Math 2163, Exam III, Nov. 19, 2007

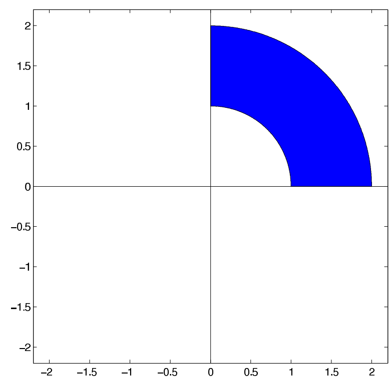
Name: ____________________

Score:

1. (10 points) Sketch the region whose area is given by the integral and evaluate the integral:

$$\int_0^{\pi/2} \int_1^2 r\, dr\, d\theta$$

Solution:



$$\begin{aligned}\int_0^{\pi/2} \int_1^2 r\, dr\, d\theta &= \int_0^{\pi/2} \frac{1}{2} r^2 \Big|_{r=1}^{r=2} d\theta \\ &= \int_0^{\pi/2} \frac{3}{2}\, d\theta \\ &= \frac{3}{2}\theta\Big|_{\theta=0}^{\theta=\pi/2} = \frac{3}{4}\pi.\end{aligned}$$

2. (10 points) Evaluate the triple integral $\iiint_E dV$ where E is bounded by planes $x = 0$, $y = 0$, $z = 0$ and $2x + 2y + z = 4$.

Solution: Notice that the plane $2x+2y+z = 4$ intersects the xy-plane at $2x+2y = 4$. Consider E to be a type I 3-D region, then we have

$$E = \{(x, y, z) \,|\, (x, y) \in D,\ 0 \leq z \leq 4 - 2x - 2y\},$$

where D is the triangle

$$D = \{(x, y) \,|\, 0 \leq x \leq 2,\ 0 \leq y \leq 2 - x\}.$$

Therefore

$$\begin{aligned}
\iiint_E dV &= \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} dz\, dy\, dx \\
&= \int_0^2 \int_0^{2-x} (4 - 2x - 2y)\, dy\, dx \\
&= \int_0^2 ((4 - 2x)y - y^2)|_{y=0}^{y=2-x}\, dx = \int_0^2 (2 - x)^2\, dx \\
&= -\frac{1}{3}(2 - x)^3|_{x=0}^{x=2} = \frac{8}{3}.
\end{aligned}$$

3. (10 points) Find the area of the part of surface $z = xy$ that lies in the cylinder $x^2 + y^2 = 4$.

Solution: We have $z = f(x, y) = xy$ and $f_x = y$, $f_y = x$. The easiest way is to use polar coordinates, and theregion is

$$D = \{(r, \theta) \,|\, 0 \leq \theta \leq 2\pi,\, 0 \leq r \leq 2\}.$$

Then the surface area is

$$\begin{aligned} A &= \iint_D \sqrt{f_x^2 + f_y^2 + 1}\, dA = \iint_D \sqrt{y^2 + x^2 + 1}\, dA \\ &= \int_0^{2\pi} \int_0^2 \sqrt{r^2 + 1}\, r\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{1}{3}(r^2 + 1)^{3/2}|_{r=0}^{r=2}\, d\theta \\ &= \int_0^{2\pi} \frac{1}{3}(5\sqrt{5} - 1)\, d\theta \\ &= \frac{2\pi}{3}(5\sqrt{5} - 1). \end{aligned}$$

4. (10 points) Find the mass of the solid bounded by the paraboloid $z = 6x^2 + 6y^2$ and the plane $z = 6$ if the solid has density $\rho(x, y, z) = 3\sqrt{x^2 + y^2}$.

Solution: First we calculate the intersection of $z = 6x^2 + 6y^2$ and $z = 6$:

$$\begin{cases} z = 6x^2 + 6y^2 \\ z = 6 \end{cases} \qquad \Rightarrow \qquad x^2 + y^2 = 1$$

Then the region is

$$E = \{(x, y, z) \,|\, (x, y) \in D,\ 6x^2 + 6y^2 \leq z \leq 6\}$$

where $D = \{(x, y) \,|\, x^2 + y^2 \leq 1\}$. The easiest way is to use cylindrical coordinates:

$$E = \{(r, \theta, z) \,|\, 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1,\ 6r^2 \leq z \leq 6\}.$$

Then the mass is

$$\begin{aligned} mass &= \iiint_E \rho(x, y, z)\, dV \\ &= \int_0^{2\pi} \int_0^1 \int_{6r^2}^6 (3r)(r)\, dz\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 3r^2 z|_{z=6r^2}^{z=6}\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (18r^2 - 18r^4)\, dr\, d\theta \\ &= \int_0^{2\pi} \frac{12}{5}\, d\theta = \frac{24\pi}{5}. \end{aligned}$$

5. (10 points) Use spherical coordinates to evaluate the triple integral

$$\iiint_E e^{\left(\sqrt{x^2+y^2+z^2}\right)^3}\, dV$$

where E is the solid between two spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

Solution: Under spherical coordinates, the region is

$$E = \{(\rho,\theta,\phi) \,|\, 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi,\ 1 \le \rho \le 2\}.$$

So the integral is

$$\begin{aligned}
\iiint_E e^{\left(\sqrt{x^2+y^2+z^2}\right)^3}\, dV &= \int_0^{\pi}\int_0^{2\pi}\int_1^2 e^{\rho^3}\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} \frac{1}{3}e^{\rho^3} \sin\phi|_{\rho=1}^{\rho=2}\, d\theta\, d\phi \\
&= \int_0^{\pi}\int_0^{2\pi} \frac{e^8-e}{3} \sin\phi\, d\theta\, d\phi \\
&= \int_0^{\pi} \frac{e^8-e}{3} 2\pi \sin\phi\, d\phi \\
&= \frac{e^8-e}{3} 2\pi(-\cos\phi)|_{\phi=0}^{\phi=\pi} = \frac{e^8-e}{3} 4\pi.
\end{aligned}$$