Math 2153, Exam II, Mar. 6, 2008

Name: ____________________

Score:

Each problem is worth 5 points. The total is 50 points.

1. Evaluate the integral

$$\int \frac{dx}{x^2 + 2x + 2}$$

Solution By completing the squares, and then set change of variables $u = x + 1$, $du = dx$,

$$\begin{aligned}\int \frac{dx}{x^2 + 2x + 2} &= \int \frac{dx}{(x+1)^2 + 1} \\ &= \int \frac{du}{u^2 + 1} \\ &= \arctan u + C \\ &= \arctan(x + 1) + C\end{aligned}$$

2. Evaluate the integral

$$\int \frac{x^2}{x+5}dx$$

Solution Using the long division, we have

$$\begin{array}{r|l} & x \quad - 5 \\ \hline x+5 & x^2 \\ & -x^2 - 5x \\ \cline{2-2} & \qquad -5x \\ & \qquad 5x + 25 \\ \cline{2-2} & \qquad\qquad 25 \end{array}$$

Therefore

$$\begin{aligned} \int \frac{x^2}{x+5}dx &= \int \left(x - 5 + \frac{25}{x+5} \right) dx \\ &= \frac{1}{2}x^2 - 5x + 25 \ln |x+5| + C \end{aligned}$$

3. Evaluate the following improper integral
(No credit if you treat it as a normal integral.)

$$\int_0^1 \frac{5}{x^{0.2}} dx$$

Solution This is a type II improper integral and $\frac{5}{x^{0.2}} \to \infty$ as $x \to 0$.

$$\begin{aligned}
\int_0^1 \frac{5}{x^{0.2}} dx &= \lim_{t \to 0} \int_t^1 \frac{5}{x^{0.2}} dx \\
&= \lim_{t \to 0} \frac{5x^{0.8}}{0.8} |_t^1 \\
&= \lim_{t \to 0} \left(\frac{5}{0.8} - \frac{5t^{0.8}}{0.8} \right) \\
&= \frac{5}{0.8}
\end{aligned}$$

4. Evaluate the integral

$$\int \frac{10}{(x-1)(x^2+9)} dx$$

Solution Use partial fractions,

$$\begin{aligned}
& \frac{10}{(x-1)(x^2+9)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+9} \\
\Rightarrow \quad & \frac{10}{(x-1)(x^2+9)} = \frac{A(x^2+9)+(Bx+C)(x-1)}{(x-1)(x^2+9)} \\
\Rightarrow \quad & 10 = A(x^2+9) + (Bx+C)(x-1)
\end{aligned}$$

By seting $x = 1$, we have

$$10 = 10A + 0$$

Therefore $A = 1$. Substitute it into the equation for A, B and C and simplify:

$$\begin{aligned}
& 10 = (x^2+9) + (Bx+C)(x-1) \\
\Rightarrow \quad & 10 = (x^2+9) + (Bx^2 + (C-B)x - C) \\
\Rightarrow \quad & 10 = (1+B)x^2 + (C-B)x + (9-C) \\
\Rightarrow \quad & 0x^2 + 0x + 10 = (1+B)x^2 + (C-B)x + (9-C)
\end{aligned}$$

Hence

$$1 + B = 0, \qquad C - B = 0, \qquad 9 - C = 10$$

We can easily solve them and

$$B = -1, \qquad C = -1$$

Combining the above,

$$\frac{10}{(x-1)(x^2+9)} = \frac{1}{x-1} + \frac{-x-1}{x^2+9} = \frac{1}{x-1} - \frac{x+1}{x^2+9}$$

Finally

$$\begin{aligned}
\int \frac{10}{(x-1)(x^2+9)} dx &= \int \left(\frac{1}{x-1} - \frac{x+1}{x^2+9} \right) dx \\
&= \int \frac{1}{x-1} dx - \int \frac{x}{x^2+9} dx - \int \frac{1}{x^2+9} dx \\
&= \ln|x-1| - \frac{1}{2}\ln|x^2+9| - \frac{1}{3}\arctan\frac{x}{3} + C
\end{aligned}$$

5. Evaluate the improper integral
(No credit if you treat it as a normal integral.)

$$\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$$

Solution This is a Type I improper integral. We need to break it into the integral from $-\infty$ to 0, and the integral from 0 to ∞.

$$\begin{aligned} \int_{-\infty}^{0} \frac{x}{1+x^2} dx &= \lim_{t\to\infty} \int_{t}^{0} \frac{x}{1+x^2} dx \\ &= \lim_{t\to-\infty} \frac{1}{2} \ln(1+x^2)|_t^0 \\ &= \lim_{t\to-\infty} (\frac{1}{2}\ln 1 - \frac{1}{2}\ln t^2) \\ &= 0 - \infty = -\infty \end{aligned}$$

Because the integral from $-\infty$ to 0 is already divergent, the integral $\int_{-\infty}^{\infty} \frac{x}{1+x^2} dx$ diverges. It is not necessary to compute the integral from 0 to ∞.

6. Find a formula for the general term a_n of the sequence

$$\{\frac{1}{e^2}, -\frac{4}{e^3}, \frac{9}{e^4}, -\frac{16}{e^5}, \cdots\}$$

Then evaluate the limit of this sequence.

Solution $a_n = (-1)^{n+1}\frac{n^2}{e^{n+1}}$, for $n = 1, 2, 3, \cdots$.

Notice that the limit of $|a_n|$ is

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^2}{e^{n+1}} \overset{H}{=} \lim_{n\to\infty} \frac{2n}{e^{n+1}} \overset{H}{=} \lim_{n\to\infty} \frac{2}{e^{n+1}} = 0$$

Then we have

$$\lim_{n\to\infty} a_n = 0.$$

7. Use the integral test to determine whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{1}{8n+1}$$

Solution Let $f(x) = \frac{1}{8x+1}$. Clearly, it is continuous, positive and decreasing. Since

$$\int_1^{\infty} \frac{1}{8x+1} dx = \lim_{t\to\infty} \int_1^t \frac{1}{8x+1} dx = \lim_{t\to\infty} (\frac{\ln(8t+1)}{8} - \frac{\ln 1}{8}) = \infty$$

According to the integral test, the series is also divergent.

8. Determine whether the series is convergent or divergent. If it is convergent, find its sum. *(Hint: telescoping series.)*

$$\sum_{n=1}^{\infty} \frac{2}{n^2 + 3n + 2}$$

Solution By using partial fractions, we can break each term as

$$a_n = \frac{2}{n^2 + 3n + 2} = \frac{2}{n+1} - \frac{2}{n+2}$$

Hence the partial sum is

$$\begin{aligned} S_n &= a_1 + a_2 + a_3 + \cdots + a_n \\ &= (\frac{2}{2} - \frac{2}{3}) + (\frac{2}{3} - \frac{2}{4}) + (\frac{2}{4} - \frac{2}{5}) + \cdots + (\frac{2}{n+1} - \frac{2}{n+2}) \\ &= \frac{2}{2} - \frac{2}{n+2} \\ &= 1 - \frac{2}{n+2} \end{aligned}$$

Then

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty} (1 - \frac{2}{n+2}) = 1$$

So the series is convergent and the sum is 1.

9. Express the number as a ratio of integers:

$$3.\bar{2} = 3.2222\cdots$$

Solution

$$\begin{aligned} 3.\bar{2} &= 3 + 0.2 + 0.02 + 0.002 + \cdots \\ &= 3 + \frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots \end{aligned}$$

The geometric series

$$\frac{2}{10} + \frac{2}{100} + \frac{2}{1000} + \cdots = \frac{a}{1-r} = \frac{\frac{2}{10}}{1 - \frac{1}{10}} = \frac{2}{9}$$

Then

$$3.\bar{2} = 3 + \frac{2}{9} = \frac{29}{9}$$

10. Test whether the series is convergent or divergent:

$$\sum_{n=1}^{\infty} \frac{n}{n^4+1}$$

Solution 1 Use the integral test and set $f(x) = \frac{x}{x^4+1}$. Since

$$\int_1^{\infty} \frac{x}{x^4+1} dx = \lim_{t\to\infty} \int_1^t \frac{x}{x^4+1} dx = \lim_{t\to\infty} \frac{\arctan x^2}{2}|_1^t$$
$$= \lim_{t\to\infty} (\frac{\arctan t^2}{2} - \frac{\arctan 1}{2}) = \pi/4 - \pi/8 = \pi/8$$

Hence the series is convergent.

Solution 2 Use the comparison test and set $a_n = \frac{n}{n^4+1}$, $b_n = \frac{1}{n^3}$. Notice that both a_n and b_n are positive and

$$a_n = \frac{n}{n^4+1} < \frac{n}{n^4} = \frac{1}{n^3} = b_n$$

Since $\sum b_n = \sum \frac{1}{n^3}$ is a p-series with $p = 3$, it is convergent. Hence $\sum a_n = \sum \frac{n}{n^4+1}$ is also convergent

Solution 3 Use the limit comparision test and set $a_n = \frac{n}{n^4+1}$, $b_n = \frac{1}{n^3}$, we have

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^4+1}}{\frac{1}{n^3}} = \lim_{n\to\infty} \frac{n^4}{n^4+1}$$
$$= \lim_{n\to\infty} \frac{1}{1+1/n^4} = 1$$

Therefore, series $\sum a_n$ and $\sum b_n$ are either both convergent or both divergent. Since $\sum b_n = \sum \frac{1}{n^3}$ is convergent, $\sum a_n = \sum \frac{n}{n^4+1}$ is also convergent.