Math 2153, Exam II, Oct. 28, 2011

Name: ____________________

Score:

Read the problems carefully before you begin. Show all your work neatly and concisely, and indicate your final answer clearly.

1. (15 points) Set up, but do not evaluate, an integral for the area of the suface obtained by rotating the curve (a) about the x-axis and (b) about the y-axis.

$$y = e^{-x}, \qquad 1 \le x \le 3$$

Solution Recall the area of surface obtained by rotation can be written as

$$\text{rotating about the x axis} \qquad S = \int 2\pi y \, ds$$

$$\text{rotating about the y axis} \qquad S = \int 2\pi x \, ds$$

where

$$ds = \begin{cases} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx \\ \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy \end{cases}$$

Also, notice that $y = e^{-x}$ implies that $x = -\ln y$. Hence

$$\frac{dy}{dx} = -e^{-x}, \qquad \frac{dx}{dy} = -\frac{1}{y}$$

For $1 \le x \le 3$, we have $e^{-3} \le y \le e^{-1}$.

The solution to this problem is:

(a) rotating about the x-axis

$$\int_1^3 2\pi e^x \sqrt{1 + (-e^x)^2} \, dx$$

or

$$\int_{e^{-3}}^{e^{-1}} 2\pi y \sqrt{1 + \left(-\frac{1}{y}\right)^2} \, dy$$

(b) rotating about the y-axis

$$\int_1^3 2\pi x \sqrt{1 + (-e^x)^2} \, dx$$

or

$$\int_{e^{-3}}^{e^{-1}} 2\pi(-\ln y) \sqrt{1 + \left(-\frac{1}{y}\right)^2} \, dy$$

2. (10 points) Find a formula for the general term a_n of the sequence. Then determine whether the **sequence** is convergent or divergent. If it converges, find the limit.

$$\{0, \frac{3}{4}, -\frac{8}{9}, \frac{15}{16}, -\frac{24}{25}, \frac{35}{36}, \dots\}$$

Solution Notice that the first term 0 can be written as $-\frac{0}{1}$. If we start from $n = 1$, that is $a_1 = 0$, $a_2 = \frac{3}{4}$, $a_3 = -\frac{8}{9}$, ..., the rule is

(a) The signs are alternating. All odd terms are negative and all even terms are positive. Theis can be expressed as $(-1)^n$

(b) The denominators are 1, 4, 9, 16 ..., which can be expressed as n^2.

(c) The numerators are equal to the denominator minus 1, which can be expressed as n^2-1.

Combine the above, we have

$$a_n = (-1)^n \frac{n^2-1}{n^2}.$$

Next, we check the limit of a_n. It is clear that

$$\lim_{n\to\infty} \frac{n^2-1}{n^2} = \lim_{n\to\infty} \left(1 - \frac{1}{n^2}\right) = 1$$

However, since the sign of a_n is alternating, the value of a_n oscillates between -1 and 1. Therefore, the sequence is divergent.

3. (15 points) Use the integral test to show that the **series** $\sum_{n=1}^{\infty} \frac{1}{(n+4)^2}$ is convergent. Then, find an upper bound for $|S - S_{10}|$.

Solution To use the integral test, we set $f(x) = \frac{1}{(x+4)^2}$. First, notice that $f(x)$ is

(a) Continuous as long as $x \neq -4$. Indeed, we only need $f(x)$ to be continuous for all $x > 1$, which is true.

(b) Positive.

(c) Decreasing. This is true because when x increases, $(x+4)^2$ increases and consequently $\frac{1}{(x+4)^2}$ decreases.

Combine the above, we know that the integral test can be applied to this problem. Next, compute

$$\int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{(x+4)^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x+4}\right)\Bigg|_1^t = \frac{1}{5}$$

which is finite. According to the integral test, the series $\sum_{n=1}^{\infty} \frac{1}{(n+4)^2}$ is convergent.

Now, let's estimate $|S - S_{10}|$. We know that

$$\int_{n+1}^{\infty} \frac{1}{(x+4)^2}\,dx \leq |S - S_n| \leq \int_n^{\infty} \frac{1}{(x+4)^2}$$

Therefore

$$|S - S_{10}| \leq \int_{10}^{\infty} \frac{1}{(x+4)^2} = \lim_{t\to\infty} \int_{10}^t \frac{1}{(x+4)^2}\,dx = \lim_{t\to\infty} \left(-\frac{1}{x+4}\right)\Bigg|_{10}^t = \frac{1}{14}$$

4. (15 points) Express the number as a ratio of integers.

$$0.\bar{5} = 0.5555\ldots$$

Solution

$$\begin{aligned} 0.5555\ldots &= 0.5 + 0.05 + 0.005 + 0.0005 + \cdots \\ &= 0.5 + 0.5 * (0.1) + 0.5 * (0.1)^2 + 0.5 * (0.1)^3 + \cdots \end{aligned}$$

It is a geometric series with $a = 0.5$ and $r = 0.1$. Since $-1 < r < 1$, the series is convergent. Furthermore, we know it converges to $\frac{a}{1-r}$. Therefore

$$0.5555\ldots = \frac{a}{1-r} = \frac{0.5}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}$$

5. (15 points) Determine whether the **series** is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=2}^{\infty} \frac{4}{n^2 - n}$$

Solution Use partial fractions, we know that

$$\frac{4}{n^2 - n} = \frac{4}{n(n-1)} = \frac{4}{n-1} - \frac{4}{n}$$

Therefore the series can be written as

$$\begin{aligned} \sum_{n=2}^{\infty} \frac{4}{n^2 - n} &= \sum_{n=2}^{\infty} \left(\frac{4}{n-1} - \frac{4}{n} \right) \\ &= \left(\frac{4}{1} - \frac{4}{2} \right) + \left(\frac{4}{2} - \frac{4}{3} \right) + \cdots + \left(\frac{4}{n-1} - \frac{4}{n} \right) + \cdots \end{aligned}$$

This is a typical telescoping series, and after cancelling a lot of terms we have

$$\sum_{n=2}^{\infty} \frac{4}{n^2 - n} = \lim_{n \to \infty} \left(\frac{4}{1} - \frac{4}{n} \right) = 4$$

The series converges to 4.

6. (15 points) Show that the following **series** is convergent. You need to explain which test you are using and how to use this test. How many terms of the series do we need to add in order to find the sum with an error $|S - S_n| < 0.01$?

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

Solution First, notice this is an alternating series. Use the alternating series test, we need to check the following two things:

(a) $b_n = \frac{1}{\sqrt{n+1}}$. When n increases, $\sqrt{n+1}$ increases and hence b_n decreases.

(b) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0$

Combine the above, the series converges.

Next, to estimate the error, recall the formula

$$|S - S_n| \leq b_{n+1} = \frac{1}{\sqrt{(n+1)+1}} = \frac{1}{\sqrt{n+2}}$$

To ensure that $|S - S_n| < 0.01$, we only need to set

$$\frac{1}{\sqrt{n+2}} < 0.01$$

which implies that

$$\sqrt{n+2} > 100 \quad \Rightarrow \quad n+2 > 10000 \quad \Rightarrow \quad n > 9998 \text{ or } n \geq 9999$$

So we need to add 9999 terms to guarantee that $|S - S_n| < 0.01$.

7. (15 points) Determine whether the **series** is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$$

Solution The easiest way to solve this problem is to use the ratio test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(-100)^{n+1}}{(n+1)!}}{\frac{(-100)^n}{n!}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-100)^{n+1}}{(-100)^n} \cdot \frac{n!}{(n+1)!} \right| \\
&= \lim_{n\to\infty} \left| (-100) \cdot \frac{1}{n+1} \right| \\
&= \lim_{n\to\infty} \frac{100}{n+1} \\
&= 0
\end{aligned}$$

Since the limit is less than 1, according to the ratio test, the series is absolutely convergent.