Math 2153, Exam III, Apr. 17, 2008

Name: ____________________

Score:

Each problem is worth 5 points. The total is 50 points.
For series convergence or divergence, please write down the name of the test you are using and details of using the test. Otherwise no credit will be given.

1. Test the following series for convergence or divergence. Then gives an estimate of $|R_{10}| = |S - S_{10}|$.

$$\frac{7}{\ln 2} - \frac{7}{\ln 3} + \frac{7}{\ln 4} - \frac{7}{\ln 5} + \frac{7}{\ln 6} - \cdots$$

Solution This is an alternating series and can be written as

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{7}{\ln(n+1)}$$

By using the alternating series test and notice that the absolute value of the general term satisfies

(a) $\frac{7}{\ln(n+1)}$ is decreasing as n increases;

(b) $\lim_{n\to\infty} \frac{7}{\ln(n+1)} = 0$,

we know the series is convergent.

For alternating series,

$$|R_{10}| \leq |a_{11}| = \frac{7}{\ln 12}.$$

2. Determine whether $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ is absolutely convergent, conditionally convergent, or divergent.

Solution Notice that

$$0 \leq \sum_{n=1}^{\infty} |\frac{\sin(4n)}{4^n}| \leq \sum_{n=1}^{\infty} \frac{1}{4^n}$$

and because $\sum_{n=1}^{\infty} \frac{1}{4^n}$ is a convergent geometric series. By using the comparison test, we know that $\sum_{n=1}^{\infty} |\frac{\sin(4n)}{4^n}|$ is convergent. Hence $\sum_{n=1}^{\infty} \frac{\sin(4n)}{4^n}$ is absolutely convergent by definition.

3. Determine whether $\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent, conditionally convergent, or divergent.

Solution By using the root test

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^n} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+1} = \lim_{n\to\infty} \frac{1+1/n^2}{2+1/n^2} = \frac{1}{2} < 1.$$

Therefore the given series is absolutely convergent.

4. Find the radius of convergence and interval of convergence of $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n+4}$.
(You need to specify whether the series converges or not at the endpoints of the interval.)

Solution Using the Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}x^{n+1}}{(n+1)+4}}{\frac{(-1)^n x^n}{n+4}} \right| = \lim_{n\to\infty} \left| x \frac{n+4}{n+5} \right| = \lim_{n\to\infty} \left| x \frac{1+4/n}{1+5/n} \right| = |x|.$$

The power series converges when $|x| < 1$ and diverges when $|x| > 1$. Hence the radius of convergence is 1.

To find the interval of convergence, we need to know whether the series converges or not when $x = 1$ or $x = -1$. On these two endpoints, the ratio test will give $|x| = 1$ and hence inconclusive. Other tests need to be used in order to draw the conclusion.

(a) when $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$ which is an alternating series. By using the alternating series test, it can be shown that the series is convergent;

(b) when $x = -1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{n+4}$. By using limit comparision test with a divergent p-series $\sum_{n=1}^{\infty} \frac{1}{n}$, we know that $\sum_{n=1}^{\infty} \frac{1}{n+4}$ is also divergent.

Combine the above, we can see that the series converges on $I = (-1, 1]$.

5. Find a power series representation of centered at 0 for $f(x) = \frac{x}{9+x^2}$.
(You need to give the general form of the series instead of writing only the first several terms. For example, $\sum_{n=0}^{\infty} 2^n x^n$ is a correct form while $1 + 2x + 4x^2 + 8x^3 + \cdots$ will not be accepted.)

Solution We will use the formula $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$.

$$\begin{aligned} f(x) = \frac{x}{9+x^2} &= \frac{x}{9} \cdot \frac{1}{1+\frac{x^2}{9}} \\ &= \frac{x}{9} \cdot \frac{1}{1-(-\frac{x^2}{9})} \\ &= \frac{x}{9} \sum_{n=0}^{\infty} (-\frac{x^2}{9})^n \\ &= \sum_{n=0}^{\infty} \frac{x}{9} (-\frac{x^2}{9})^n \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{9^{n+1}} \end{aligned}$$

6. Find the first 5 terms in the Taylor series representation centered at $a = 1$ for $f(x) = \sqrt{x}$.

Solution

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\cdots$	$\cdots$	$\cdots$

So the first five terms of the Taylor series are

$$
\begin{aligned}
&f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 \cdots \\
=&1 + \frac{1}{2}(x-1) - \frac{1}{4 \cdot 2!}(x-1)^2 + \frac{3}{8 \cdot 3!}(x-1)^3 - \frac{15}{16 \cdot 4!}(x-1)^4 + \cdots \\
=&1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3 - \frac{5}{128}(x-1)^4 + \cdots
\end{aligned}
$$

7. Use Taylor series to evaluate the integral

$$\int \frac{\sin x}{x}\, dx$$

Solution Since we know the Taylor series for $\sin x$ is $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$,

$$\begin{aligned}
\int \frac{\sin x}{x}\, dx &= \int \frac{\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}}{x}\, dx \\
&= \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n+1)!}\, dx \\
&= \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2n}}{(2n+1)!}\, dx \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!(2n+1)} + C
\end{aligned}$$

8. Eliminate the parameter t to find a Cartesian equation of the curve:

$$\begin{cases} x = 10\ln(9t) \\ y = \sqrt{t} \end{cases}$$

Solution 1 From the first equation, we have

$$\frac{x}{10} = \ln(9t) \;\Rightarrow\; 9t = e^{x/10} \;\Rightarrow\; t = \frac{e^{x/10}}{9}$$

Substitute it into the second equation gives

$$y = \sqrt{\frac{e^{x/10}}{9}}$$

Solution 2 From the second equation, we have

$$t = y^2$$

Substitute it into the first equation gives

$$x = 10\ln(9y^2)$$

9. Find an equation of the tangent line to the parametric curve at the point corresponding to $t = 1$.

$$\begin{cases} x = e^{\sqrt{t}} \\ y = t - \ln(t^9) \end{cases}$$

Solution First, we need to find the slope at $t = 1$. The derivative $\frac{dy}{dx}$ is

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1 - \frac{1}{t^9} 9t^8}{e^{\sqrt{t}} \frac{1}{2} t^{-1/2}} = \frac{1 - \frac{9}{t}}{\frac{e^{\sqrt{t}}}{2\sqrt{t}}}$$

So at $t = 1$, the slope is

$$\left.\frac{dy}{dx}\right|_{t=1} = \frac{1 - \frac{9}{1}}{\frac{e^{\sqrt{1}}}{2\sqrt{1}}} = -\frac{16}{e}$$

Next, we need to find the xy coordinates at $t = 1$.

$$x|_{t=1} = e^{\sqrt{1}} = 1, \qquad y|_{t=1} = 1 - \ln(1^9) = 1 - 0 = 1$$

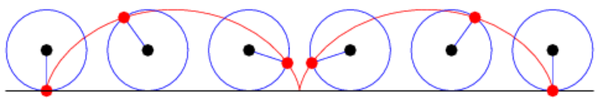
Finally, the tangent line is

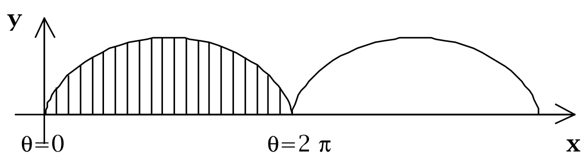
$$(y - 1) = -\frac{16}{e}(x - e)$$

10. A cycloid is the curve defined by the path of a point on the edge of circular wheel as the wheel rolls along a straight line. If the rotating wheel has radius 2, the equation of the cycloid is

$$\begin{cases} x = 2(\theta - \sin\theta) \\ y = 2(1 - \cos\theta) \end{cases}$$

Find the area of the shaded region.





Solution

$$\begin{aligned}
Area &= \int_0^{2\pi} y\frac{dx}{d\theta}\,d\theta \\
&= \int_0^{2\pi} 2(1-\cos\theta)(2-2\cos\theta)\,d\theta \\
&= \int_0^{2\pi} 4(1-\cos\theta)^2\,d\theta \\
&= \int_0^{2\pi} (4-8\cos\theta+4\cos^2\theta)\,d\theta \\
&= \int_0^{2\pi} (4-8\cos\theta+4\frac{1+\cos(2\theta)}{2})\,d\theta \\
&= \int_0^{2\pi} (6-8\cos\theta+2\cos(2\theta))\,d\theta \\
&= (6\theta - 8\sin\theta + \sin(2\theta))|_0^{2\pi} \\
&= 12\pi
\end{aligned}$$

Trigonometry

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \sin(\pi/2 - \theta) = \cos\theta$$

$$\cos(\pi/2 - \theta) = \sin\theta \qquad \tan(\pi/2 - \theta) = \cot\theta$$

$$\sin(x+y) = \sin x\cos y + \cos x\sin y \qquad \cos(x+y) = \cos x\cos y - \sin x\sin y$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y \qquad \cos(x-y) = \cos x\cos y + \sin x\sin y$$

$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x\tan y} \qquad \tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x\tan y}$$

$$\sin 2x = 2\sin x\cos x \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin A\cos B = \frac{1}{2}[\sin(A-B) + \sin(A+B)]$$

$$\sin A\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$\cos A\cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$$

Differentiation rules

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x\cot x \qquad \frac{d}{dx}(\sec x) = \sec x\tan x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x\ln a$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x\ln a}$$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

Table of integrals

$$\int u\,dv = u\,v - \int v\,du \qquad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C \qquad \int \csc u \cot u\,du = -\csc u + C$$

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u + \sqrt{u^2+a^2}) + C \qquad \int \frac{du}{\sqrt{u^2-a^2}} = \ln|u + \sqrt{u^2-a^2}| + C$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C \qquad \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

$$\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C \qquad \int \ln u\,du = u\ln u - u + C$$

Infinite sequences and series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots = \begin{cases} \frac{a}{1-r} & \text{for } |r| < 1 \\ \text{divergent} & \text{otherwise} \end{cases}$$

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le S \le S_n + \int_{n}^{\infty} f(x)\,dx$$

Taylor series

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots
\end{aligned}
$$

Some Maclaurin series and interval of convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad (-1,\, 1)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad (-\infty,\, \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad (-\infty,\, \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (-\infty,\, \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad [-1,\, 1]$$

Area, arc length, and surface area

$$
\begin{aligned}
\text{area } &= \int_{\alpha}^{\beta} y \left(\frac{dx}{dt}\right) dt \\
\text{arc length } &= \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
\text{surface area } &= \begin{cases} \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt & \text{rotate around x-axis} \\ \int_{\alpha}^{\beta} 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt & \text{rotate around y-axis} \end{cases}
\end{aligned}
$$