Math 2163, Practice Final Exam

1. **Please read through previous practice exams. Problem types that have appeared in previous practice exams will not be repeated here!**

2. The angle formed by edges AB and AC is given as the angles between vectors:

$$\begin{aligned} AB &: \ (3,\, 6) - (1,\, 0) = <2,\, 6> \\ AC &: \ (-1,\, 4) - (1,\, 0) = <-2,\, 4> \end{aligned}$$

Therefore the angle is

$$\angle A = \arccos \frac{<2,\, 6> \cdot <-2,\, 4>}{|<2,\, 6>|\,|<-2,\, 4>|} = \arccos \frac{20}{\sqrt{40}\sqrt{20}} = \arccos \frac{1}{\sqrt{2}} = \pi/4$$

Similarly, we can find the angle between BA and BC is

$$\angle B = \arccos \frac{<-2,\, -6> \cdot <-4,\, -2>}{|<-2,\, -6>|\,|<-4,\, -2>|} = \arccos \frac{20}{\sqrt{40}\sqrt{20}} = \pi/4$$

and the angle between CA and CB is

$$\angle C = \arccos \frac{<2,\, -4> \cdot <4,\, 2>}{|<2,\, -4>|\,|<4,\, 2>|} = \arccos \frac{0}{\sqrt{20}\sqrt{20}} = \arccos 0 = \pi/2$$

Therefore, the triangle is right-angled.

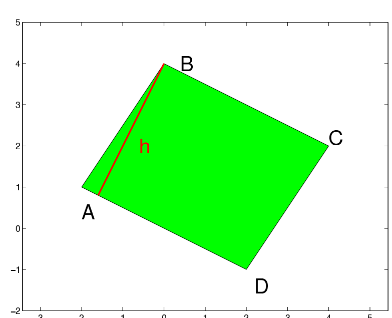
3. The vectors are orthogonal if

$$<-6,\, b,\, 2> \cdot <b,\, b^2,\, b> = 0$$

which means

$$\begin{aligned} & -6b + b^3 + 2b = 0 \\ \Longrightarrow \quad & b^3 - 4b = 0 \\ \Longrightarrow \quad & b(b^2 - 4) = 0 \\ \Longrightarrow \quad & b(b-2)(b+2) = 0 \\ \Longrightarrow \quad & b = 0 \text{ or } b = 2 \text{ or } b = -2 \end{aligned}$$

4. **Solution 1:**



First we have vectors

$$AB = <0,4> - <-2,1> = <2,3>$$
$$AD = <2,-1> - <-2,1> = <4,-2>$$

Then $\cos\angle A$ can be calculated by

$$\cos\angle A = \frac{<2,3>\cdot<4,-2>}{|<2,3>|\,|<4,-2>|} = \frac{2}{\sqrt{13}\sqrt{20}} = \frac{1}{\sqrt{65}}$$

Hence

$$\sin\angle A = \sqrt{1-\cos^2\angle A} = \sqrt{1-\frac{1}{65}} = \frac{8}{\sqrt{65}}$$

The area of the parallelogram is equal to length$\times$height, where the length of AD is

$$|AD| = |<4,-2>| = \sqrt{20}$$

and the height h is

$$h = |AB|\sin\angle A = |<2,3>|\frac{8}{\sqrt{65}} = \sqrt{13}\frac{8}{\sqrt{65}} = \frac{8}{\sqrt{5}}$$

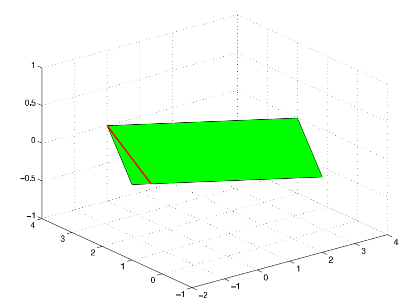
Then we have

$$AREA = |AD|h = \sqrt{20}\frac{8}{\sqrt{5}} = 16$$

Solution 2: Recall that for three-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$,

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin\theta$$

which is exactly the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$. However, here we only have two-dimensional vectors AB and AD. To make them 3-D, we assume they both lie in the xy-plane.



Then 3-D vectors AB and AD are

$$AB = <2,3,0>$$
$$AD = <4,-2,0>$$

and hence

$$AREA = |<2,3,0>\times<4,-2,0>| = |<0,0,-16>| = 16$$

5. The line segment between two points is given by

$$\begin{aligned}\mathbf{r}(t) &= A + t(B-A) = (1-t)A + tB \\ &= (1-t) < 10,3,1 > +t < 5,6,-3 > \\ &=< 10-10t, 3-3t, 1-t > + < 5t, 6t, -3t > \\ &=< 10-5t, 3+3t, 1-4t >\end{aligned}$$

for $0 \le t \le 1$. So the parametric equation is

$$\begin{cases} x = 10-5t \\ y = 3+3t \qquad\qquad 0 \le t \le 1 \\ z = 1-4t \end{cases}$$

6. We will randomly take two different point from the given line. Notice both $(1,2,3)$, $(-1,-2,-3)$ are on the line $x = y/2 = z/3$. Now we have three points on the plane, this gives us two vectors

$$\begin{aligned}&(1,2,3) - (-1,2,-1) =< 2,0,4 > \\ &(-1,-2,-3) - (-1,2,-1) =< 0,-4,-2 >\end{aligned}$$

Then the normal vector of the plane is

$$\mathbf{n} =< 2,0,4 > \times < 0,-4,-2 >=< 16,4,-8 >$$

So the plane is

$$\begin{aligned}&\mathbf{n} \cdot (< x,y,z > - < -1,2,-1 >) = 0 \\ \Longrightarrow\quad &16(x+1) + 4(y-2) - 8(z+1) = 0\end{aligned}$$

7. θ in the cylindrical and spherical coordinates are the same. z in the rectangular and cylindrical coordinates are the same.

   (a) Cylindrical: $r = \sqrt{x^2+y^2} = 3\sqrt{2}$, $\theta = \arctan(y/x) = \arctan 1 = \pi/4$, $z = -2$.
   Spherical: $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{22}$, $\theta = \arctan(y/x) = \arctan 1 = \pi/4$, $\phi = \arccos(z/\rho) = \arccos \frac{-2}{\sqrt{22}}$

   (b) Rectangular: $x = r\cos\theta = \frac{\sqrt{3}}{2}$, $y = r\sin\theta = 1/2$, $z = \sqrt{3}$.
   Spherical: $\rho = \sqrt{x^2+y^2+z^2} = 2$, $\theta = \pi/6$, $\phi = \arccos(z/\rho) = \arccos\sqrt{3}/2 = \pi/6$.

   (c) Rectangular: $x = \rho\sin\phi\cos\theta = 4\sin\pi/3\cos\pi/4 = \sqrt{6}$, $y = \rho\sin\phi\sin\theta = \sqrt{6}$, $z = \rho\cos\phi = 2$.
   Cylindrical: $r = \sqrt{x^2+y^2} = \sqrt{12}$, $\theta = \pi/4$, $z = 2$.

8. Since $x^2+xy+y^2 \geq 0$ and it is equal to 0 only when $(x,y)=(0,0)$, so we only need to check the continuity at the origin. Notice that

$$\lim_{x=0,\, y\to 0} \frac{xy}{x^2+xy+y^2} = \lim_{x=0,\, y\to 0} 0 = 0$$
$$\lim_{y=0,\, x\to 0} \frac{xy}{x^2+xy+y^2} = \lim_{y=0,\, x\to 0} 0 = 0$$
$$\lim_{x=y,\, x,y\to 0} \frac{xy}{x^2+xy+y^2} = \lim_{x=y,\, x,y\to 0} \frac{x^2}{x^2+xx+x^2} = \frac{1}{3}$$

We get different limits when approaching $(0,0)$ along different pathes. So the limit does not exist. And hence The function is not continuous at $(0,0)$. The function is continuous everywhere else except $(0,0)$.

9. Clearly, we have

$$u_t = -4e^{-4t}\sin x$$
$$u_x = e^{-4t}\cos x$$
$$u_{xx} = (u_x)_x = -e^{-4t}\sin x$$

Then it is clear to see that $u_t = -4e^{-4t}\sin x = 4u_{xx}$.

10. Given a surface $z=f(x,y)$, then the the tangent plane at point (x_0,y_0,z_0) is

$$f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)-(z-z_0)=0$$

Now the given point is $(x_0,y_0,z_0)=(3,6,5)$, all we need to do is to calculate $f_x = \frac{\partial z}{\partial x}$ and $f_y = \frac{\partial z}{\partial y}$ at point (x_0,y_0,z_0). Define $F(x,y,z)=5x^2+3y^2+8z^2-353=0$, by the Implicit Function Theorem,

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{10x}{16z}$$
$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{6y}{16z}$$

Hence at point $(x_0,y_0,z_0)=(3,6,5)$,

$$f_x(3,6) = -\frac{10(3)}{16(5)} = -\frac{3}{8}$$
$$f_y(3,6) = -\frac{6(6)}{16(5)} = -\frac{9}{20}$$

And the tangent plane is

$$-\frac{3}{8}(x-3)-\frac{9}{20}(y-6)-(z-5)=0$$

11. Let $F(t)$ be the antiderivative of $\cos t^8$, that is, $F'(t)=\cos t^8$. Then $f(x,y)=F(x)-F(y)$ and hence

$$f_x = F'(x)-0 = \cos x^8$$
$$f_y = 0-F'(y) = -\cos y^8$$

12. First we calculate the critical points.

$$\begin{cases} f_x = 6xy - 6x = 0 \\ f_y = 3x^2 + 3y^2 - 6y = 0 \end{cases}$$

By solving the first equation, we have either $x = 0$ or $y = 1$. First, if $x = 0$, substitute it into the second equation gives $3y^2 - 6y = 0$ which implies $y = 0$ or $y = 2$. So we have two critical points $(0, 0)$ and $(0, 2)$. Second, if $y = 1$, substitute it into the second equation gives $3x^2 - 3 = 0$ which implies $x = 1$ or $x = -1$. This gives another two critical points $(1, 1)$ and $(-1, 1)$. Combine all the above, we have four critical points $(0, 0)$, $(0, 2)$, $(1, 1)$ and $(-1, 1)$.

Now we classify these critical points. Clearly

$$f_{xx} = 6y - 6, \quad f_{xy} = 6x, \quad f_{yy} = 6y - 6$$

By the formula

$$D(x_0, y_0) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = (6y_0 - 6)^2 - (6x_0)^2$$

We have

$D(0, 0) = 36 > 0, \qquad f_{xx}(0, 0) = -6 < 0 \qquad \Rightarrow f(0, 0)$ is local maximum
$D(0, 2) = 36 > 0, \qquad f_{xx}(0, 2) = 12 > 0 \qquad \Rightarrow f(0, 2)$ is local minimum
$D(1, 1) = -36 < 0, \qquad (1, 1)$ is a saddle point
$D(-1, 1) = -36 < 0, \qquad (-1, 1)$ is a saddle point

Finally, calculate the local maximum $f(0, 0) = 2$, and local minimum $f(0, 2) = -2$.

13. We need to calculate the maximum value of

$$P(p, q, r) = 2pq + 2pr + 2rq$$

under the constraint

$$g(p, q, r) = p + q + r = 1$$

Use the Lagangre multiplier method,

$$\begin{cases} \nabla P = \lambda \nabla g \\ g(p, q, r) = 1 \end{cases} \quad \Rightarrow \begin{cases} 2q + 2r = \lambda \\ 2p + 2r = \lambda \\ 2p + 2q = \lambda \\ p + q + r = 1 \end{cases}$$

It is easy to see that the solution is $p = q = r = 1/3$ and $\lambda = 4/3$. Hence the maximum value is

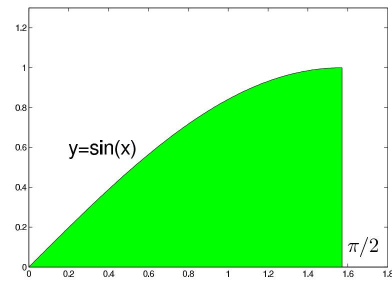
$$P(1/3, 1/3, 1/3) = 2/9 + 2/9 + 2/9 = 2/3$$

14. Think of $(1-x^2-y^2-z^2)$ as the "weight" function. When the "weight" is positive, it will add to the total integral. And when the "weight" is negative, it will lower the total integral. To achieve maximum value of the integral, we only want to integrate on regions with positive "weight", which is

$$\{(x,y,z)|\, 1-x^2-y^2-z^2>0\} = \{(x,y,z)|\, x^2+y^2+z^2<1\}$$

In other words, the region is bounded inside the unit ball.

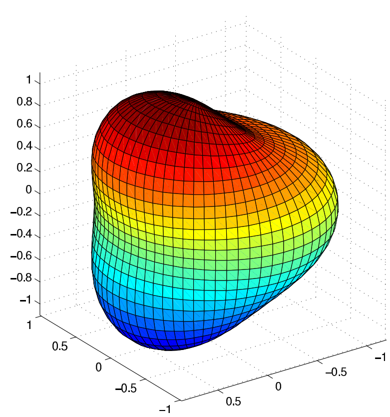
15. The region of the integral is shown in the graph.



Then

$$\begin{aligned}
&\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x\sqrt{3+\cos^2 x}\, dxdy \\
&= \int_0^{\pi/2} \int_0^{\sin x} \cos x\sqrt{3+\cos^2 x}\, dydx \\
&= \int_0^{\pi/2} \cos x\sqrt{3+\cos^2 x}\, y|_0^{\sin x}\, dx \\
&= \int_0^{\pi/2} \cos x\sqrt{3+\cos^2 x}\, \sin x\, dx \\
&\quad (\text{ Let } u=\cos x, \text{ then } du=-\sin x\, dx) \\
&= \int_1^0 u\sqrt{3+u^2}(-du) \\
&= -\frac{1}{3}(3+u^2)^{3/2}|_1^0 = \frac{8}{3}-\sqrt{3}
\end{aligned}$$

16. Using the spherical coordinates, the volumn is

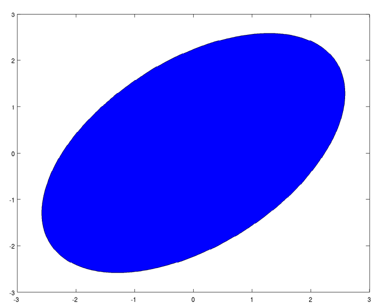


$$
\begin{aligned}
\iiint_E dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^{1+\frac{1}{5}\sin\theta\sin 3\phi} \rho^2 \sin\phi \, d\rho d\theta d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}(1+\frac{1}{5}\sin\theta\sin 3\phi)^3 \sin\phi \, d\theta d\phi \\
&= \int_0^{\pi} \frac{\sin\phi}{3}(\theta - \frac{3}{5}\cos\theta\sin 3\phi + \frac{3}{50}(\theta - \frac{\sin 2\theta}{2})\sin^2 3\phi \\
&\qquad - \frac{1}{125}\sin^3 3\phi\sin\phi(\cos\theta - \frac{\cos^3\theta}{3}))|_0^{2\pi} \, d\phi \\
&= \cdots = \frac{3608}{2625}\pi
\end{aligned}
$$

17. The Jacobian is

$$
\begin{aligned}
\frac{\partial(x,y)}{\partial(\alpha,\beta)} &= \begin{vmatrix} x_\alpha & x_\beta \\ y_\alpha & y_\beta \end{vmatrix} \\
&= \begin{vmatrix} 5\sin\beta & 5\alpha\cos\beta \\ 4\cos\beta & -4\alpha\sin\beta \end{vmatrix} = -20\alpha\sin^2\beta - 20\alpha\cos^2\beta = -20\alpha
\end{aligned}
$$

18. Substitute $x = \sqrt{5}u - \sqrt{\frac{5}{3}}v$, $y = \sqrt{5}u + \sqrt{\frac{5}{3}}v$ into the ellipse $x^2 - xy + y^2 = 5$, and then simplify



$$
\begin{aligned}
&(\sqrt{5}u - \sqrt{\frac{5}{3}}v)^2 - (\sqrt{5}u - \sqrt{\frac{5}{3}}v)(\sqrt{5}u + \sqrt{\frac{5}{3}}v) + (\sqrt{5}u + \sqrt{\frac{5}{3}}v)^2 = 5 \\
&\Rightarrow 5u^2 + 5v^2 = 5 \\
&\Rightarrow u^2 + v^2 = 1
\end{aligned}
$$

The Jacobian is

$$
\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \sqrt{5} & -\sqrt{\frac{5}{3}} \\ \sqrt{5} & \sqrt{\frac{5}{3}} \end{vmatrix} = \frac{10}{\sqrt{3}}
$$

Hence

$$\begin{aligned}\iint_D (x^2 - xy + y^2)\,dA &= \iint_{u^2+v^2\le 1} (5u^2+5v^2)\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,dudv \\ &= \frac{10}{\sqrt{3}}\iint_{u^2+v^2\le 1}(5u^2+5v^2)\,dudv \\ &\text{(using polar coordinates)} \\ &= \frac{10}{\sqrt{3}}\int_0^{2\pi}\int_0^1 (5r^2)r\,drd\theta \\ &= \frac{25}{\sqrt{3}}\pi\end{aligned}$$

19. $\nabla f = < \frac{1}{x+8y}, \frac{8}{x+8y} >$

20.

$$\begin{aligned}\int_C xy^4\,ds &= \int_{-\pi/2}^{\pi/2}\cos t(\sin t)^4\sqrt{(-\sin t)^2+(\cos t)^2}\,dt \\ &= \int_{-\pi/2}^{\pi/2}\cos t(\sin t)^4\,dt = \frac{1}{5}(\sin t)^5|_{-\pi/2}^{\pi/2} = \frac{2}{5}\end{aligned}$$

21.

$$\begin{aligned}mass &= \int_C \rho(x,y)\,ds = \int_C (x+y)\,ds \\ &= \int_0^{\pi/2}(3\cos t+3\sin t)\sqrt{(-3\sin t)^2+(3\cos t)^2}\,dt \\ &= 9\int_0^{\pi/2}(\cos t+\sin t)\,dt = 9(\sin t-\cos t)|_0^{\pi/2} = 18\end{aligned}$$

22. **Solution 1:** The parametric equation for the line segment C_1 from $(0,0)$ to $(1,2)$ is $x = t$, $y = 2t$, $0 \le t \le 1$. Therefore

$$\int_{C_1} x^2\,dx + y^2\,dy = \int_0^1 t^2\,dt + \int_0^1 (2t)^2\,2dt = 3$$

The parametric equation for the line segment C_2 from $(1,2)$ to $(3,2)$ is $x = 1+2t$, $y = 2$, $0 \le t \le 1$. Therefore

$$\int_{C_2} x^2\,dx + y^2\,dy = \int_0^1 (1+2t)^2\,2dt + \int_0^1 (2)^2\,0\,dt = 9 - \frac{1}{3}$$

Hence

$$\int_C x^2\,dx + y^2\,dy = \int_{C_1} x^2\,dx + y^2\,dy + \int_{C_2} x^2\,dx + y^2\,dy = 9 + \frac{8}{3}$$

Solution 2: Notice that $< x^2, y^2 >$ is a conservative vector field, and the potential function is $f(x,y) = (x^3 + y^3)/3$. Applying the Fundamental Theorem on C_1 and C_2 (both are smooth curves) seperately, then

$$\int_C x^2\,dx + y^2\,dy = \int_{C_1} x^2\,dx + y^2\,dy + \int_{C_2} x^2\,dx + y^2\,dy$$
$$=[f(1,2) - f(0,0)] + [f(3,2) - f(1,2)] = f(3,2) - f(0,0) = 9 + \frac{8}{3}$$

23.
$$\int_C yz\,dy + xy\,dz = \int_0^1 (5t)(2t^2)\,5dt + \int_0^1 (4\sqrt{t})(5t)\,4tdt$$
$$= \frac{25}{2} + \frac{160}{7}$$

24. (a) $\frac{\partial(8x\cos y - y\cos x)}{\partial y} = -8x\sin y - \cos x = \frac{\partial(-4x^2\sin y - \sin x)}{\partial x}$ and clearly both are continuous everywhere. Therefore it is conservative vector field. To find the potential f, we use

$$f_x = 8x\cos y - y\cos x$$
$$f_y = -4x^2\sin y - \sin x$$

Integrate the first equation with respect to x gives

$$f = 4x^2\cos y - y\sin x + g(y)$$

Then taking derivative with respect to y gives

$$f_y = -4x^2\sin y - \sin x + g'(y)$$

Compare it with the the known condition $f_y = -4x^2\sin y - \sin x$, we immediately see that $g'(y) = 0$. Therefore $g(y) = C$ and

$$f = 4x^2\cos y - y\sin x + C$$

(b) $\frac{\partial(x^3+4xy)}{\partial y} = 4x$ and $\frac{\partial(4xy-y^3)}{\partial x} = 4y$. There are not equal and hence it is not a conversative field.

25. Since it is stated "Use the fundamental theorem of line integrals", the given vector fields must be conservative. You can skip the step of checking $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$. Of course it is always safer to check this condition before you start to calculate the potential.

(a) $f_x = \frac{y^2}{1+x^2}$ implies $f = y^2\arctan x + g(y)$. Taking partial derivative with respect to y gives $f_y = 2y\arctan x + g'(y)$. Clearly $g'(y)$ must be 0 and consequestly $g(y) = C$. We just need to pick a value for C to continue the calculation. The easiest way is to set $C = 0$. Then $f = y^2\arctan x$. The starting point is $A = \mathbf{r}(0) =< 0, 0 >$ and the ending point is $B = \mathbf{r}(1) =< 1, 2 >$. By the Fundamental theorem:

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(B) - f(A) = 4\arctan 1 - 0 = \pi$$

(b) $f_x = (2xz + y^2)$ implies $f = x^2 z + xy^2 + g(y,z)$. Then by comparing

$$\begin{cases} f_y = 2xy + g_y(y,z) \\ f_z = x^2 + g_z(y,z) \end{cases} \quad \text{and} \quad \begin{cases} f_y = 2xy \\ f_z = x^2 + 3z^2 \end{cases}$$

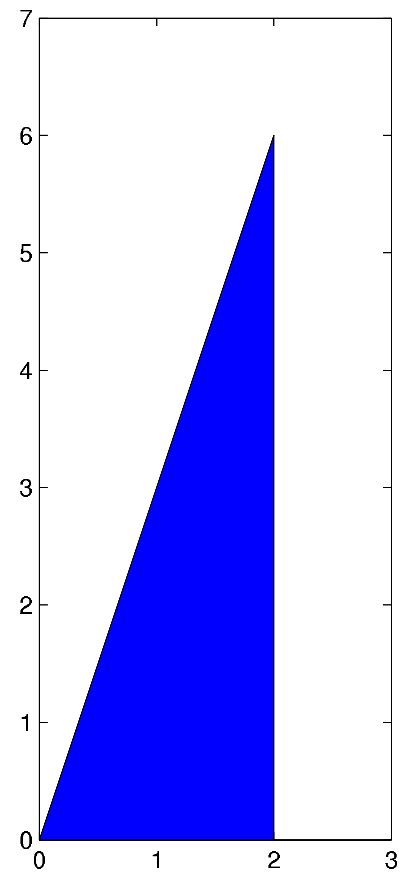
We have $g_y(y,z) = 0$ and $g_z(y,z) = 3z^2$. Therefore $g(y,z) = z^3 + C$. Combine these together and set $C = 0$, we have one potential function

$$f = x^2 z + xy^2 + z^3$$

Now the starting point is $A = \mathbf{r}(0) =< 0, 1, -1 >$ and the ending point is $B = \mathbf{r}(1) =< 1, 2, 1 >$. Therefore

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A) = 6 - (-1) = 7$$

26. Notice that the line segments in C are arranged in the negative orientation. According to the Green's Theorem



$$\begin{aligned} \int_C y^2\,dx + 2x\,dy &= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= -\iint_D (2 - 2y)\,dA = -\int_0^2 \int_0^{3x} (2 - 2y)\,dydx \\ &= -\int_0^2 (6x - 9x^2)\,dx = 12 \end{aligned}$$