Math 2153, Exam I, Feb. 7, 2008

Name: ____________________

Score:

Each problem is worth 5 points. The total is 50 points.

1. Solve the equation for x.

$$2\ln x = \ln 2 + \ln(3x-4)$$

Solution Using the property of logarithmic functions, we have

$$\begin{aligned} & \ln x^2 = \ln[2(3x-4)] \\ \Rightarrow \quad & x^2 = 2(3x-4) = 6x-8 \\ \Rightarrow \quad & x^2 - 6x + 8 = 0 \\ \Rightarrow \quad & (x-2)(x-4) = 0 \\ \Rightarrow \quad & x = 2, \quad x = 4 \end{aligned}$$

Finally, notice that both x and $3x-2$ are positive for $x=2$ and $x=4$. Therefore, we have two solutions $x=2$ and $x=4$.

2. Differentiate the following function $f(x) = \ln(x \cos 3x)$.

   Solution By using the chain rule:

$$f'(x) = \frac{1}{x \cos 3x}(x \cos 3x)' = \frac{\cos 3x - 3x \sin 3x}{x \cos 3x}$$

3. Use logarithmic differentiation to find the derivative of $y = (2x+1)^7(x^4-3)^3$.
   (You get no point if not using logarithmic differentiation.)

   Solution Take nature log of both sides of the equation and use the property of logarithmic functions

$$\ln y = \ln(2x+1)^7(x^4-3)^3 = 7\ln(2x+1) + 3\ln(x^4-3)$$

   Differentiate both sides:

$$\frac{y'}{y} = 7\frac{2}{2x+1} + 3\frac{4x^3}{x^4-3} = \frac{14}{2x+1} + \frac{12x^3}{x^4-3}$$

   Therefore

$$\begin{aligned} y' &= y\left(\frac{14}{2x+1} + \frac{12x^3}{x^4-3}\right) \\ &= (2x+1)^7(x^4-3)^3\left(\frac{14}{2x+1} + \frac{12x^3}{x^4-3}\right) \\ &= 14(2x+1)^6(x^4-3)^3 + 12x^3(2x+1)^7(x^4-3)^2 \end{aligned}$$

4. Find the limit

$$\lim_{x\to\infty} e^{2-x^2}$$

Solution As x goes to ∞, $2-x^2$ goes to $-\infty$. Since $\lim_{u\to-\infty} e^u = 0$, so the answer should be 0.

5. Find the limit

$$\lim_{x\to 0} \frac{e^x - 1}{x^5 + 3x}$$

Solution Notice the limit is of type $\frac{0}{0}$, we need to use the L'Hospital's rule.

$$\begin{aligned}
\lim_{x\to 0} \frac{e^x - 1}{x^5 + 3x} &\overset{H}{=} \lim_{x\to 0} \frac{(e^x - 1)'}{(x^5 + 3x)'} \\
&= \lim_{x\to 0} \frac{e^x}{5x^4 + 3} \\
&= \frac{e^0}{0 + 3} \\
&= 1/3
\end{aligned}$$

6. Simplify the expression $\tan(\arcsin x)$.
(Your answer should be an algebraic function of x.)

Solution Let $\theta = \arcsin x$, then we know that $-\pi/2 \leq \theta \leq \pi/2$ and

$$\sin\theta = x$$

$\cos\theta$ must be non-negative for $-\pi/2 \leq \theta \leq \pi/2$. It is easy to calculate that

$$\cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{1-x^2}$$

Finally

$$\tan(\arcsin x) = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{\sqrt{1-x^2}}$$

7. Evaluate the integral $\int_0^1 t\,5^{-t}\,dt$.

Solution We will use integration by parts. Let $u = t$ and $v' = 5^{-t}$, then we have $u' = 1$ and $v = \frac{-5^{-t}}{\ln 5}$.

$$\begin{aligned}
\int_0^1 t\,5^{-t}\,dt &= t\,\frac{-5^{-t}}{\ln 5}\bigg|_0^1 - \int_0^1 \frac{-5^{-t}}{\ln 5}\,dt \\
&= -\frac{1}{5\ln 5} + \frac{1}{\ln 5}\int_0^1 5^{-t}\,dt \\
&= -\frac{1}{5\ln 5} + \frac{1}{\ln 5}\,\frac{-5^{-t}}{\ln 5}\bigg|_0^1 \\
&= -\frac{1}{5\ln 5} + \frac{4}{5(\ln 5)^2}
\end{aligned}$$

8. Evaluate the integral $\int \sin^3 x \cos^3 x \, dx$.

Solution Set $u = \sin x$, then $du = \cos x \, dx$ and we have

$$\begin{aligned}
\int \sin^3 x \cos^3 x \, dx &= \int \sin^3 x \cos^2 x (\cos x \, dx) \\
&= \int \sin^3 x (1 - \sin^2 x)(\cos x \, dx) \\
&= \int u^3 (1 - u^2) \, du \\
&= \frac{1}{4} u^4 - \frac{1}{6} u^6 + C \\
&= \frac{1}{4} \sin^4 x - \frac{1}{6} \sin^6 x + C
\end{aligned}$$

Remark: This problem can also be solved by setting $u = \cos x$.

9. Evaluate the integral

$$\int \frac{1}{x^2\sqrt{x^2-16}}\,dx$$

Solution Use trigonometric substitution and set $x = 4\sec\theta$, where $\theta \in [0, \pi/2) \cup [\pi, 3\pi/2)$. Then we know that $\tan\theta$ must be non-negative and $dx = 4\tan\theta\sec\theta\,d\theta$.

$$\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2-16}}\,dx &= \int \frac{1}{16\sec^2\theta\sqrt{16\sec^2\theta-16}} 4\tan\theta\sec\theta\,d\theta \\
&= \int \frac{1}{16\sec^2\theta\,(4\tan\theta)} 4\tan\theta\sec\theta\,d\theta \\
&= \int \frac{1}{16\sec\theta}\,d\theta \\
&= \frac{1}{16}\int \cos\theta\,d\theta \\
&= \frac{1}{16}\sin\theta + C \\
&= \frac{1}{16}\sin(\sec^{-1}\frac{x}{4}) + C \\
&= \frac{\sqrt{x^2-16}}{16x} + C
\end{aligned}$$

10. Evaluate the integral *(You do NOT need to simplify the answer.)*

$$\int \frac{\cos\theta}{4+\sin^2\theta}\,d\theta$$

Solution Set $u = \sin\theta$, then $du = \cos\theta\,d\theta$ and

$$\begin{aligned}\int \frac{\cos\theta}{4+\sin^2\theta}\,d\theta &= \int \frac{1}{4+u^2}\,du \\ &= \frac{1}{2}\tan^{-1}\frac{u}{2} + C \\ &= \frac{1}{2}\tan^{-1}\frac{\sin\theta}{2} + C\end{aligned}$$

Trigonometry

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} \qquad \sin^2\theta + \cos^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

$$\sin(-\theta) = -\sin\theta \qquad \cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta \qquad \sin(\pi/2 - \theta) = \cos\theta$$

$$\cos(\pi/2 - \theta) = \sin\theta \qquad \tan(\pi/2 - \theta) = \cot\theta$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \qquad \cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y \qquad \cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \qquad \tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

$$\sin 2x = 2 \sin x \cos x \qquad \begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 = 1 - 2\sin^2 x \end{aligned}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\sin^2 x = \frac{1 - \cos 2x}{2} \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

Differentiation rules

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(e^x) = e^x \qquad \frac{d}{dx}(a^x) = a^x \ln a$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x} \qquad \frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}(\csc^{-1} x) = -\frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$$

Table of integrals

$$\int u\,dv = u\,v - \int v\,du \qquad \int \frac{1}{u}\,du = \ln|u| + C$$

$$\int e^u\,du = e^u + C \qquad \int a^u\,du = \frac{a^u}{\ln a} + C$$

$$\int \sin u\,du = -\cos u + C \qquad \int \cos u\,du = \sin u + C$$

$$\int \sec^2 u\,du = \tan u + C \qquad \int \csc^2 u\,du = -\cot u + C$$

$$\int \sec u \tan u\,du = \sec u + C \qquad \int \csc u \cot u\,du = -\csc u + C$$

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \cot u\,du = \ln|\sin u| + C$$

$$\int \sec u\,du = \ln|\sec u + \tan u| + C \qquad \int \csc u\,du = \ln|\csc u - \cot u| + C$$

$$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C \qquad \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + C$$

$$\int \frac{du}{\sqrt{u^2+a^2}} = \ln(u + \sqrt{u^2+a^2}) + C \qquad \int \frac{du}{\sqrt{u^2-a^2}} = \ln|u + \sqrt{u^2-a^2}| + C$$

$$\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C \qquad \int \frac{du}{u^2-a^2} = \frac{1}{2a}\ln\left|\frac{u-a}{u+a}\right| + C$$

$$\int ue^{au}\,du = \frac{1}{a^2}(au-1)e^{au} + C \qquad \int \ln u\,du = u\ln u - u + C$$