Math 2153, Exam III, Nov. 30, 2011

Name: ____________________

Score:

Read the problems carefully before you begin. Show all your work neatly and concisely, and indicate your final answer clearly.

1. (15 points) Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty}(-1)^n\frac{(2x-3)^n}{n+4}$$

Solution. By using the ratio test, we have

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\frac{(2x-3)^{n+1}}{(n+1)+4}}{(-1)^n\frac{(2x-3)^n}{n+4}}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2x-3)(n+4)}{n+5}\right| \\ &= |2x-3|\end{aligned}$$

Therefore,

$$|2x-3|<1 \quad \Rightarrow \quad |x-\frac{3}{2}|<\frac{1}{2}$$

Hence the radius of convergence is $\frac{1}{2}$. And the interval goes from $x=1$ to $x=2$. Next, we need to examing the convergence at the two end points $x=1$ and $x=2$.

(a) at $x=1$, the series becomes

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^n}{n+4}=\sum_{n=0}^{\infty}\frac{1}{n+4}$$

By using either the integral test or the limit comparison test, we know this series is divergent.

(b) at $x=2$,

the series becomes

$$\sum_{n=0}^{\infty}(-1)^n\frac{(1)^n}{n+4}=\sum_{n=0}^{\infty}\frac{(-1)^n}{n+4}$$

This is an alternating series, and using the alternating series test, we know it is convergent.

Combine the above, the interval of convergence is $(1,2]$.

2. (15 points) Find a power series representation for $f(x) = \frac{3x}{1-x^3}$ centered at $a = 0$, and determine the interval of convergence.

Solution By using $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, for $|x| < 1$, we have

$$\frac{1}{1-x^3} = \sum_{n=0}^{\infty} (x^3)^n = \sum_{n=0}^{\infty} x^{3n}, \qquad \text{for } |x^3| < 1$$

Therefore,

$$\frac{3x}{1-x^3} = 3x \sum_{n=0}^{\infty} x^{3n} = \sum_{n=0}^{\infty} 3x^{3n+1}$$

for all $|x^3| < 1$, which is equivalent to say that the interval of convergence is $(-1, 1)$.

3. (10 points) Use power series to evaluate the indefinite integral

$$\int x^4 \arctan(x^4)\,dx$$

Solution

$$\begin{aligned}
\int x^4 \arctan(x^4)\,dx &= \int x^4 \sum_{n=0}^{\infty} (-1)^n \frac{(x^4)^{2n+1}}{2n+1}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n x^4 \frac{x^{8n+4}}{2n+1}\,dx \\
&= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+8}}{2n+1}\,dx \\
&= \sum_{n=0}^{\infty} (-1)^n \int \frac{x^{8n+8}}{2n+1}\,dx \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{x^{8n+9}}{(8n+9)(2n+1)} + C
\end{aligned}$$

4. (15 points) Find the Taylor series for $f(x) = 2 + x + 3x^2 - x^3$ centered at $a = 1$.

Solution Clearly, we have

$$\begin{aligned} f(x) &= 2 + x + 3x^2 - x^3 & f(1) &= 5 \\ f'(x) &= 1 + 6x - 3x^2 & f'(1) &= 4 \\ f''(x) &= 6 - 6x & f''(1) &= 0 \\ f'''(x) &= -6 & f'''(1) &= -6 \\ f^{(4)}(x) &= 0 & f^{(4)}(1) &= 0 \end{aligned}$$

We can stop at $f^{(4)}(x)$ because all higher order derivatives will be 0. Then, the Taylor series is

$$\begin{aligned} f(x) &= f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4 + \cdots \\ &= 5 + 4(x-1) + 0 + \frac{-6}{3!}(x-1)^3 + 0 \\ &= 5 + 4(x-1) - (x-1)^3 \end{aligned}$$

5. (15 points) Use the series to evaluate the limit

$$\lim_{x \to 0} \frac{\sin 5x - 5x}{x^3}$$

Solution

$$\begin{aligned}
\lim_{x \to 0} \frac{\sin 5x - 5x}{x^3} &= \lim_{x \to 0} \frac{\left((5x) - \frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \cdots\right) - 5x}{x^3} \\
&= \lim_{x \to 0} \frac{-\frac{(5x)^3}{3!} + \frac{(5x)^5}{5!} - \cdots}{x^3} \\
&= \lim_{x \to 0} \left(-\frac{5^3}{3!} + \frac{5^5 x^2}{5!} - \cdots\right) \\
&= -\frac{5^3}{3!}
\end{aligned}$$

6. (15 points) Use the binomial series to expand the function as a power series centered at $a = 0$. Write the first 4 terms of the series.

$$\frac{x^2}{\sqrt{2+x}}$$

Solution Note that

$$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2}\sqrt{1+\frac{x}{2}}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2}$$

By using the binomial series, we have

$$\left(1+\frac{x}{2}\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x}{2}\right)^n$$

Hence

$$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x}{2}\right)^n$$

and the first four terms are

$$\frac{x^2}{\sqrt{2}}\left(1+\left(-\frac{1}{2}\right)\frac{x}{2}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3\right)$$

$$=\frac{x^2}{\sqrt{2}}+\frac{x^2}{\sqrt{2}}\left(-\frac{1}{2}\right)\frac{x}{2}+\frac{x^2}{\sqrt{2}}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{x^2}{\sqrt{2}}\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{x}{2}\right)^3$$

7. (15 points) (The statement in the original version is not rigorous, and hence has been changed here). Find an equation of the tangent line to the curve $x = 3t^2 + 1$, $y = 2t^3 + 1$ at the point $(13,\ 17)$.

Solution First, notice that

$$\begin{cases} \frac{dx}{dt} = 6t \\ \frac{dy}{dt} = 6t^2 \end{cases} \qquad \Rightarrow \quad \frac{dy}{dx} = \frac{6t^2}{6t} = t$$

At point $(13, 17)$, we have

$$\begin{cases} 3t^2 + 1 = 13 & \Rightarrow \quad t = \pm 2 \\ 2t^3 + 1 = 17 & \Rightarrow \quad t = 2 \end{cases} \qquad \Rightarrow \quad t = 2$$

Hence the slope is 2. The equation of the tangent line can be written as

$$y - 17 = 2(x - 13)$$

Formula

$$
\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \\
&= f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots
\end{aligned}
$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots , \qquad (R = \infty)$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots , \qquad (R = \infty)$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots , \qquad (R = \infty)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots , \qquad (R = 1)$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad (R = 1)$$