Math 2153, Exam I, Feb. 17, 2010

Name: ____________________

Score:

Read the problems carefully before you begin. Show all your work neatly and concisely, and indicate your final answer clearly. Total points = 50.

1. (8 points) Evaluate the integral $\int (x^2+1)e^{-x}\,dx$

   Solution 1 Use integration by parts twice:

$$\begin{aligned}
& & &\int (x^2+1)e^{-x}\,dx \\
&\left(\text{set } \begin{matrix} u = x^2+1 & v' = e^{-x} \\ u' = 2x & v = -x^{-x} \end{matrix} \Rightarrow\right) & &= (x^2+1)(-e^{-x}) - \int 2x(-e^{-x})\,dx \\
&(\text{simplify } \Rightarrow) & &= -(x^2+1)e^{-x} + \int 2xe^{-x}\,dx \\
&\left(\text{set } \begin{matrix} u = 2x & v' = e^{-x} \\ u' = 2 & v = -e^{-x} \end{matrix} \Rightarrow\right) & &= -(x^2+1)e^{-x} + \left[2x(-e^{-x}) - \int 2(-e^{-x})\,dx\right] \\
&(\text{simplify } \Rightarrow) & &= -(x^2+1)e^{-x} - 2xe^{-x} + \int 2e^{-x}\,dx \\
& & &= -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C
\end{aligned}$$

   Solution 2 Use tabular integration by parts:

	differentiate u	integrate v'
$+$	x^2+1	e^{-x}
$-$	$2x$	$-e^{-x}$
$+$	2	e^{-x}
$-$	0	$-e^{-x}$

   We have

$$\int (x^2+1)e^{-x}\,dx = -(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x} + C$$

2. (8 points) Evaluate the integral $\int \sin^2 x \cos^5 x \, dx$

Solution Using the identity

$$\sin^2 x + \cos^2 x = 1,$$

we have

$$\begin{aligned}\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x \cos^4 x \cos x \, dx \\ &= \int \sin^2 x (1 - \sin^2 x)^2 \cos x \, dx\end{aligned}$$

Now, use the substitution

$$u = \sin x, \qquad du = \cos x \, dx,$$

then,

$$\begin{aligned}\int \sin^2 x \cos^5 x \, dx &= \int \sin^2 x (1 - \sin^2 x)^2 \cos x \, dx \\ &= \int u^2 (1 - u^2)^2 \, du \\ &= \int u^2 (1 - 2u^2 + u^4) \, du \\ &= \int (u^2 - 2u^4 + u^6) \, du \\ &= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + C \\ &= \frac{\sin^3 x}{3} - \frac{2 \sin^5 x}{5} + \frac{\sin^7 x}{7} + C\end{aligned}$$

3. (8 points) Evaluate the integral (simplify your answer so that it does not contain inverse trigonometric functions)

$$\int \frac{1}{x^2\sqrt{4x^2-1}}\,dx$$

Solution Use the trigonometric substitution

$$x = \frac{1}{2}\sec\theta, \qquad dx = \frac{1}{2}\tan\theta\sec\theta\,d\theta$$

we have

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{4x^2-1}}\,dx &= \int \frac{1}{\left(\frac{1}{2}\sec\theta\right)^2\sqrt{4\left(\frac{1}{2}\sec\theta\right)^2-1}}\,\frac{1}{2}\tan\theta\sec\theta\,d\theta \\
&= \int \frac{\frac{1}{2}\tan\theta\sec\theta}{\frac{1}{4}\sec^2\theta\sqrt{\sec^2\theta-1}}\,d\theta \\
&= \int \frac{2\tan\theta\sec\theta}{\sec^2\theta\tan\theta}\,d\theta \\
&= \int \frac{2}{\sec\theta}\,d\theta \\
&= \int 2\cos\theta\,d\theta \\
&= 2\sin\theta + C
\end{aligned}$$

Since we have set $x = \frac{1}{2}\sec\theta$, clearly,

$$\begin{aligned}
& \sec\theta = 2x \\
\Rightarrow\quad & \cos\theta = \frac{1}{2x} \\
\Rightarrow\quad & \sin\theta = \sqrt{1-\cos^2\theta} = \sqrt{1-\left(\frac{1}{2x}\right)^2}
\end{aligned}$$

Hence we have

$$\int \frac{1}{x^2\sqrt{4x^2-1}}\,dx = 2\sin\theta + C = 2\sqrt{1-\left(\frac{1}{2x}\right)^2} + C$$

4. (10 points) Evaluate the integral

$$\int \frac{x^3 + 4x^2 - 12x + 1}{x^2 + 4x - 12}\, dx$$

Solution First, by long division, we have

$$\frac{x^3 + 4x^2 - 12x + 1}{x^2 + 4x - 12} = x + \frac{1}{x^2 + 4x - 12}$$

So

$$\int \frac{x^3 + 4x^2 - 12x + 1}{x^2 + 4x - 12}\, dx = \int \left(x + \frac{1}{x^2 + 4x - 12} \right) dx$$

Next, we use partial fraction for $\frac{1}{x^2+4x-12}$, that is,

$$\frac{1}{x^2 + 4x - 12} = \frac{1}{(x-2)(x+6)} = \frac{A}{x-2} + \frac{B}{x+6} = \frac{A(x+6) + B(x-2)}{(x-2)(x+6)}$$

Then we have

$$A(x+6) + B(x-2) = 1$$

By setting $x = 2$, we compute that $A = 1/8$. By setting $x = -6$, we have $B = -1/8$. Therefore

$$\frac{1}{x^2 + 4x - 12} = \frac{1/8}{x-2} - \frac{1/8}{x+6}$$

Finally

$$\begin{aligned}
\int \frac{x^3 + 4x^2 - 12x + 1}{x^2 + 4x - 12}\, dx &= \int \left(x + \frac{1}{x^2 + 4x - 12} \right) dx \\
&= \int \left(x + \frac{1/8}{x-2} - \frac{1/8}{x+6} \right) dx \\
&= \int x\, dx + \frac{1}{8} \int \frac{1}{x-2}\, dx - \frac{1}{8} \int \frac{1}{x+6}\, dx \\
&= \frac{x^2}{2} + \frac{1}{8} \ln|x-2| - \frac{1}{8} \ln|x+6| + C
\end{aligned}$$

5. (8 points) Determine whether the following integral is convergent or divergent. Evaluate it if convergent.

$$\int_2^4 \frac{1}{\sqrt[4]{x-2}}\,dx$$

Solution This integral is an improper integral of type 2, since the value of $\frac{1}{\sqrt[4]{x-2}}$ goes to $+\infty$ as x goes to 2. Therefore

$$\begin{aligned}
\int_2^4 \frac{1}{\sqrt[4]{x-2}}\,dx &= \lim_{t\to 2^+} \int_t^4 (x-2)^{-1/4}\,dx \\
&= \lim_{t\to 2^+} \frac{4}{3}(x-2)^{3/4}|_t^4 \\
&= \lim_{t\to 2^+} \left(\frac{4}{3}\,2^{3/4} - \frac{4}{3}(t-2)^{3/4}\right) \\
&= \frac{4}{3}\,2^{3/4}
\end{aligned}$$

6. (8 points) Find the area of the surface obtained by rotating the curve about the x-axis.

$$y = \sqrt{1+2x}, \qquad 1 \le x \le 5$$

Solution 1

$$Area = \int_1^5 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$= \int_1^5 2\pi \sqrt{1+2x}\sqrt{1 + \left(\frac{2}{2\sqrt{1+2x}}\right)^2}\, dx$$

$$= \int_1^5 2\pi \sqrt{1+2x}\sqrt{1 + \frac{1}{1+2x}}\, dx$$

$$= \int_1^5 2\pi \sqrt{(1+2x)\left(1 + \frac{1}{1+2x}\right)}\, dx$$

$$= \int_1^5 2\pi \sqrt{2+2x}\, dx = 2\sqrt{2}\pi \frac{2}{3}(x+1)^{3/2}|_1^5$$

$$= \frac{4\sqrt{2}\pi}{3}6^{3/2} - \frac{4\sqrt{2}\pi}{3}2^{3/2} = \frac{16\pi}{3}(3\sqrt{3}-1)$$

Solution 2 Notice that $y = \sqrt{1+2x}$, $1 \le x \le 5$ is equivalent to $x = \frac{y^2-1}{2}$, $\sqrt{3} \le y \le \sqrt{11}$, we have

$$Area = \int_{\sqrt{3}}^{\sqrt{11}} 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

$$= \int_{\sqrt{3}}^{\sqrt{11}} 2\pi y \sqrt{1+y^2}\, dy$$

$$= \pi \frac{2}{3}(1+y^2)^{3/2}|_{\sqrt{3}}^{\sqrt{11}}$$

$$= \frac{2\pi}{3}12^{3/2} - \frac{2\pi}{3}4^{3/2} = \frac{16\pi}{3}(3\sqrt{3}-1)$$