Math 2163, Practice Exam II, Solution

1. (a) $\nabla f =< f_s,\ f_t >=< 2s\,e^t,\ s^2e^t >$, and $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{<1,1>}{\sqrt{2}}$, so

$$D_{\mathbf{v}}f(2,0) =< 2(2)e^0,\ 2^2e^0 > \cdot \frac{<1,1>}{\sqrt{2}} =< 4,4 > \cdot \frac{<1,1>}{\sqrt{2}} = 4\sqrt{2}.$$

(b) $\nabla f =< 2xy^3,\ 3x^2y^2 - 4y^3 >$ and $\mathbf{v} =< \cos\frac{\pi}{4},\ \sin\frac{\pi}{4} >=< \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >$, so

$$D_{\mathbf{v}}f(2,1) =< 4,8 > \cdot < \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} >= 6\sqrt{2}.$$

(c) $\nabla f =< \frac{1}{y+z},\ -\frac{x}{(y+z)^2},\ -\frac{x}{(y+z)^2} >$ and $\frac{\mathbf{v}}{|\mathbf{v}|} = \frac{<1,2,3>}{\sqrt{14}}$,

$$D_{\mathbf{v}}f(4,1,1) =< \frac{1}{2},\ -1,\ -1 > \cdot \frac{<1,2,3>}{\sqrt{14}} = \frac{-9}{2\sqrt{14}}.$$

2. (a) $\nabla f =< -\frac{y^2}{x^2},\ \frac{2y}{x} >$, then $\nabla f(2,4) =< -4,4 >$. Therefore the maximum change of rate is $|\nabla f(2,4)| = 4\sqrt{2}$ and it occurs in the same direction of $\nabla f(2,4) =< -4,4 >$.

(b) $\nabla f =< 4x^3y^3z^2,\ 3x^4y^2z^2,\ 2x^4y^3z >$, then $\nabla f(1,-1,1) =< -4,3,-2 >$. Therefore the maximum change of rate is $|\nabla f(1,-1,1)| = \sqrt{29}$ and it occurs in the same direction of $\nabla f(1,-1,1) =< -4,3,-2 >$.

3. (a) By simplifying

$$\begin{cases} f_x &= 2xe^{y^2-x^2} + (x^2+y^2)(-2x)e^{y^2-x^2} = 0 \\ f_y &= 2ye^{y^2-x^2} + (x^2+y^2)(2y)e^{y^2-x^2} = 0 \end{cases}$$

we have

$$\begin{cases} 2e^{y^2-x^2}x(1-x^2-y^2) = 0 \\ 2e^{y^2-x^2}y(1+x^2+y^2) = 0 \end{cases} \Rightarrow \begin{cases} x(1-x^2-y^2) = 0 \\ y(1+x^2+y^2) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x = 0 \quad \text{or} \quad 1-x^2-y^2 = 0 \\ y = 0 \end{cases}$$

So we have the following solutions

i. $x = 0$, $y = 0$

ii. Solve

$$\begin{cases} 1-x^2-y^2 = 0 \\ y = 0 \end{cases}$$

gives two points: $(1,0)$ and $(-1,0)$.

Combine all together, there are three critical points $(0,0)$, $(1,0)$ and $(-1,0)$. To classify the critical points, we need to calculate

$$D = f_{xx}f_{yy} - f_{xy}^2$$

Since

$$\begin{aligned} f_{xx} &= 2e^{y^2-x^2}((1-x^2-y^2)(1-2x^2)-2x^2), \\ f_{xy} &= -4xye^{y^2-x^2}(x^2+y^2), \\ f_{yy} &= 2e^{y^2-x^2}((1+x^2+y^2)(1+2y^2)+2y^2). \end{aligned}$$

It is easy to see that

$$\begin{aligned} D(0,0) &= 4 > 0, \qquad f_{xx}(0,0) = 2 > 0, \\ D(1,0) &= -16e^{-2} < 0, \\ D(-1,0) &= -16e^{-2} < 0. \end{aligned}$$

So $f(0,0) = 0$ is a local minimum and $(1,0)$, $(-1,0)$ are saddle points.

(b) First, we have

$$\begin{cases} f_x &= 6x^2 + y^2 + 10x = 0, \\ f_y &= 2xy + 2y = 0 \end{cases}$$

By solving $f_y = 2xy + 2y = 2y(x+1) = 0$, we have either $y = 0$ or $x = -1$.

i. If $y = 0$, substituting it into $f_x = 6x^2 + y^2 + 10x = 0$ gives $x = 0$ or $x = -5/3$.

ii. If $x = -1$, substituting it into $f_x = 6x^2 + y^2 + 10x = 0$ gives $y = 2$ or $y = -2$.

Combine the above, there are four critical points: $(0,0)$, $(-5/3,0)$, $(-1,2)$, $(-1,-2)$. Next we need to calculate D. Since

$$\begin{aligned} f_{xx} &= 12x + 10, \\ f_{xy} &= 2x + 2, \\ f_{yy} &= 2y. \end{aligned}$$

It is easy to see that

$$\begin{aligned} D(0,0) &= 20 > 0, \qquad f_{xx}(0,0) = 10 > 0, \\ D(-5/3,0) &> 0, \qquad f_{xx}(-5/3,0) < 0, \\ D(-1,2) &< 0, \\ D(-1,-2) &< 0 \end{aligned}$$

Hence $f(0,0) = 0$ is a local minimum, $f(-5/3,0) = 125/27$ is a local maximum, and $(-1,2)$, $(-1,-2)$ are saddle points.

4. Step 1: Find the local minimums and maximums.

$$\begin{cases} f_x = 4 - 2x = 0 \\ f_y = 6 - 2y = 0 \end{cases} \qquad \Rightarrow \qquad \begin{cases} x = 2 \\ y = 3 \end{cases}$$

Then $f(2,3) = 13$ is a candidate for local minimums and maximums.

Step 2: find minimums and maximums on the boundary edges. On the bottom boundary $y = 0$, we have $f(x,0) = 4x - x^2$ and it has minimum at $f(0,0) = f(4,0) = 0$ and maximum at $f(2,0) = 4$. On the left boundary $x = 0$, we have $f(0,y) = 6y - y^2$ and it has minimum at $f(0,0) = 0$ and maximum at $f(0,3) = 9$. On the right boundary $x = 4$, we have $f(4,y) = 6y - y^2$ and it has minimum at $f(4,0) = 0$ and maximum at $f(4,3) = 9$. On the top boundary $y = 5$, we have $f(x,5) = 4x - x^2 + 5$ and it has minimum at $f(0,5) = f(4,5) = 5$ and maximum at $f(2,5) = 9$.

Compare step 1 and 2, we have the absolute minimum at $f(0,0) = f(4,0) = 0$ and the absolute maximum at $f(2,3) = 13$.

5. The distance is given by $f(x,y,z) = \sqrt{(x-8)^2 + (y-10)^2 + (z-8)^2}$ and the constraint is given by $g(x,y,z) = 8x - 10y + 4z = 16$. To avoid long notations, we denote $d = \sqrt{(x-8)^2 + (y-10)^2 + (z-8)^2}$, then

$$\nabla f =< \frac{x-8}{d}, \frac{y-10}{d}, \frac{z-8}{d} >,$$
$$\nabla g =< 8, -10, 4 >$$

Using the Lagrange multiplier method, we have

$$\begin{cases} \frac{x-8}{d} = 8\lambda \\ \frac{y-10}{d} = -10\lambda \\ \frac{z-8}{d} = 4\lambda \\ 8x - 10y + 4z = 16 \end{cases}$$

Noticing that

$$(\frac{x-8}{d})^2 + (\frac{y-10}{d})^2 + (\frac{z-8}{d})^2 = 1,$$

Therefore $(8\lambda)^2 + (-10\lambda)^2 + (4\lambda)^2 = 1$, which implies $\lambda = \pm 1/\sqrt{180}$. Now, we have

$$x = 8\lambda d + 8, \qquad y = -10\lambda d + 10, \qquad z = 4\lambda d + 8$$

substitute them into the constraint $g(x,y,z) = 8x - 10y + 4z = 16$, we have

$$64\lambda d + 64 + 100\lambda d - 100 + 16\lambda d + 32 = 16$$

and hence $180\lambda d = 20$. Since we know that $\lambda = \pm 1/\sqrt{180}$, it is easy to see that $d = \pm 20/\sqrt{180}$.

Finally, recall that we defined $d = \sqrt{(x-8)^2 + (y-10)^2 + (z-8)^2}$, which is exactly the distance. So at this point we already have the answer that the shortest distance is $d = 20/\sqrt{180}$. Of course one can solve for (x, y, z) by plug in values of λ and d back into

$$x = 8\lambda d + 8, \qquad y = -10\lambda d + 10, \qquad z = 4\lambda d + 8$$

6. Let x, y, z be the length of three sides. The volume is given by $f(x, y, z) = xyz$ and the constraint is surface area $g(x, y, z) = 2xy + 2yz + 2xz = 150$. Using the Lagrange multiplier we have

$$\begin{cases} yz = \lambda(2y + 2z) \\ xz = \lambda(2x + 2z) \\ xy = \lambda(2x + 2y) \\ 2xy + 2yz + 2xz = 150 \end{cases}$$

Multiplying the first equation by x, the second equation by y and the third equation by z, we have

$$\begin{cases} xyz = \lambda(2xy + 2xz) \\ xyz = \lambda(2xy + 2yz) \\ xyz = \lambda(2xz + 2yz) \end{cases}$$

$$\Rightarrow \quad \lambda(2xy + 2xz) = \lambda(2xy + 2yz) = \lambda(2xz + 2yz)$$

$$\Rightarrow \quad \lambda = 0 \quad \text{or} \quad (2xy + 2xz) = (2xy + 2yz) = (2xz + 2yz)$$

It is not hard to see that λ can not be 0 since otherwise one of x, y, z has to be zero. Now solving

$$(2xy + 2xz) = (2xy + 2yz) = (2xz + 2yz)$$

gives $x = y = z$. Substitute it into the constraint $2xy + 2yz + 2xz = 150$ gives $x = y = z = 5$.

7. The approximation is

$$\sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i, y_j)\Delta A = 8f(4,2) + 8f(8,2) + 8f(4,4) + 8f(8,4) = -816$$

8. (a)

$$\int_0^6 \frac{2}{3}(x+y)^{3/2}\Big|_{x=0}^{x=10}\, dy = \int_0^6 [\frac{2}{3}(y+10)^{3/2} - \frac{2}{3}y^{3/2}]\, dy = \frac{4}{15}16^{5/2} - \frac{4}{15}6^{5/2} - \frac{4}{15}10^{5/2}.$$

(b)

$$\int_0^9 \int_0^{\sqrt{x}} \frac{3y}{x^2+1} dy\, dx = \int_0^9 \frac{3y^2/2}{x^2+1}\Big|_{y=0}^{y=\sqrt{x}}\, dx = \int_0^9 \frac{3x/2}{x^2+1} dx$$
$$= \frac{3}{4}\ln(x^2+1)\Big|_0^9 = \frac{3}{4}\ln 82$$

(c) first change the order of the integral:

$$\begin{aligned}\int_0^1 \int_1^2 \frac{x}{x^2+y^2} dx\, dy &= \int_0^1 \frac{1}{2} \ln(x^2+y^2)|_{x=1}^{x=2} dy \\ &= \int_0^1 [\frac{1}{2}\ln(y^2+4) - \frac{1}{2}\ln(y^2+1)] dy\end{aligned}$$

Using integration by parts, we have

$$\begin{aligned}\int_0^1 \frac{1}{2}\ln(y^2+4) dy &= \frac{1}{2} y \ln(y^2+4)|_0^1 - \int_0^1 \frac{1}{2} y \frac{2y}{y^2+4} dy \\ &= \frac{\ln 5}{2} - \int_0^1 \frac{y^2}{y^2+4} dy \\ &= \frac{\ln 5}{2} - \int_0^1 (1 - \frac{4}{y^2+4}) dy \\ &= \frac{\ln 5}{2} - [y - 2\tan^{-1}\frac{y}{2}]|_0^1 \\ &= \frac{\ln 5}{2} - [1 - 2\tan^{-1}\frac{1}{2}] = \frac{\ln 5}{2} - 1 + 2\tan^{-1}\frac{1}{2}\end{aligned}$$

Similarly, one can calculate that

$$\int_0^1 \frac{1}{2}\ln(y^2+1) dy = \frac{\ln 2}{2} - 1 + \tan^{-1} 1 = \frac{\ln 2}{2} - 1 + \frac{\pi}{4}$$

Combine the above, we have the final result

$$\int_1^2 \int_0^1 \frac{x}{x^2+y^2} dy\, dx = \frac{\ln 5}{2} + 2\tan^{-1}\frac{1}{2} - \frac{\ln 2}{2} - \frac{\pi}{4}.$$

(d) We would like to first shange the order of the integral. Since

$$\begin{aligned}D &= \{(x,y) \mid 0 \le y \le 1,\ \sqrt{y} \le x \le 1\} \\ &= \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le x^2\}\end{aligned}$$

so we have

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\, dx\, dy &= \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\, dy\, dx = \int_0^1 (\sqrt{x^3+1}) y|_{y=0}^{y=x^2} dx \\ &= \int_0^1 x^2 \sqrt{x^3+1}\, dx = \frac{2}{9}(x^3+1)^{3/2}|_0^1 = \frac{2}{9}(\sqrt{8}-1).\end{aligned}$$