Quiz 8 – Math 2153, Calculus II – Nov. 4, 2011

1. Find the radius of convergence and the interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$$

Solution We use the ratio test to determine the radius of convergence.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(4x+1)^{n+1}}{n+1^2}}{\frac{(4x+1)^n}{n^2}}\right| = \lim_{n\to\infty}\left|\frac{(4x+1)^{n+1}}{n+1^2}\cdot\frac{n^2}{(4x+1)^n}\right|$$

$$= \lim_{n\to\infty}\left|(4x+1)\frac{n^2}{(n+1)^2}\right| = \lim_{n\to\infty}\left|(4x+1)\frac{1}{(1+\frac{1}{n})^2}\right|$$

$$= |4x+1|$$

The series is convergent for all

$$|4x+1| < 1 \qquad \Rightarrow \qquad 4|x+\frac{1}{4}| < 1 \qquad \Rightarrow \qquad |x+\frac{1}{4}| < 1/4$$

Therefore the radius of convergence is $\frac{1}{4}$.

Next, notice that

$$|x+\frac{1}{4}| < 1/4 \qquad \Rightarrow \qquad -1/4 < x+1/4 < 1/4$$

$$\Rightarrow \qquad -1/4-1/4 < x < 1/4-1/4 \qquad \Rightarrow \qquad -1/2 < x < 0$$

We need to determine whether the series is convergent or not at $x=-1/2$ and $x=0$.

(a) For $x=-1/2$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(4(-1/2)+1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

It is convergent by the alternating series test, because $\lim_{n\to\infty}\frac{1}{n^2}=0$ and $\frac{1}{n^2}$ is decreasing when n increases.

(b) For $x=0$, the series becomes

$$\sum_{n=1}^{\infty}\frac{(4(0)+1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

It is a p-series with $p=2$ and hence is convergent.

Combine the above, the interval of convergence is

$$[-1/2,\, 0]$$