Math 2163, Exam I, Sept. 21, 2007

Name: ____________________

Score:

1. (12 points) Calculate the following (please write your final answer on the line):

   (a) $3 < 2, 1, -1 > -2 < 0, 4, 2 >= \underline{< 6, -5, -7 >}$

   (b) $| < -1, 4, 3 > | = \underline{\sqrt{(-1)^2 + 4^2 + 3^2} = \sqrt{26}}$

   (c) $(\mathbf{i} - \mathbf{j} + \mathbf{k}) \times (3\mathbf{j} + 2\mathbf{k}) = \underline{< -5, -2, 3 >}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & 3 & 2 \end{vmatrix} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 0 & 3 \end{vmatrix} =< -5, -2, 3 > .$$

   (d) the angle between $< 5, 0 >$ and $< 5, 5 >$ is $\underline{\pi/4}$

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \frac{25}{5\sqrt{50}} = \frac{5}{\sqrt{50}} = \frac{1}{\sqrt{2}} \implies \theta = \arccos\frac{1}{\sqrt{2}} = \pi/4.$$

2. (10 points) Given $f(x,y) = \sqrt{10 - x^2 - 2y^2}$.

(a) Find the domain of $f(x,y)$;

(b) Calculate $f_{xy}(1,2)$.

Solution:

(a) $\{(x,y) \,|\, 10 - x^2 - 2y^2 \geq 0\}$ or $\{(x,y) \,|\, x^2 + 2y^2 \leq 10\}$;

(b)

$$
\begin{aligned}
f &= (10 - x^2 - 2y^2)^{1/2}, \\
f_x &= \frac{1}{2}(10 - x^2 - 2y^2)^{-1/2}(-2x) = -x(10 - x^2 - 2y^2)^{-1/2}, \\
f_{xy} &= (-x)(-1/2)(10 - x^2 - 2y^2)^{-3/2}(-4y) = -2xy(10 - x^2 - 2y^2)^{-3/2},
\end{aligned}
$$

$$f_{xy}(1,2) = -4(10 - 1 - 8)^{-3/2} = -4.$$

3. (8 points) Calculate $\partial z/\partial x$ and $\partial z/\partial y$, if $z = f(x, y)$ is defined implicitly by

$$z + 5 = xe^y \cos z.$$

Solution 1

$$\begin{aligned}
\frac{\partial(z+5)}{\partial x} &= \frac{\partial(xe^y \cos z)}{\partial x} \\
&\Longrightarrow \frac{\partial z}{\partial x} = e^y \cos z - xe^y \sin z \frac{\partial z}{\partial x} \\
&\Longrightarrow \frac{\partial z}{\partial x} = \frac{e^y \cos z}{1 + xe^y \sin z};
\end{aligned}$$

$$\begin{aligned}
\frac{\partial(z+5)}{\partial y} &= \frac{\partial(xe^y \cos z)}{\partial y} \\
&\Longrightarrow \frac{\partial z}{\partial y} = xe^y \cos z - xe^y \sin z \frac{\partial z}{\partial y} \\
&\Longrightarrow \frac{\partial z}{\partial y} = \frac{xe^y \cos z}{1 + xe^y \sin z};
\end{aligned}$$

Solution 2 Define $w = F(x, y, z) = z + 5 - xe^y \cos z = 0$. By the implicit function theorem,

$$\begin{aligned}
\frac{\partial z}{\partial x} &= -\frac{F_x}{F_z} = -\frac{-e^y \cos z}{1 + xe^y \sin z} = \frac{e^y \cos z}{1 + xe^y \sin z}; \\
\frac{\partial z}{\partial y} &= -\frac{F_y}{F_z} = -\frac{-xe^y \cos z}{1 + xe^y \sin z} = \frac{xe^y \cos z}{1 + xe^y \sin z};
\end{aligned}$$

4. (8 points) Use the chain rule to find $\partial u/\partial p$ and write it as a function of p, r, and t.

$$u = \frac{x+y}{y+z}$$
$$x = p + 6r + 5t, \quad y = p - 6r + 5t, \quad z = p + 6r - 5t.$$

Solution: By the chain rule, $\dfrac{\partial u}{\partial p} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial p} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial p} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial p}$, where

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{1}{y+z}, & \frac{\partial x}{\partial p} &= 1; \\
\frac{\partial u}{\partial y} &= \frac{1}{y+z} - \frac{x+y}{(y+z)^2} = \frac{z-x}{(y+z)^2}, & \frac{\partial y}{\partial p} &= 1; \\
\frac{\partial u}{\partial z} &= -\frac{x+y}{(y+z)^2}, & \frac{\partial z}{\partial p} &= 1.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial u}{\partial p} &= \frac{1}{y+z}(1) + \frac{z-x}{(y+z)^2}(1) + \left(-\frac{x+y}{(y+z)^2}\right)(1) \\
&= \frac{y+z+z-x-x-y}{(y+z)^2} \\
&= \frac{2(z-x)}{(y+z)^2} \\
&= \frac{2((p+6r-5t)-(p+6r+5t))}{((p-6r+5t)+(p+6r-5t))^2} \\
&= \frac{-20t}{(2p)^2} = -\frac{5t}{p^2}.
\end{aligned}$$

5. (a) (4 points) For what values of c will the following two lines be parallel?

$$\frac{x+1}{2} = \frac{y+4}{-1} = \frac{z-1}{3}, \qquad \frac{x-1}{6} = \frac{y-1}{c} = \frac{z-1}{9}.$$

(b) (4 points) For what values of c will the following two planes be orthogonal?

$$6x + y - 5z = 5, \qquad x + c^2 y + cz = 0.$$

(c) (4 points) Find an equation of the plan which passes through the point $(4, 1, -3)$ and is perpendicular to the line $x = 3t$, $y = 1$, $z = 2 - t$.

Solution:

(a) These two given lines have directions $\mathbf{v}_1 =< 2, -1, 3 >$ and $\mathbf{v}_2 =< 6, c, 9 >$, respectively. Since they are parallel, we have $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{0}$. Because

$$\mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 3 \\ 6 & c & 9 \end{vmatrix} =< -9 - 3c, 0, 2c + 6 >,$$

so c must be equal to -3 to make it a zero vector. (An easier way to solve this problem is by observing that $< 6, c, 9 >= 3 < 2, -1, 3 >$).

(b) The normal vectors for the given planes are $\mathbf{n}_1 =< 6, 1, -5 >$ and $\mathbf{n}_2 =< 1, c^2, c >$. The two planes are orthogonal implies that $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. So we have

$$< 6, 1, -5 > \cdot < 1, c^2, c >= 6 + c^2 - 5c = c^2 - 5c + 6 = (c-2)(c-3) = 0,$$

which tells us that $c = 2$ or $c = 3$.

(c) The normal vector of the plane is given by the direction of the line, which should be $< 3, 0, -1 >$. Then the plane is

$$\begin{aligned} &< 3, 0, -1 > \cdot (< x, y, z > - < 4, 1, -3 >) = 0, \\ \Longrightarrow\ & 3(x-4) - (z+3) = 0, \\ \Longrightarrow\ & 3x - z - 15 = 0. \end{aligned}$$