Solution to practice problems for midterm 3

1. If the rectangle has dimension x and y, then its perimeter is $2x + 2y = 100$. So $y = 50 - x$. Thus, the area is $A = xy = x(50 - x) = 50x - x^2$, where $0 \leq x \leq 50$. We wish to maximize the function $A(x)$. Therefoe, we first need to find all critical points of $A(x)$. Notice that $A'(x) = 50 - 2x = 0$ only at $x = 25$. The area $A(25) = 25(50 - 25) = 625$, while $A(0) = A(50 = 0$. Clearly, the maximum area occurs at $x = 25$ and $y = 50 - x = 25$, and the maximum area is $A(25) = 625$.

2. Let $x > 0$ and $f(x) = x + \frac{1}{x}$. We wish to minimize $f(x)$. Notice that $f'(x) = 1 - \frac{1}{x^2} = 0$ implies $\frac{1}{x^2} = 1$. Hence $x^2 = 1$. We have two critical points, $x = 1$ and $x = -1$. However, since we only consider positive numbers $x > 0$, here we only take on critical point $x = 1$. Use the first derivative test, notice that $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$, so f has an absolute minimum at $x = 1$. At this point, $f(1) = 2$.

3. Let b be the length of the base of the box and h be the height. The surface area is $1200 = b^2 + 4hb$. Solve it for h, we have $h = \frac{1200-b^2}{4b}$. The volume is $V = b^2h = b^2\frac{1200-b^2}{4b} = 300b - \frac{b^3}{4}$. Therefore $V'(b) = 300 - \frac{3}{4}b^2 = 0$ implies $b^2 = 400$. Since b must be positive, we have $b = 20$. Furthermore, notice that $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, by the first derivative test $b = 20$ is an absolute maximum. When $b = 20$, then $h = \frac{1200-b^2}{4b} = 10$. In this case we have the largest possible volume $b^2h = 4000$.

4. The square of distance from a point (x, y) on the line $6x + y = 9$ to the point $(-3, 1)$ is $D(x) = (x + 3)^2 + (y - 1)^2$. From the line equation, we have $y = 9 - 6x$, hence $D(x) = (x+3)^2 + (9-6x-1)^2 = 37x^2 - 90x + 73$. Then $D'(x) = 74x - 90 = 0$ only when $x = 45/37$. Since $D''(x) = 74 > 0$, so D is concave upward for all x. Thus D has an absolute minimum at $x = 45/37$. The point on the line closest to $(-3, 1)$ is $(45/37,\ 63/37)$.

5. Let (x, y) be the upper-right corner of the rectangle ($x > 0$, $y > 0$). The other three corners are $(-x, y)$, $(-x, -y)$ and $(x, -y)$. The area of the rectangle is $A = (2x)(2y) = 4xy$. Since $x^2/a^2 + y^2/b^2 = 1$, we have $y = \frac{b}{a}\sqrt{a^2 - x^2}$. So the area is $A = 4\frac{b}{a}x\sqrt{a^2 - x^2}$. Notice that $A'(x) = \frac{4b}{a}[x \cdot \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x) + (a^2 - x^2)^{1/2}] = \frac{4b}{a\sqrt{a^2-x^2}}[a^2 - 2x^2] = 0$ implies that $a^2 - 2x^2 = 0$. Therefore, the only critical point is $x = \frac{1}{\sqrt{2}}a$, since $x > 0$. It clearly gives a maximum. Then $y = \frac{1}{\sqrt{2}}b$ and the maximum area is $A = 2ab$.

6. (a) $F(x) = \frac{1}{2}x + \frac{1}{4}x^3 - \frac{1}{5}x^4 + C$
   (b) $F(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x + C$
   (c) $F(x) = 4x^{5/4} - 4x^{7/4} + C$
   (d) $F(x) = 4x^{3/2} - \frac{6}{7}x^{7/6} + C$
   (e) $F(u) = \frac{1}{3}u^3 - \frac{6}{\sqrt{u}} + C$
   (f) $F(x) = 3e^x + 7\tan x + C$
   (g) $F(t) = -\cos t + 2\cosh t + C$

7. (a) $f'(x) = 3x^2 + 4x^3 + C$, and $f(x) = x^3 + x^4 + Cx + D$
   (b) The general antiderivative is $f(x) = x - 3x^2 + C$. Since $f(0) = 8$, the particular antiderivative is $f(x) = x - 3x^2 + 8$.

8. (a) $A = \lim_{n\to\infty} R_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt[4]{1 + \frac{15i}{n}} \cdot \frac{15}{n}$

   (b) $A = \lim_{n\to\infty} L_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\ln(3 + 7(i-1)/n)}{3 + 7(i-1)/n} \cdot \frac{7}{n}$

   (c) $A = \lim_{n\to\infty} M_n = \lim_{n\to\infty} \sum_{i=1}^{n} f((x_{i-1} + x_i)/2)\Delta x = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{(2i-1)\pi}{4n} \cos\left(\frac{(2i-1)\pi}{4n}\right) \cdot \frac{\pi}{2n}$

9. (a) The area under $f(x) = x^{10}$ on $5 \leq x \leq 7$.

   (b) The area under $y = \tan x$ on $0 \leq x \leq \pi/4$.

10. $\int_0^5 f(x)\,dx = \int_0^3 3\,dx + \int_3^5 x\,dx = 17$

11. Since $1 \leq \sqrt{1+x^2} \leq \sqrt{2}$ for $-1 \leq x \leq 1$, we have

$$2 = \int_{-1}^{1} 1\,dx \leq \int_{-1}^{1} \sqrt{1+x^2}\,dx \leq \int_{-1}^{1} \sqrt{2}\,dx = 2\sqrt{2}$$

12. By the Fundamental Theorem pf Calculus, part 1,

   (a) $F'(x) = -\sqrt{1 + \sec x}$

   (b) $h'(x) = -\frac{\arctan(1/x)}{x^2}$

   (c) $y' = -e^x \sin^3(e^x)$

13. (a) $3/4$

   (b) $5/9$

   (c) $40/3$

   (d) 1

   (e) $e^2 - 1$

   (f) $2t - t^2 + \frac{1}{3}t^3 - \frac{1}{4}t^4 + C$

   (g) $\tan t + \sec t + C$

   (h) $-4\cos\theta - 3\sin\theta + C$

   (i) $-7/2$

14. (a) $-\frac{1}{2}\cos(x^2) + C$

   (b) $\frac{1}{3}(2x + x^2)^{3/2} + C$

   (c) $-\frac{1}{3}\ln|5 - 3x| + C$

   (d) $-\frac{1}{\pi}\cos(\pi t) + C$

   (e) $\frac{1}{2}\ln(x^2 + 1) + C$

   (f) $\frac{1}{3}(\ln x)^3 + C$

   (g) $\frac{2}{3}(1 + e^x)^{3/2} = C$

   (h) 26

   (i) $182/9$

(j) $e - \sqrt{e}$

(k) 16/15

15. The Graph is omitted.

(a)

$$A = \int_{-1}^{1} [(1 - y^2) - (y^2 - 1)]\, dy = 8/3$$

(b)

$$A = \int_{-1}^{1} |\sin(\pi x/2) - x|\, dx = 2\int_{0}^{1} [\sin(\pi x/2) - x]\, dx = \frac{4}{\pi} - 1$$

16. (a) A cross-section is a washer(annulus) with inner radius x^3 and out radius x, so its area is

$$A(x) = \pi(x)^2 - \pi(x^3)^2 = \pi(x^2 - x^6)$$

The volume is

$$V = \int_0^1 A(x)\, dx = \ldots = \frac{4}{21}\pi$$

(b) A cross-section is a washer with inner radius y^2 and outer radius $2y$, so its area is

$$A(y) = \pi(4y^2 - y^4)$$

The volume is

$$V = \int_0^1 A(y)\, dy = \ldots = \frac{64}{15}\pi$$