Math 2144, Exam I, Sept. 20, 2010

Name: ____________________

Score:

Please read the instructions on each problem carefully, and indicate answers as directed. Show details in your work. If you simply give a solution without steps of how you derive this solution, you may not get credit for it.

The total is 50 points

1. (5 points) A box with an open top has volume 2 m^3. The box has a square base and the length of a side of the base is a. Express the surface area of the box S as a function of a.

   Solution The area of the base, which is a square, is a^2. Assume the height of the box is h, the the total volume is $V = a^2 h$. We have

   $$a^2 h = 2 \quad \Rightarrow \quad h = \frac{2}{a^2}$$

   The box has a bottom and four sides (no top lid). Therefore the surface area is

   $$A = \text{(area of the bottom)} + \text{(area of four sides)} = a^2 + 4ah$$

   Since $h = \frac{2}{a^2}$, we have

   $$S = a^2 + 4a\frac{2}{a^2} = a^2 + \frac{8}{a}$$

2. (6 points) Find the domain of the function

$$f(x) = \log_4 x^2 + \log_4(x+5) - \log_4 \sqrt{8-x}$$

Then express the function in a single logarithm.

Solution The domain is defined by

$$\begin{array}{rclcl} \log_4 x^2 & \Rightarrow & x \neq 0 & & \\ \log_4(x+5) & \Rightarrow & x+5>0 & \Rightarrow & x>-5 \\ \log_4 \sqrt{8-x} & \Rightarrow & 8-x>0 & \Rightarrow & x<8 \end{array}$$

Combine the above, we have the domain

$$D = \{x| -5 < x < 8 \text{ and } x \neq 0\} = (-5,0) \cup (0,8)$$

By the laws of logarithm, we have

$$f(x) = \log_4 x^2 + \log_4(x+5) - \log_4 \sqrt{8-x} = \log_4 \frac{x^2(x+5)}{\sqrt{8-x}}$$

3. (5 points) Find the limit.

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1}$$

Solution

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} &= \lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} \\
&= \lim_{x\to 1} \frac{(\sqrt{x}-1)(\sqrt{x}+1)}{(x-1)(\sqrt{x}+1)} \\
&= \lim_{x\to 1} \frac{x-1}{(x-1)(\sqrt{x}+1)} \\
&= \lim_{x\to 1} \frac{1}{\sqrt{x}+1} \\
&= \frac{1}{2}
\end{aligned}$$

4. (5 points) Find the limit.

$$\lim_{x\to\infty}(\sqrt{4x^2+3}-2x)$$

Solution

$$\begin{aligned}
\lim_{x\to\infty}(\sqrt{4x^2+3}-2x) &= \lim_{x\to\infty}(\sqrt{4x^2+3}-2x)\cdot\frac{\sqrt{4x^2+3}+2x}{\sqrt{4x^2+3}+2x} \\
&= \lim_{x\to\infty}\frac{(\sqrt{4x^2+3}-2x)(\sqrt{4x^2+3}+2x)}{\sqrt{4x^2+3}+2x} \\
&= \lim_{x\to\infty}\frac{(4x^2+3)-(2x)^2}{\sqrt{4x^2+3}+2x} \\
&= \lim_{x\to\infty}\frac{3}{\sqrt{4x^2+3}+2x} \\
&= 0
\end{aligned}$$

5. (6 points) Find the limit.

$$\lim_{x \to 0} x^2 \sin \frac{10}{x^2}$$

Solution Notice that for all values of x, we have

$$\begin{aligned} & -1 \le \sin \frac{10}{x^2} \le 1 \\ \Rightarrow \quad & -x^2 \le x^2 \sin \frac{10}{x^2} \le x^2 \end{aligned}$$

By the squeezing theorem,

$$0 = \lim_{x \to 0} (-x^2) \le \lim_{x \to 0} x^2 \sin \frac{10}{x^2} \le \lim_{x \to 0} x^2 = 0$$

Therefore

$$\lim_{x \to 0} x^2 \sin \frac{10}{x^2} = 0$$

6. (6 points) Find the horizontal and vertical asymptotes of

$$y = \frac{x^2+4}{x^2-x+6}$$

Solution To find the vertical asymptotes, we need to find the root of $x^2 - x + 6 = 0$. Notice that $x^2 - x + 6$ can not be factored. In other words, $x^2 - x + 6 = 0$ does not have a real root. Therefore, the function does not have a vertical asymptote.

To find the horizontal asymptotes, we need to compute the following:

$$\begin{aligned}
\lim_{x\to\infty} \frac{x^2+4}{x^2-x+6} &= \lim_{x\to\infty} \frac{(x^2+4)/x^2}{(x^2-x+6)/x^2} \\
&= \lim_{x\to\infty} \frac{1+\frac{4}{x^2}}{1-\frac{1}{x}+\frac{6}{x^2}} \\
&= \frac{1+0}{1-0+0} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to-\infty} \frac{x^2+4}{x^2-x+6} &= \lim_{x\to-\infty} \frac{(x^2+4)/x^2}{(x^2-x+6)/x^2} \\
&= \lim_{x\to-\infty} \frac{1+\frac{4}{x^2}}{1-\frac{1}{x}+\frac{6}{x^2}} \\
&= \frac{1+0}{1-0+0} \\
&= 1
\end{aligned}$$

Combine the above, the horizontal asymptote is $y = 1$.

7. (6 points) Find equation of the tangent line for $y = x^2 + 2x$ at the point $(1,\ 3)$.

Solution First, we need to find $f'(1)$, where $f(x) = x^2 + 2x$. By the definition

$$
\begin{aligned}
f'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{[(a+h)^2 + 2(a+h)] - [a^2 + 2a]}{h} \\
&= \lim_{h \to 0} \frac{(a^2 + 2ah + h^2 + 2a + 2h) - (a^2 + 2a)}{h} \\
&= \lim_{h \to 0} \frac{2ah + h^2 + 2h}{h} \\
&= \lim_{h \to 0} (2a + h + 2) \\
&= 2a + 2
\end{aligned}
$$

Therefore

$$f'(1) = 2 \cdot 1 + 2 = 4$$

This gives the slope of the tangent line.

Use the point-slope formula, the line is

$$y - 3 = 4(x - 1)$$

8. (6 points) Find the derivative of $f(x) = \frac{2}{x+5}$.

Solution By definition

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{2}{x+h+5} - \frac{2}{x+5}}{h} \\
&= \lim_{h\to 0} \frac{\frac{2(x+5)}{(x+5)(x+h+5)} - \frac{2(x+h+5)}{(x+5)(x+h+5)}}{h} \\
&= \lim_{h\to 0} \frac{2(x+5)-2(x+h+5)}{(x+5)(x+h+5)h} \\
&= \lim_{h\to 0} \frac{-2h}{(x+5)(x+h+5)h} \\
&= \lim_{h\to 0} \frac{-2}{(x+5)(x+h+5)} \\
&= \frac{-2}{(x+5)^2}
\end{aligned}$$

9. (5 points) Show that the equation $x^5 - 2x^3 - x - 4 = 0$ has a root between 0 and 2.

Solution Let

$$f(x) = x^5 - 2x^3 - x - 4$$

Then we have

$$f(0) = 0^5 - 2 \cdot 0^3 - 0 - 4 = -4$$
$$f(2) = 2^5 - 2 \cdot 2^3 - 2 - 4 = 32 - 16 - 2 - 4 = 10$$

Since the function $f(x)$ changes sign from 0 to 2, by the intermediate value theorem, $f(x)$ must have a root between 0 and 2.