{x^2+y^2} \\
& = \lim_{r\to 0} \frac{e^{-r^2}-1}{r^2} \\
(L'Hospital) \quad & = \lim_{r\to 0} \frac{-2re^{-r^2}}{2r} \\
& = \lim_{r\to 0} (-e^{-r^2}) \\
& = -1
\end{aligned}$$

7. (5 points) Find all first order derivative of $f(x, y) = \ln(x^2 - 2y)$.

Solution

$$f_x = \frac{2x}{x^2 - 2y}$$

$$f_x = \frac{-2}{x^2 - 2y}$$

8. (5 points) Calculate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, where z is defined implicitly by

$$x^2 + y^2 + z^2 = 3xyz$$

Solution For $\frac{\partial z}{\partial x}$, we first take derivative of both side of the equation with respect to x,

$$\begin{aligned} & \frac{\partial(x^2+y^2+z^2)}{\partial x} = \frac{\partial(3xyz)}{\partial x} \\ \Rightarrow \quad & 2x + 2z\frac{\partial z}{\partial x} = 3yz + 3xy\frac{\partial z}{\partial x} \\ \Rightarrow \quad & \frac{\partial z}{\partial x} = \frac{2x - 3yz}{3xy - 2z} \end{aligned}$$

For $\frac{\partial z}{\partial y}$, similarly we have

$$\begin{aligned} & \frac{\partial(x^2+y^2+z^2)}{\partial y} = \frac{\partial(3xyz)}{\partial y} \\ \Rightarrow \quad & 2y + 2z\frac{\partial z}{\partial y} = 3xz + 3xy\frac{\partial z}{\partial y} \\ \Rightarrow \quad & \frac{\partial z}{\partial y} = \frac{2y - 3xz}{3xy - 2z} \end{aligned}$$

9. (5 points) Find an equation of the tangent plane to the surface $z = y\cos(x - y)$ at point $(2, 2, 2)$.

Solution The normal vector is defined by

$$\mathbf{n} =< f_x,\, f_y,\, -1 >=< -y\sin(x - y),\, cos(x - y) + y\sin(x - y),\, -1 >$$

At point $(2, 2, 2)$, the normal vector is

$$\mathbf{n} =< -2\sin(2 - 2),\, cos(2 - 2) + 2\sin(2 - 2),\, -1 >=< 0,\, 1,\, -1 >$$

Therefore, the tangent plane is

$$0 \times (x - 2) + 1 \times (y - 2) - 1 \times (z - 2) = 0$$

which can be simplified into

$$y - z = 0$$

10. (5 points) Given $z = u\sqrt{v-w}$, find $\frac{\partial^3 z}{\partial u \partial v \partial w}$.

Solution

$$\frac{\partial z}{\partial u} = \sqrt{v-w}$$

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial \sqrt{v-w}}{\partial v} = \frac{1}{2}(v-w)^{-1/2}$$

$$\frac{\partial^3 z}{\partial u \partial v \partial w} = \frac{\partial \frac{1}{2}(v-w)^{-1/2}}{\partial w} = \frac{1}{4}(v-w)^{-3/2}$$