Math 2163, Exam II, Oct. 24, 2007

Name: ____________________

Score:

1. (8 points) Find the directional derivative of $f(x, y) = xe^{-4y}$ at the point $(2, 0)$ in the direction indicated by the angle $\theta = \pi/2$.

   Solution: First we calculate the gradient vector

$$\nabla f =< f_x,\ f_y >=< e^{-4y},\ -4xe^{-4y} > .$$

   Therefore

$$\nabla f(2, 0) =< e^0 \ \ -4 \times 2 \times e^0 >=< 1,\ -8 > .$$

   Then the dirctional derivative is

$$\begin{aligned} D_{\mathbf{u}} f &= \nabla f(2, 0) \cdot < \cos\theta, \sin\theta > \\ &=< 1, -8 > \cdot < \cos\pi/2, \sin\pi/2 >=< 1, -8 > \cdot < 0, 1 > \\ &= -8. \end{aligned}$$

2. (8 points) Find the maximum rate of change of $f(x,y)=\sin(xy)$ at the point $(1,0)$.

Solution: The gradient vector is

$$\nabla f =< y\cos(xy),\ x\cos(xy) > .$$

Therefore

$$\nabla f(1,0) =< 0\cos 0,\ 1\cos 0 >=< 0,1 > .$$

Hence the maximum rate of change is

$$|\nabla f(2,0)| = |<0,1>| = \sqrt{0^2+1^2} = 1,$$

and it occurs in the direction of $<0,1>$.

3. (8 points) Find all critical points of (you do NOT need to classify them)

$$f(x, y) = (2x - x^2)(2y - y^2).$$

Solution: We start from solving the following system of equations:

$$\begin{cases} f_x = (2-2x)(2y-y^2) = 0 \\ f_y = (2x-x^2)(2-2y) = 0 \end{cases} \Rightarrow \begin{cases} x = 1 \text{ or } y = 0 \text{ or } y = 2 \\ x = 0 \text{ or } x = 2 \text{ or } y = 1 \end{cases}$$

Since we need both equations to be true at the same time, taking combinations of one solution to the first equation and one solution to the second equation gives 5 critical points:

$$(1, 1), \quad (0, 0), \quad (0, 2), \quad (2, 0), \quad (2, 2).$$

4. (8 points) Use Lagrange multipliers to find the dimensions of a rectangular box of maximum volume such that the sum of the lengths of its all 12 edges is 24.

Solution: Let x, y, z be the length, width and height respectively. We want to find the maximum volume

$$f(x, y, z) = xyz$$

while subjecting to the constraint

$$g(x, y, z) = 4x + 4y + 4z = 24.$$

Clearly

$$\nabla f =< yz,\, xz,\, xy >, \qquad \nabla g =< 4,\, 4,\, 4 > .$$

By using the Lagrange multipliers method, we have

$$\begin{cases} \nabla f = \lambda \nabla g \\ 4x + 4y + 4z = 24 \end{cases} \qquad \Rightarrow \qquad \begin{cases} yz = 4\lambda \\ xz = 4\lambda \\ xy = 4\lambda \\ 4x + 4y + 4z = 24 \end{cases}$$

By multiplying the first, second and third equation with x, y, z, respectively, we have

$$\begin{cases} xyz = 4\lambda x \\ xyz = 4\lambda y \\ xyz = 4\lambda z \end{cases}$$

Since $\lambda \neq 0$(otherwise one of x, y, z has to be 0), it is not hard to see that $x = y = z$. Substitute it into the last equation $4x + 4y + 4z = 24$, we have $x = y = z = 2$. And the maximum volume is $2 \cdot 2 \cdot 2 = 8$.

5. (10 points) Calculate the iterated integral

$$\int_0^1 \int_1^2 \frac{x}{x^2 + y}\, dx\, dy$$

Solution:

$$\int_0^1 \int_1^2 \frac{x}{x^2+y}\, dx\, dy = \int_0^1 \frac{1}{2} \ln(x^2+y)|_{x=1}^{x=2}\, dy$$

(Change of variables $u = x^2, \quad du = 2x\, dx$)

$$= \int_0^1 [\frac{1}{2} \ln(y+4) - \frac{1}{2} \ln(y+1)]\, dy$$
$$= \frac{1}{2} \int_0^1 \ln(y+4)\, dy - \frac{1}{2} \int_0^1 \ln(y+1)\, dy$$
$$= \frac{1}{2}[(y+4)\ln(y+4) - (y+4)]|_0^1 - \frac{1}{2}[(y+1)\ln(y+1) - (y+1)]|_0^1$$

(Change of variables $u = y+4,\ v = y+1,$

and then use the formula $\int \ln u\, du = u \ln u - u + C$)

$$= [(\frac{5}{2} \ln 5 - 5) - (2 \ln 4 - 4)] - [(\ln 2 - 2) - (\frac{1}{2} \ln 1 - 1)]$$
$$= \frac{5}{2} \ln 5 - \ln 2 - 2 \ln 4$$
$$= \frac{5}{2} \ln 5 - 5 \ln 2$$

6. (8 points) Evaluate the double integral

$$\iint_D x \cos y \, dA$$

where D is bounded by $y = 0$, $y = x^2$ and $x = 2$.

Solution: First we have

$$D = \{(x, y) \mid 0 \leq x \leq 2, \quad 0 \leq y \leq x^2\}.$$

Then

$$\begin{aligned}
\iint_D x \cos y \, dA &= \int_0^2 \int_0^{x^2} c \cos y \, dy \, dx \\
&= \int_0^2 (x \sin y)|_{y=0}^{y=x^2} \, dx \\
&= \int_0^2 x \sin x^2 \, dx \\
&= -\frac{1}{2} \cos x^2|_0^2 \\
&= -\frac{1}{2} \cos 4 + \frac{1}{2}.
\end{aligned}$$