Quiz 7 – Math 2153, Calculus II – Oct.21, 2011

1. Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$$

Solution To determine whether a series is absolutely convergent or not, an easy way is to use the ratio test or the root test. However, for this problem, unfortunately, both the ratio test and the root test are inconclusive. So we have to find other ways.

First, notice that the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$ is an alternating series and

(a) $\lim_{n\to\infty} \left|(-1)^n \frac{n}{\sqrt{n^3+2}}\right| = \lim_{n\to\infty} \frac{n}{\sqrt{n^3+2}} = 0$

(b) When n increases, $\left|(-1)^n \frac{n}{\sqrt{n^3+2}}\right| = \frac{n}{\sqrt{n^3+2}} = \frac{1}{\sqrt{n+\frac{2}{n^2}}}$ decreases, because it is easy to see that $n + \frac{2}{n^2}$ increases as n increases.

Therefore, by the alternating series test, $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$ is convergent.

Next, we shall look at

$$\sum_{n=1}^{\infty}\left|(-1)^n \frac{n}{\sqrt{n^3+2}}\right| = \sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3+2}}$$

By using the limit comparison test and compare it with $\sum_{n=1}^{\infty}\frac{n}{\sqrt{n^3}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, we have

$$\lim_{n\to\infty}\frac{\frac{n}{\sqrt{n^3+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty}\frac{\sqrt{n^3}}{\sqrt{n^3+2}} = \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{2}{n^3}}} = 1$$

Therefore, these two series either both converge or both diverge. The series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a p-series with $p = 1/2$ and it diverges. So the series $\sum_{n=1}^{\infty}\left|(-1)^n \frac{n}{\sqrt{n^3+2}}\right|$ also diverges.

Combine the above, we know that the series $\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{n^3+2}}$ is conditionally convergent.