Math 2153, Exam II, March 24, 2010

Name: ____________________

Score:

Read the problems carefully before you begin. Show all your work neatly and concisely, and indicate your final answer clearly. Total points = 50.

1. (8 points) Find the centroid of the region bounded by the given curves:

$$y = x^2, \qquad x = y^2.$$

Solution Notice that the two curves intersect at $(0,0)$ and $(1,1)$. For $0 \le x \le 1$, we also know that $y = \sqrt{x}$ is greater than $y = x^2$. Therefore we set

$$f(x) = \sqrt{x}, \qquad g(x) = x^2.$$

Then

$$A = \int_0^1 (f(x) - g(x))\,dx = \int_0^1 (\sqrt{x} - x^2)\,dx = (\frac{2}{3}x^{3/2} - \frac{1}{3}x^3)|_0^1 = \frac{1}{3},$$

and

$$\bar{x} = \frac{1}{A}\int_0^1 x(f(x) - g(x))\,dx = 3\int_0^1 x(\sqrt{x} - x^2)\,dx = 3(\frac{2}{5}x^{5/2} - \frac{1}{4}x^4)|_0^1 = \frac{9}{20},$$

$$\bar{y} = \frac{1}{A}\int_0^1 \frac{1}{2}(f^2(x) - g^2(x))\,dx = 3\int_0^1 \frac{1}{2}(x - x^4)\,dx = \frac{3}{2}(\frac{1}{2}x^2 - \frac{1}{5}x^5)|_0^1 = \frac{9}{20}.$$

The centroid is located at $(\frac{9}{20}, \frac{9}{20})$.

2. (10 points) Find a formula for the general term a_n, $n = 1, \ldots$, of the following sequences, then evaluate the limit of the sequence.

   (a) $\{1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \cdots\}$,

   (b) $\{2, \sqrt[3]{4}, \sqrt[5]{8}, \sqrt[7]{16}, \sqrt[9]{32} \cdots\}$.

   Solution

   (a) $a_n = (-\frac{1}{3})^{n-1}$, and
   $$\lim_{n\to\infty} a_n = 0$$

   (b) $a_n = (2^n)^{\frac{1}{2n-1}} = 2^{\frac{n}{2n-1}}$, and
   $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 2^{\frac{n}{2n-1}} = \lim_{n\to\infty} 2^{\frac{1}{2-1/n}} = 2^{\frac{1}{2-0}} = \sqrt{2}$$

3. (8 points) Using the Integral Test to determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)^3}$$

Solution By the integral test, we need to check the integral

$$\int_1^{\infty} \frac{1}{(2x+1)^3}\, dx$$

Notice that

$$\begin{aligned}
\int_1^{\infty} \frac{1}{(2x+1)^3}\, dx &= \lim_{t\to\infty} \int_1^{\infty} \frac{1}{(2x+1)^3}\, dx \\
&= \lim_{t\to\infty} \left(\frac{1}{-4(2x+1)^2} \right)\Bigg|_0^1 \\
&= \lim_{t\to\infty} \left(\frac{1}{-4(2t+1)^2} - \frac{1}{-4(2+1)^2} \right) \\
&= 0 - \frac{1}{-4(2+1)^2} \\
&= \frac{1}{36}
\end{aligned}$$

The improper integral converges to a finite number. Therefor, by the integral test, the series converges.

4. (8 points) Express the number as a ratio of integers:

$$1.\overline{73} = 1.73737373\ldots$$

Solution

$$1.73737373\ldots = 1 + (0.73 + 0.0073 + 0.000073 + \ldots)$$

Notice that $0.73 + 0.0073 + 0.000073 + \ldots$ is a geometric series that can be written as

$$0.73 + 0.0073 + 0.000073 + \ldots = \sum_{n=1}^{\infty} 0.73 \times (0.01)^{n-1} = \frac{0.73}{1 - 0.01} = \frac{73}{99},$$

we have

$$1.73737373\ldots = 1 + \frac{73}{99} = \frac{172}{99}$$

5. (8 points) Test whether the following series is convergent or divergent. Give the name of the test you are using and the details of how you use it.

$$\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$$

Solution We will use the limit comparison test and compare the series with

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6}}$$

Clearly, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\frac{1+n+n^2}{\sqrt{1+n^2+n^6}}}{\frac{n^2}{\sqrt{n^6}}} &= \lim_{n\to\infty} \left(\frac{1+n+n^2}{n^2} \frac{\sqrt{n^6}}{\sqrt{1+n^2+n^6}} \right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n} + 1 \right) \sqrt{\frac{n^6}{1+n^2+n^6}} \\
&= \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n} + 1 \right) \sqrt{\frac{1}{\frac{1}{n^6} + \frac{1}{n^4} + 1}} \\
&= (0+0+1) \sqrt{\frac{1}{0+0+1}} \\
&= 1
\end{aligned}$$

By the limit comparison test, these two series are either both convergent or both divergent. Notice that

$$\sum_{n=1}^{\infty} \frac{n^2}{\sqrt{n^6}} = \sum_{n=1}^{\infty} \frac{n^2}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n}$$

is the Harmonic series and is divergent, so the series

$$\sum_{n=1}^{\infty} \frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$$

is also divergent.

6. (8 points) Is the following series convergent? Is it absolutely convergent? Give the name of the test you are using and the details of how you use it.

$$\sum_{n=1}^{\infty} \frac{n^2(-2)^n}{n!}$$

Solution We use the ratio test. Notice that

$$\begin{aligned}
\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{\frac{(n+1)^2(-2)^{n+1}}{(n+1)!}}{\frac{n^2(-2)^n}{n!}}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^2(-2)^{n+1}n!}{(n+1)!\,n^2(-2)^n}\right| \\
&= \lim_{n\to\infty} \left|\frac{(n+1)^2}{n^2}\,\frac{(-2)^{n+1}}{(-2)^n}\,\frac{n!}{(n+1)!}\right| \\
&= \lim_{n\to\infty} \left|\left(1+\frac{2}{n}+\frac{1}{n^2}\right)(-2)\,\frac{1\cdot 2\cdots n}{1\cdot 2\cdots n\cdot(n+1)}\right| \\
&= \lim_{n\to\infty} \left|\left(1+\frac{2}{n}+\frac{1}{n^2}\right)(-2)\,\frac{1}{(n+1)}\right| \\
&= (1+0+0)\,(-2)\,0 \\
&= 0
\end{aligned}$$

By the ratio test, the series is absolutely convergent, hence also convergent.