Math 2144, Exam II, Oct. 28, 2010

Name: ____________________

Score:

Please read the instructions on each problem carefully, and indicate answers as directed. Show details in your work. If you simply give a solution without steps of how you derive this solution, you may not get credit for it.

The total is 50 points

1. (5 points) If $f(t) = \frac{4t}{t^2+6}$, find $f'(t)$.

   Solution

$$\begin{aligned} f'(t) &= \frac{(4t)'(t^2+6) - (4t)(t^2+6)'}{(t^2+6)^2} \\ &= \frac{(4)(t^2+6) - (4t)(2t)}{(t^2+6)^2} \\ &= \frac{-4t^2+24}{(t^2+6)^2} \end{aligned}$$

2. (6 points) Use implicit differentiation to compute $\frac{dy}{dx}$

$$e^y = \cos(x+y)$$

Solution By taking derivative of both sides with respect to x, we have

$$
\begin{aligned}
& \frac{de^y}{dx} = \frac{d\cos(x+y)}{dx} \\
\Rightarrow \quad & e^y \frac{dy}{dx} = -\sin(x+y)\left(1 + \frac{dy}{dx}\right) \\
\Rightarrow \quad & e^y \frac{dy}{dx} = -\sin(x+y) - \sin(x+y)\frac{dy}{dx} \\
\Rightarrow \quad & e^y \frac{dy}{dx} + \sin(x+y)\frac{dy}{dx} = -\sin(x+y) \\
\Rightarrow \quad & (e^y + \sin(x+y))\frac{dy}{dx} = -\sin(x+y) \\
\Rightarrow \quad & \frac{dy}{dx} = \frac{-\sin(x+y)}{e^y + \sin(x+y)}
\end{aligned}
$$

3. (6 points) Use logarithmic differentiation to compute $\frac{dy}{dx}$

$$y = \left(\sqrt{x}\right)^x$$

Solution First, take logarithm of both sides of the equation, we have

$$\ln y = \ln\left(\sqrt{x}\right)^x = x \ln\sqrt{x} = x \ln x^{1/2} = \frac{x}{2}\ln x$$

Next, take derivative of both sides of the equation with respect to x,

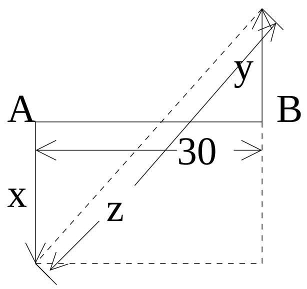
$$\begin{aligned}
&\frac{d\ln y}{dx} = \frac{d(\frac{x}{2}\ln x)}{dx} \\
\Rightarrow &\frac{1}{y}\frac{dy}{dx} = \left(\frac{x}{2}\right)'\ln x + \frac{x}{2}(\ln x)' = \frac{1}{2}\ln x + \frac{x}{2}\frac{1}{x} = \frac{1}{2}\ln x + \frac{1}{2} \\
\Rightarrow &\frac{dy}{dx} = y\left(\frac{1}{2}\ln x + \frac{1}{2}\right)
\end{aligned}$$

Finally, substitute $y = (\sqrt{x})^x$ back, we have

$$\frac{dy}{dx} = \left(\sqrt{x}\right)^x\left(\frac{1}{2}\ln x + \frac{x}{2}\right)$$

4. (8 points) At noon, ship A is 30 km west of ship B. Ship A is sailing south at 13 km/h and ship B is sailing north at 7 km/h. How fast is the distance between the ships changing at 2:00pm?

Solution Set x, y and z as shown in the graph. Then



$$z^2 = (30)^2 + (x+y)^2$$

Taking derivative of both sides of the equation with respect to t, we have

$$\begin{aligned}
\frac{dz^2}{dt} &= \frac{d(30)^2}{dt} + \frac{d(x+y)^2}{dt} \\
\Rightarrow \qquad 2z\frac{dz}{dt} &= 0 + 2(x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right) \\
\Rightarrow \qquad z\frac{dz}{dt} &= (x+y)\left(\frac{dx}{dt} + \frac{dy}{dt}\right)
\end{aligned}$$

At 2pm, we have

$$\frac{dx}{dt} = 13 \qquad \frac{dy}{dt} = 7$$

and

$$x = 13 \times 2 = 26 \qquad y = 7 \times 2 = 14 \qquad z = \sqrt{(30)^2 + (26+14)^2} = 50$$

Therefore, the equation becomes

$$\begin{aligned}
50\frac{dz}{dt} &= (26+14)(13+7) \\
\Rightarrow \qquad \frac{dz}{dt} &= \frac{(40)(20)}{50} = 16
\end{aligned}$$

5. (8 points) Find the absolute maximum and absolute minimum values of $y = \cos t + \sin^2 t$ for $0 \leq t \leq \pi$.

Solution First, we compute the critical points of f.

$$\begin{aligned} & f'(t) = -\sin t + 2\sin t \cos t = 0 \\ \Rightarrow \qquad & \sin t(2\cos t - 1) = 0 \\ \Rightarrow \qquad & \sin t = 0 \quad \text{or} \quad \cos t = \frac{1}{2} \end{aligned}$$

For $0 \leq t \leq \pi$, we have

$$\begin{aligned} \sin t = 0 \quad &\Rightarrow \quad t = 0,\ \pi \\ \cos t = \frac{1}{2} \quad &\Rightarrow \quad t = \frac{\pi}{3} \end{aligned}$$

Therefore, we get three critical points 0, π, and $\frac{\pi}{3}$. Notice that two critiacal points are the ends of the interval $[0,\ \pi]$.

Compare the values of f on all critical points and also on the ends of the interval, we have

$$\begin{aligned} f(0) &= \cos 0 + \sin^2 0 = 1 \\ f(\pi) &= \cos \pi + \sin^2 \pi = -1 \\ f(\frac{\pi}{3}) &= \cos \frac{\pi}{3} + \sin^2 \frac{\pi}{3} = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{5}{4} \end{aligned}$$

Clearly, f achieves maximum at $t = \frac{\pi}{3}$ and the maximum value is $\frac{5}{4}$; f achieves minimum at $t = \pi$ and the minimum value is -1.

6. (6 points) Find the intervals of concavity and inflection points for

$$f(x) = 12x^3 + 14x^2 - 7x - 9$$

Solution

$$\begin{aligned} f'(x) &= 36x^2 + 28x - 7 \\ f''(x) &= 72x + 28 \end{aligned}$$

- For x in $(-\infty, -\frac{28}{72})$, we have $f''(x) < 0$. The function is concave downward.
- For x in $(-\frac{28}{72}, \infty)$, we have $f''(x) > 0$. The function is concave upward.

The point $x = -\frac{28}{72}$ is the inflection point.

7. (6 points) Find all vertical, horizontal, and slant asymptotes of $f(x) = \frac{x^2+7}{x+1}$.

Solution

- Vertical asymptotes: $x = -1$
- Horizontal asymptotes: None (because $\lim_{x\to\infty} \frac{x^2+7}{x+1} = \infty$ and $\lim_{x\to-\infty} \frac{x^2+7}{x+1} = -\infty$)
- Slant asymptotes: by the long division, we can find

$$\frac{x^2+7}{x+1} = x - 1 + \frac{8}{x+1}$$

Therefore the slane asymptote is $y = x - 1$

8. (5 points) Compute

$$\lim_{t\to 0} \frac{e^t - 1}{t^3}$$

Solution This is a type $\frac{0}{0}$ limit, by the L'Hospital's rule

$$\lim_{t\to 0} \frac{e^t - 1}{t^3} = \lim_{t\to 0} \frac{(e^t - 1)'}{(t^3)'} = \lim_{t\to 0} \frac{e^t}{3t^2}$$

Notice that $e^0 = 1$ and $3(0)^2 = 0$, the new limit is of type $\frac{1}{0}$. Do not use the L'Hospital's rule anymore. Indeed,

$$\lim_{t\to 0} \frac{e^t}{3t^2} = \frac{1}{0} = \infty$$