Math 2153, Exam III, April 23, 2010

Name: ____________________

Score:

Read the problems carefully before you begin. Show all your work neatly and concisely, and indicate your final answer clearly. Total points = 50.

1. (8 points) Find the interval of convergence of the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{(x+8)^n}{n6^n}$$

Solution By the ratio test

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(x+8)^{n+1}}{(n+1)6^{n+1}}}{(-1)^n\frac{(x+8)^n}{n6^n}}\right| = \lim_{n\to\infty}\left|\frac{(-1)n(x+8)}{6(n+1)}\right|$$

$$= \left|\frac{x+8}{6}\right| < 1$$

$$\Rightarrow \quad |x+8| < 6$$

$$\Rightarrow \quad -14 < x < -2$$

When $x = -14$, the series becomes

$$\sum_{n=1}^{\infty}(-1)^n\frac{(-6)^n}{n6^n} = \sum_{n=1}^{\infty}\frac{1}{n}$$

and is divergent (by the integral test).

When $x = -2$, the series becomes

$$\sum_{n=1}^{\infty}(-1)^n\frac{6^n}{n6^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$$

and is convergent (alternating series test).

Combining the above, the interval of convergence is

$$(-14,\ -2]$$

2. (8 points) Find the first 5 terms of the Taylor series centered at 0 for the function $f(x) = \ln(10 - x)$.

Solution Clearly

$$
\begin{aligned}
f(x) &= \ln(10-x) & f(0) &= \ln 10 \\
f'(x) &= -\frac{1}{10-x} = \frac{1}{x-10} & f'(0) &= -\frac{1}{10} \\
f''(x) &= -\frac{1}{(x-10)^2} & f''(0) &= -\frac{1}{100} \\
f'''(x) &= \frac{2}{(x-10)^3} & f'''(0) &= -\frac{2}{1000} \\
f^{(4)}(x) &= -\frac{6}{(x-10)^4} & f^{(4)}(0) &= -\frac{6}{10000}
\end{aligned}
$$

Therefore, the first 5 terms of the Taylor series are

$$
\begin{aligned}
f(x) &= f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots \\
&= \ln 10 - \frac{1}{10}x - \frac{1}{200}x^2 - \frac{1}{3000}x^3 - \frac{1}{40000}x^4 + \cdots
\end{aligned}
$$

3. (8 points) Evaluate the indefinite integral using power series

$$\int \frac{t}{1-t^8}\,dt$$

Solution Notice that

$$\frac{t}{1-t^8} = t\sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1}.$$

Then

$$\int \frac{t}{1-t^8}\,dt = \int \sum_{n=0}^{\infty} t^{8n+1}\,dt = \sum_{n=0}^{\infty} \int t^{8n+1}\,dt = \sum_{n=0}^{\infty} \frac{t^{8n+2}}{8n+2}.$$

4. (8 points) Find the equation of the tangent line to the curve at the point corresponding to $t = 11\pi$,

$$x = t\sin t, \qquad y = t\cos t.$$

Solution Clearly

$$\frac{dx}{dt} = \sin t + t\cos t, \qquad \frac{dy}{dt} = \cos t - t\sin t.$$

Therefore

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\cos t - t\sin t}{\sin t + t\cos t}$$

When $t = 11\pi$, we have $\sin 11\pi = 0$ and $\cos 11\pi = -1$. Hence,

$$\begin{aligned} x &= 11\pi \sin 11\pi = 0, \\ y &= 11\pi \cos 11\pi = -11\pi, \\ \frac{dy}{dx} &= \frac{\cos 11\pi - 11\pi \sin 11\pi}{\sin 11\pi + 11\pi \cos 11\pi} = \frac{-1-0}{0-11\pi} = \frac{1}{11\pi}. \end{aligned}$$

The tangent line is

$$y - (-11\pi) = \frac{1}{11\pi}(x - 0).$$

Or, it can be written as

$$y + 11\pi = \frac{1}{11\pi}x.$$

5. (8 points) Find the length of the curve:

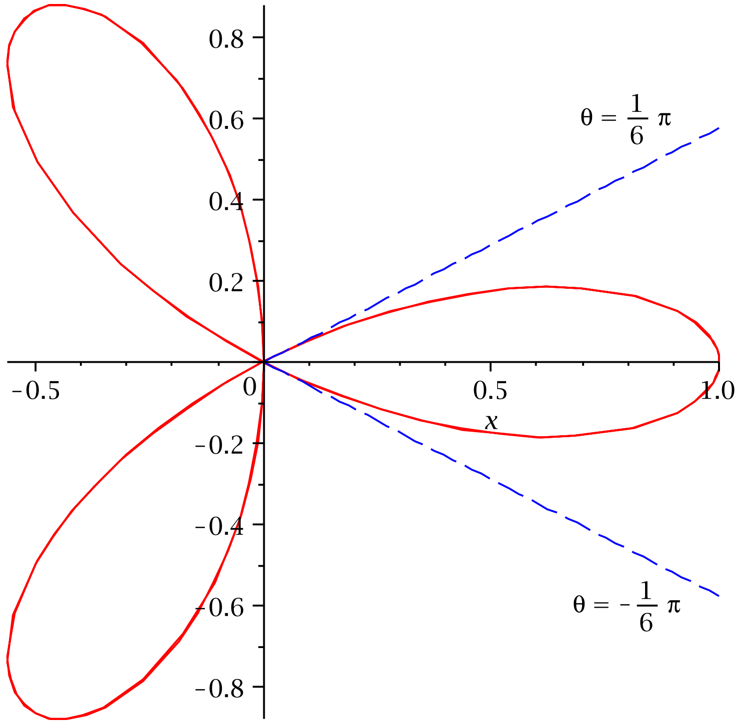
$$x = 1 + 3t^2, \qquad y = 4 + 2t^3, \qquad 0 \le t \le 1.$$

Solution By the arc length formula,

$$\begin{aligned}
L &= \int_0^1 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^1 \sqrt{(6t)^2 + (6t^2)^2}\, dt \\
&= \int_0^1 \sqrt{36t^2 + 36t^4}\, dt \\
&= \int_0^1 \sqrt{36t^2(1+t^2)}\, dt \\
&= \int_0^1 6t\sqrt{1+t^2}\, dt \\
&\quad (\text{set } u = 1 + t^2,\ du = 2t\, dt) \\
&= \int_1^2 3\sqrt{u}\, du \\
&= 3\frac{2}{3}u^{3/2}|_1^2 \\
&= 2(2)^{3/2} - 2(1)^{3/2} \\
&= 4\sqrt{2} - 2.
\end{aligned}$$

6. (10 points) Sketch the polar curve defined by $r = \cos 3\theta$. Then find the area that it encloses.

Solution



By symmetry, the area can be computed by

$$\begin{aligned} A &= 6\int_0^{\pi/6} \frac{1}{2} r^2\, d\theta \\ &= 6\int_0^{\pi/6} \frac{1}{2}\cos^2 3\theta\, d\theta \\ &= 6\int_0^{\pi/6} \frac{1}{2}\frac{\cos 6\theta + 1}{2}\, d\theta \\ &= \frac{3}{2}\left(\frac{1}{6}\sin 6\theta + \theta\right)\Big|_0^{\pi/6} \\ &= \frac{\pi}{4}. \end{aligned}$$