Math 2144, Practice Final Exam

Part I: Multiple choices. Each problem is worth 5 points. **Please enter your solution in the parentheses in front of each problem.**

1. (A) A box with an open top is to be constructed from a rectangular piece of cardboard with dimensions $5\text{in}\times 28\text{in}$, by cutting out equal squares of sides x at each corner and then folding up the sides. Express the volume V of the box as a function of x

   (A) $4x^3 - 66x^2 + 140x$;

   (B) $x^3 - 33x^2 + 140x$;

   (C) $x^3 - 65x^2 + 140x$;

   (D) $x^3 - 66x^2 + 140x$;

   (E) $x^3 + 66x^2 + 140x$.

2. (E) A rectangle has perimeter 10 m. Express the area of the rectangle as a function of the length l of one of its sides.

   (A) $5l + l^2$;

   (B) $10l - l^2$;

   (C) $5 - l$;

   (D) $10l + l^2$;

   (E) $5l - l^2$.

3. (C) Which of the following function is an even function

   (A) $f(x) = x^2 + 2x + 1$;

   (B) $f(x) = \sin x$;

   (C) $f(x) = e^{x^2}$;

   (D) $f(x) = \dfrac{\ln x}{x}$;

   (E) $f(x) = \sqrt{x+4}$.

4. (A) Find $f + g$ where $f(x) = x^3 + 6x^2$ and $g(x) = 3x^2 - 5$

   (A) $x^3 + 9x^2 - 5$;

   (B) $x^3 + 3x^2 + 5$;

   (C) $x^3 + 9x - 5$;

   (D) $x^2 + 9x - 5$;

   (E) $x^2 + 9x$.

5. (B) Find $f \circ g$ if $f(x) = x^2 + 1$ and $g(x) = 2x - 1$.

(A) $4x^2 - 4x$;

(B) $4x^2 - 4x + 2$;

(C) $2x^2 - 1$;

(D) $2x^2 + 1$;

(E) non of the above.

6. (D) Find the domain of the function $f(x) = \sqrt{4x - x^2}$

(A) $(0, 4)$;

(B) $[0, 4)$;

(C) $(-\infty, 4]$;

(D) $[0, 4]$;

(E) $[4, \infty)$.

7. (A) Find the inverse function of $g(x) = \ln \frac{x-1}{x}$

(A) $g^{-1}(x) = \dfrac{1}{1 - e^x}$;

(B) $g^{-1}(x) = \dfrac{1}{1 + e^x}$;

(C) $g^{-1}(x) = \dfrac{e^x}{1 - e^x}$;

(D) $g^{-1}(x) = \dfrac{e^x}{1 + e^x}$;

(E) $g^{-1}(x) = e^{\frac{x-1}{x}}$.

8. (B) Find the limit

$$\lim_{x \to 0} \frac{x^2 - 1}{x + 6}$$

(A) 0;

(B) $-\dfrac{1}{6}$;

(C) -1;

(D) ∞;

(E) does not exist.

9. (E) Find the limit

$$\lim_{t \to 25} \frac{t - 25}{5 - \sqrt{t}}$$

(A) 0;

(B) ∞;

(C) 5;

(D) -5;

(E) -10.

10. (C) Find the limit

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

(A) 0;

(B) ∞;

(C) $-\frac{1}{6}$;

(D) -1;

(E) ∞.

does not exist

11. (C) Find $f'(x)$ where $f(x) = \sqrt{3x+1}$

(A) 3;

(B) $\frac{1}{\sqrt{3x+1}}$;

(C) $\frac{3}{2\sqrt{3x+1}}$;

(D) $\frac{1}{2\sqrt{3x+1}}$;

(E) $\frac{3}{\sqrt{3x+1}}$.

12. (B) Find the most general antiderivative of the function $f(x) = 3e^x + 7\sec^2 x$

(A) $3e^x + 7\sec x + C$;

(B) $3e^x + 7\tan x + C$;

(C) $3e^x + 7\sin^2 x$;

(D) $3e^x + 7\cos^2 x$;

(E) none of the above.

Part II: Partial credit problems.

13. Find the horizontal, vertical and slant asymptotes of $f(x) = \frac{x^2+7}{x+1}$.

Solution

- Vertical asymptotes: $x = -1$
- Horizontal asymptotes: None (because $\lim_{x\to\infty} \frac{x^2+7}{x+1} = \infty$ and $\lim_{x\to-\infty} \frac{x^2+7}{x+1} = -\infty$)
- Slant asymptotes: by the long division, we can find

$$\frac{x^2+7}{x+1} = x - 1 + \frac{8}{x+1}$$

Therefore the slane asymptote is $y = x - 1$

14. Find an equation of the tangent line of $y = 5x - x^2$ at point (1, 1).

Solution Notice that $y' = 5 - 2x$, then $y'|_{x=1} = 3$. Therefore, the slope of tangent line is 3 and the tangent line passes through point (1, 1). It's equation can be written as $y - 1 = 3(x - 1)$.

15. Find the first and second derivative of $y = \frac{\sin x}{\cos x + 7}$.

Solution

$$\begin{aligned} y' &= \frac{(\sin x)'(\cos x+7) - (\sin x)(\cos x+7)'}{(\cos x+7)^2} = \frac{\cos x(\cos x+7) - (\sin x)(-\sin x)}{\cos^2 x + 14\cos x + 49} \\ &= \frac{\cos^2 x + 7\cos x + \sin^2 x}{\cos^2 x + 14\cos x + 49} = \frac{7\cos x + 1}{\cos^2 x + 14\cos x + 49} \\ y'' &= \frac{(7\cos x+1)'(\cos^2 x + 14\cos x + 49) - (7\cos x+1)(\cos^2 x + 14\cos x + 49)'}{(\cos^2 x + 14\cos x + 49)^2} \\ &= \frac{(-7\sin x)(\cos^2 x + 14\cos x + 49) - (7\cos x+1)(-2\cos x\sin x - 14\sin x)}{(\cos^2 x + 14\cos x + 49)^2} \end{aligned}$$

16. Use implicit differentiation to compute y' where $\sqrt{x+y} = 1 + x^2y^2$.

Solution Taking derivative with respect to x on both sides of the equation, we have

$$\begin{aligned} & \frac{1 + \frac{dy}{dx}}{2\sqrt{x+y}} = 2xy^2 + 2x^2y\frac{dy}{dx} \\ \Rightarrow \quad & \frac{1}{2\sqrt{x+y}} + \frac{1}{2\sqrt{x+y}} \cdot \frac{dy}{dx} = 2xy^2 + 2x^2y\frac{dy}{dx} \\ \Rightarrow \quad & \left(\frac{1}{2\sqrt{x+y}} - 2x^2y\right)\frac{dy}{dx} = 2xy^2 - \frac{1}{2\sqrt{x+y}} \\ \Rightarrow \quad & \frac{dy}{dx} = \frac{2xy^2 - \frac{1}{2\sqrt{x+y}}}{\frac{1}{2\sqrt{x+y}} - 2x^2y} \end{aligned}$$

17. Use logarithmic differentiation to compute y' where $y = (2x+1)^5(x^4-3)^6$

    Solution Notice that $\ln y = \ln(2x+1)^5(x^4-3)^6 = 5\ln(2x+1) + 6\ln(x^4-3)$, by implicit differentiation, we have

$$\begin{aligned} & \frac{1}{y} \cdot \frac{dy}{dx} = 5\frac{2}{2x+1} + 6\frac{4x^3}{x^4-3} \\ \Rightarrow \qquad & \frac{dy}{dx} = y\left(5\frac{2}{2x+1} + 6\frac{4x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(5\frac{2}{2x+1} + 6\frac{4x^3}{x^4-3}\right) \end{aligned}$$

18. A cylindrical tank of radius 10 m is being filled with water at a 5 m^3 per minute. how fast is the height of the water increasing?

    Solution Let t be the time, h be the height of the water and V be the total volume of water in the tank. Then

$$V = \pi(10)^2 \cdot h = 100\pi h$$

    Taking derivative with respect to time t, we have

$$\frac{dV}{dt} = 100\pi\frac{dh}{dt}$$

    Since $\frac{dV}{dt} = 5$, we can see that $\frac{dh}{dt} = \frac{5}{100\pi}$.

19. Find the critical numbers of $f(x) = x^2 e^{-3x}$

    Solution Notice that

$$f'(x) = 2xe^{-3x} + x^2(-3e^{-3x}) = (2x - 3x^2)e^{-3x} = 0$$

    implies $2x - 3x^2 = 0$. So $x = 0$ and $x = 2/3$ are two critical points.

20. Find the absolute maximum and absolute minimum of the function $f(x) = \frac{x}{x^2+1}$ on the given interval $[0, 2]$.

    Solution Notice that

$$f'(x) = \frac{1 \cdot (x^2+1) - x \cdot (2x)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = 0$$

    implies that $1 - x^2 = 0$. So $x = 1$ and $x = -1$ are two critical points. In the interval $[0, 2]$, we only need to consider $x = 1$.

$$\begin{aligned} f(1) &= \frac{1}{2} \\ f(0) &= 0 \\ f(2) &= \frac{2}{5} \end{aligned}$$

    Hence the abs. max. is $1/2$ lcoated at $x = 1$ and the abs. min. is 0 lcoated at $x = 0$.

21. For the given function $f(x)$, (a) find the intervals where f is increasing or decreasing; (b) find the local maximum and minimum values of f; (c) find the intervals of concavity and the inflection points

$$f(x) = x^2 \ln x$$

Solution

(a) First, notice that the domain of this function is $(0, \infty)$. Next, $f'(x) = 2x\ln x + x^2\frac{1}{x} = 2x\ln x + x = x(2\ln x + 1) = 0$, implies $x = 0$ and $x = e^{-1/2}$ are two critical points. However, we can not count 0 since it is not in the domain of the function. This leaves only one critical point $x = e^{-1/2}$. Clearly

$0 < x < e^{-1/2}$	$f'(x) < 0$	$f(x)$ is decreasing
$x > e^{-1/2}$	$f'(x) > 0$	$f(x)$ is increasing

(b) We only have one critical point $x = e^{-1/2}$. By the first derivative test, it is a local minimum.

(c) $f''(x) = 2\ln x + 3 = 0$ implies $x = e^{-3/2}$. Clearly

$0 < x < e^{-3/2}$	$f''(x) < 0$	$f(x)$ is concave downward
$x > e^{-3/2}$	$f''(x) > 0$	$f(x)$ is concave upward

$x = e^{-3/2}$ is the inflection point.

22. Find a positive number such that the sum of this number and its reciprocal is as small as possible.

Solution Let $x > 0$ and $f(x) = x + \frac{1}{x}$. We wish to minimize $f(x)$. Notice that $f'(x) = 1 - \frac{1}{x^2} = 0$ implies $\frac{1}{x^2} = 1$. Hence $x^2 = 1$. We have two critical points, $x = 1$ and $x = -1$. However, since we only consider positive numbers $x > 0$, here we only take on critical point $x = 1$. Use the first derivative test, notice that $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, so f has an absolute minimum at $x = 1$. At this point, $f(1) = 2$.

23. Find the point on the line $6x + y = 9$ that is closest to the point (-3,1).

Solution The square of distance from a point (x, y) on the line $6x + y = 9$ to the point $(-3, 1)$ is $D(x) = (x + 3)^2 + (y - 1)^2$. From the line equation, we have $y = 9 - 6x$, hence $D(x) = (x + 3)^2 + (9 - 6x - 1)^2 = 37x^2 - 90x + 73$. Then $D'(x) = 74x - 90 = 0$ only when $x = 45/37$. Since $D''(x) = 74 > 0$, so D is concave upward for all x. Thus D has an absolute minimum at $x = 45/37$. The point on the line closest to $(-3, 1)$ is $(45/37,\ 63/37)$.

24. Find f if $f'(x) = 1 - 6x$ and $f(0) = 8$

Solution The general antiderivative is $f(x) = x - 3x^2 + C$. Since $f(0) = 8$, the particular antiderivative is $f(x) = x - 3x^2 + 8$.

25. Express the area under f as a limit where $f(x) = \sqrt[4]{x}$, $1 \le x \le 16$, use R_n.

Solution

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + \frac{15i}{n}} \cdot \frac{15}{n}$$

26. Find the derivative of the function $y = \int_{e^x}^{0} \sin^3 t\, dt$

Solution By the Fundamental Theorem of Calculus, part 1, $y' = -e^x \sin^3(e^x)$.

27. Evaluate the integral $\int \frac{e^{1/x}}{x^2}\, dx$

Solution Set $u = 1/x$, then $du = -\frac{1}{x^2} dx$ and the integral becomes

$$\int \frac{e^{1/x}}{x^2}\, dx = \int (-e^u)\, du = -e^u + C = -e^{1/x} + C$$

28. Evaluate the integral $\int_0^{\pi/4} \sin 4t\, dt$

Solution Set $u = 4t$, then $du = 4dt$, or equivalently, $dt = \frac{1}{4} du$. This is a definite integral, so we also need to find the correct upper and lower limits. Notice that $u = 0$ when $t = 0$ and $u = \pi$ when $t = \pi/4$. Then

$$\int_0^{\pi/4} \sin 4t\, dt = \int_0^{\pi} \sin u \left(\frac{1}{4} du\right) = \frac{1}{4}(-\cos u)|_0^{\pi} = \frac{1}{2}$$

29. Sketch the region between two curves $x = 1 - y^2$ and $x = y^2 - 1$, and find its area.

Solution The graph is omitted. Notice that the two curves intersect at

$$\begin{cases} x = 1 - y^2 \\ x = y^2 - 1 \end{cases} \quad \Rightarrow \quad 1 - y^2 = y^2 - 1 \quad \Rightarrow \quad y^2 = 1$$

They intersect at points $y = 1$ and $y = -1$. Between these two points, the curve $x = y^2 - 1$ is to the left of $x = 1 - y^2$ Hence the area is

$$A = \int_{-1}^{1} [(1 - y^2) - (y^2 - 1)]\, dy = \int_{-1}^{1} (2 - 2y^2)\, dy = \frac{8}{3}$$

30. Find the volume of the solid obtained by rotating the region bounded by $y = x$ and $y = \sqrt[3]{x}$ about the axis $y = 1$.

Solution The cross section is a washer with inner radius $1 - \sqrt[3]{x}$ and outer radius $1 - x$. So its area is

$$A(x) = \pi(1 - x)^2 - \pi(1 - \sqrt[3]{x})^2 = \pi(-2x + x^2 + 2x^{1/3} - x^{2/3})$$

It is also easy to see that the curves $y = x$ and $y = \sqrt[3]{x}$ intersect with each other at $x = 0$ and $x = 1$. Then

$$V = \int_0^1 A(x)\, dx = \pi \int_0^1 (-2x + x^2 + 2x^{1/3} - x^{2/3})\, dx = \frac{7}{30}\pi$$

31. Find the volume of the solid obtained by rotating the region bounded by $y = x^2$ and $y = 2 - x^2$ about the axis $x = 1$.

Solution The shell has radius $1 - x$, circumference $2\pi(1 - x)$, and height $(2 - x^2) - x^2 = 2 - 2x^2$.

$$\begin{aligned} V &= \int_{-1}^{1} 2\pi(1 - x)(2 - 2x^2)dx \\ &= 4\pi \int_{-1}^{1} (1 - x)(1 - x^2)dx = \frac{16}{3}\pi \end{aligned}$$