A WEAK GALERKIN MIXED FINITE ELEMENT METHOD FOR BIHARMONIC EQUATIONS

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Abstract. This article introduces and analyzes a weak Galerkin mixed finite element method for solving the biharmonic equation. The weak Galerkin method, first introduced by two of the authors (J. Wang and X. Ye) in [52] for second order elliptic problems, is based on the concept of *discrete weak gradients*. The method uses completely discrete finite element functions and, using certain discrete spaces and with stabilization, it works on partitions of arbitrary polygon or polyhedron. In this article, the weak Galerkin method is applied to discretize the Ciarlet-Raviart mixed formulation for the biharmonic equation. In particular, an a priori error estimation is given for the corresponding finite element approximations. The error analysis essentially follows the framework of Babuška, Osborn, and Pitkäranta [8] and uses specially designed mesh-dependent norms. The proof is technically tedious due to the discontinuous nature of the weak Galerkin finite element functions. Some computational results are presented to demonstrate the efficiency of the method.

Key words. Weak Galerkin finite element methods, discrete gradient, biharmonic equations, mixed finite element methods.

AMS subject classifications. Primary, 65N15, 65N30.

1. Introduction. In this paper, we are concerned with numerical methods for the following biharmonic equation with clamped boundary conditions

$$\begin{aligned} \Delta^2 u &= f \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \mathbf{n}} &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a polygonal or polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$. To solve the problem (1.1) using a primal-based conforming finite element method, one would need C^1 continuous finite elements, which usually involve large degree of freedoms and hence can be computationally expensive. There are alternative numerical methods, for example, by using either nonconforming elements [2, 38, 41], the C^0 discontinuous Galerkin method [26, 14], or mixed finite element methods [11, 16, 20, 25, 32, 34, 33, 36, 37, 39, 40]. One of the earliest mixed formulation proposed for (1.1) is the Ciarlet-Raviart mixed finite element formulation [20] which decomposes (1.1) into a system of second order partial differential equations. More precisely, in this formulation, one introduces a dual variable $w = -\Delta u$ and rewrites the fourth-order biharmonic

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equation into two coupled second order equations

$$\begin{cases} w + \Delta u = 0, \\ -\Delta w = f, \end{cases} \tag{1.2}$$

In [20], the above system of second order equations is discretized by using the standard H^1 conforming elements. However, only sub-optimal error estimates are proved in [20] for quadratic or higher order of elements. Improved error estimates have been established in [8, 27, 31, 48] for quadratic or higher order of elements. In [8], Babuška, Osborn and Pitkäranta pointed out that a suitable choice of norms are L^2 for w and H^2 for u, or equivalent, in order to use the standard LBB stability analysis. In this sense, one has "optimal" order of convergence in H^2 norm for u and in L^2 norm for w, for quadratic or higher order of elements. However, when equal order approximation is used for both u and w, the "optimal" order of error estimate is restricted by the interpolation error in H^2 norm, and thus may not be really optimal. Moreover, this standard technique does not apply to the piecewise linear discretization, since in this case the interpolation error can not even be measured in H^2 norm. A solution to this has been proposed by Scholz [48]. Using an L^∞ argument, Scholz was able to improve the convergence rate in L^2 norm for w by $h^{\frac{1}{2}}$, and this theoretical result is known to be sharp. Also, Scholz's proof works for all equal-order elements including piecewise linears.

The goal of this paper is to propose and analyze a weak Galerkin discretization method for the mixed formulation (1.2). The weak Galerkin method was recently introduced in [52] for second order elliptic equations. It is an extension of the standard Galerkin finite element method where classical derivatives were substituted by weakly defined derivatives on functions with discontinuity. Optimal order of a priori error estimates has been observed and established for various weak Galerkin discretization schemes for second order elliptic equations [52, 53, 42]. A numerical implementation of weak Galerkin was discussed in [43, 42] for some model problems.

Some advantages of the weak Galerkin method has been stated in [53, 42, 43]. For example, the Weak Galerkin method using certain discrete spaces and with stabilization works on partitions of arbitrary polygon or polyhedron, and the weak Galerkin method uses completely discrete finite element spaces while it does not employ the jump/average approach as the discrete Galerkin method does. The weak Galerkin method is still a very new method and there remains a lot to explore. This is the main reason why here we would like to apply it to the biharmonic equation, with the ultimate goal of generalizing the method to other complicated, possibly nonlinear, fourth-order equations.

Applying the weak Galerkin method to both second-order equations in (1.2) appears to be trivial and straight-forward at first glance. However, the application turns out to be much more complicated than simply combining one weak Galerkin scheme with another one. The application is particularly non-trivial in the mathematical theory on error analysis. In deriving an a priori error estimate, we follow the framework as developed in [8] by using mesh-dependent norms. Many commonly used properties and inequalities for standard Galerkin finite element method need to be re-derived for weak Galerkin methods with respect to the mesh-dependent norms. Due to the discrete nature of the weak Galerkin functions, technical difficulties arise in the derivation of inequalities or estimates. The technical estimates and tools that we have developed in this paper should be essential to the analysis of weak Galerkin methods for other type of modeling equations. They should also play an important role in future developments of preconditioning techniques for weak Galerkin methods. Therefore, we believe this paper provides useful technical tools for future research, in addition to introducing an efficient new method for solving biharmonic equations.

The paper is organized as follows. In Section 2, a weak Galerkin discretization scheme for the Ciarlet-Raviart mixed formulation of the biharmonic equation is introduced and proved to be well-posed. Section 3 is dedicated to defining and analyzing several technical tools, including projections, mesh-dependent norms and some estimates. With the aid of these tools, an error analysis is presented in Section 4. Finally, in Section 5, we report some numerical results that show the efficiency of the method.

2. A Weak Galerkin Finite Element Scheme. For illustrative purpose, we consider only the two-dimensional case of (1.1) and the corresponding weak Galerkin method will be based on a shape-regular triangulation of the domain Ω. The analysis given in this paper can easily be generalized into two-dimensional rectangular meshes, and with a few adaptations, also into three-dimensional tetrahedral and cubic meshes. Another issue we would like to clarify is that, although the weak Galerkin method using certain discrete spaces and with stabilization is known to work on partitions of arbitrary polygon or polyhedron [53, 42], here we choose to concentrate on a weak Galerkin discretization without stabilization. This discretization only works for triangular, rectangular, tetrahedral and cubic meshes, but the theoretical analysis would be considerably easier since there is no stabilization involved. We expect the technique introduced in this paper can also be generalized to the stabilized weak Galerkin method on arbitrary meshes, but it remains to be confirmed in the future.

Let $D \subseteq \Omega$ be a polygon, we use the standard definition of Sobolev spaces $H^s(D)$ and $H_0^s(D)$ with $s \geq 0$ (e.g., see [1, 21] for details). The associated inner product, norm, and semi-norms in $H^s(D)$ are denoted by $(\cdot,\cdot)_{s,D}$, $\|\cdot\|_{s,D}$, and $|\cdot|_{r,D}, 0 \leq r \leq s$, respectively. When $s = 0$, $H^0(D)$ coincides with the space of square integrable functions $L^2(D)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript D is also suppressed when $D = \Omega$. For $s < 0$, the space $H^s(D)$ is defined to be the dual of $H_0^{-s}(D)$.

Occasionally, we need to use the more general Sobolev space $W^{s,p}(\Omega)$, for $1 \leq p \leq \infty$, and its norm $\|\cdot\|_{W^{s,p}(\Omega)}$. The definition simply follows the standard one given in [1, 21]. When $s = 0$, the space $W^{s,p}(\Omega)$ coincides with $L^p(\Omega)$.

The above definition/notation can easily be extended to vector-valued and matrix-valued functions. The norm, semi-norms, and inner-product for such functions shall follow the same naming convention. In addition, all these definitions can be transferred from a polygonal domain D to an edge e, a domain with lower dimension. Similar notation system will be employed. For example, $\|\cdot\|_{s,e}$ and $\|\cdot\|_e$ would denote the norm in $H^s(e)$ and $L^2(e)$ etc. We also define the $H(div)$ space as follows

$$H(div,\Omega) = \{\mathbf{q} : \ \mathbf{q} \in [L^2(\Omega)]^2, \ \nabla\cdot\mathbf{q} \in L^2(\Omega)\}.$$

Using notations defined above, the variational form of the Ciarlet-Raviart mixed formulation (1.2) seeks $u \in H_0^1(\Omega)$ and $w \in H^1(\Omega)$ satisfying

$$\begin{cases} (w,\phi) - (\nabla u, \nabla\phi) = 0 & \text{for all } \phi \in H^1(\Omega), \\ (\nabla w, \nabla\psi) = (f,\psi) & \text{for all } \psi \in H_0^1(\Omega). \end{cases} \tag{2.1}$$

For any solution w and u of (2.1), it is not hard to see that $w = -\Delta u$. In addition, by choosing $\phi = 1$ in the first equation of (2.1), we obtain

$$\int_\Omega w\, dx = 0.$$

Define $\bar{H}^1(\Omega) \subset H^1(\Omega)$ by

$$\bar{H}^1(\Omega) = \{v : \ v \in H^1(\Omega), \int_\Omega v\, dx = 0\},$$

which is a subspace of $H^1(\Omega)$ with mean-value free functions. Clearly, the solution w of (2.1) is a function in $\bar{H}^1(\Omega)$.

One important issue in the analysis is the regularity of the solution u and w. For two-dimensional polygonal domains, this has been thoroughly discussed in [12]. According to their results, the biharmonic equation with clamped boundary condition (1.1) satisfies

$$\|u\|_{4-k} \le c\|f\|_{-k}, \tag{2.2}$$

where c is a constant depending only on the domain Ω. Here the parameter k is determined by

$$\begin{aligned} k = 1 &\quad \text{if all internal angles of } \Omega \text{ are less than } 180^\circ \\ k = 0 &\quad \text{if all internal angles of } \Omega \text{ are less than } 126.283696\cdots^\circ \end{aligned}$$

The above regularity result indicates that the solution $u \in H^3(\Omega)$ when Ω is a convex polygon and $f \in H^{-1}(\Omega)$. It follows that the auxiliary variable $w \in H^1(\Omega)$. Moreover, if all internal angles of Ω are less than $126.283696\cdots^\circ$ and $f \in L^2(\Omega)$, then $u \in H^4(\Omega)$ and $w \in H^2(\Omega)$. The drawback of the mixed formulation (2.1) is that the auxiliary variable w may not possess the required regularity when the domain is non-convex. We shall explore other weak Galerkin methods to deal with such cases.

Next, we present the weak Galerkin discretization of the Ciarlet-Raviart mixed formulation. Let $\mathcal{T}_h$ be a shape-regular, quasi-uniform triangular mesh on a polygonal domain Ω, with characteristic mesh size h. For each triangle $K \in \mathcal{T}_h$, denote by K_0 and ∂K the interior and the boundary of K, respectively. Also denote by h_K the size of the element K. The boundary ∂K consists of three edges. Denote by $\mathcal{E}_h$ the collection of all edges in $\mathcal{T}_h$. For simplicity of notation, throughout the paper, we use "$\lesssim$" to denote "less than or equal to up to a general constant independent of the mesh size or functions appearing in the inequality".

Let j be a non-negative integer. On each $K \in \mathcal{T}_h$, denote by $P_j(K_0)$ the set of polynomials with degree less than or equal to j. Likewise, on each $e \in \mathcal{E}_h$, $P_j(e)$ is the set of polynomials of degree no more than j. Following [52], we define a weak discrete space on mesh $\mathcal{T}_h$ by

$$V_h = \{v : \ v|_{K_0} \in P_j(K_0), \ K \in \mathcal{T}_h; \ v|_e \in P_j(e), e \in \mathcal{E}_h\}.$$

Observe that the definition of V_h does not require any continuity of $v \in V_h$ across the interior edges. A function in V_h is characterized by its value on the interior of each element plus its value on the edges/faces. Therefore, it is convenient to represent functions in V_h with two components, $v = \{v_0, v_b\}$, where v_0 denotes the value of v on all K_0 and v_b denotes the value of v on $\mathcal{E}_h$.

We further define an L^2 projection from $H^1(\Omega)$ onto V_h by setting $Q_h v \equiv \{Q_0 v, Q_b v\}$, where $Q_0 v|_{K_0}$ is the local L^2 projection of v in $P_j(K_0)$, for $K \in \mathcal{T}_h$, and $Q_b v|_e$ is the local L^2 projection in $P_j(e)$, for $e \in \mathcal{E}_h$. To take care of the homogeneous Dirichlet boundary condition, define

$$V_{0,h} = \{v \in V_h \ : \ v = 0 \text{ on } \mathcal{E}_h \cap \partial\Omega\}.$$

It is not hard to see that the L^2 projection Q_h maps $H_0^1(\Omega)$ onto $V_{0,h}$.

The weak Galerkin method seeks an approximate solution $[u_h;\, w_h] \in V_{0,h} \times V_h$ to the mixed form of the biharmonic problem (1.2). To this end, we first introduce a discrete L^2-equivalent inner-product and a discrete gradient operator on V_h. For any $v_h = \{v_0, v_b\}$ and $\phi_h = \{\phi_0, \phi_b\}$ in V_h, define an inner-product as follows

$$(\!(v_h, \phi_h)\!) \triangleq \sum_{K\in\mathcal{T}_h} (v_0, \phi_0)_K + \sum_{K\in\mathcal{T}_h} h_K \langle v_0 - v_b, \phi_0 - \phi_b\rangle_{\partial K}.$$

It is not hard to see that $(\!(v_h, v_h)\!) = 0$ implies $v_h \equiv 0$. Hence, the inner-product is well-defined. Notice that the inner-product $(\!(\cdot,\cdot)\!)$ is also well-defined for any $v \in H^1(\Omega)$ for which $v_0 = v$ and $v_b|_e = v|_e$ is the trace of v on the edge e. In this case, the inner-product $(\!(\cdot,\cdot)\!)$ is identical to the standard L^2 inner-product.

The discrete gradient operator is defined element-wise on each $K \in \mathcal{T}_h$. To this end, let $RT_j(K)$ be a space of Raviart-Thomas element [44] of order j on triangle K. That is,

$$RT_j(K) = (P_j(K))^2 + \mathbf{x}P_j(K).$$

The degrees of freedom of $RT_j(K)$ consist of moments of normal components on each edge of K up to order j, plus all the moments in the triangle K up to order $(j-1)$. Define

$$\Sigma_h = \{\mathbf{q} \in (L^2(\Omega))^2 :\ \mathbf{q}|_K \in RT_j(K),\ K \in \mathcal{T}_h\}.$$

Note that Σ_h is not necessarily a subspace of $H(div, \Omega)$, since it does not require any continuity in the normal direction across any edge. A discrete weak gradient [52] of $v_h = \{v_0, v_b\} \in V_h$ is defined to be a function $\nabla_w v_h \in \Sigma_h$ such that on each $K \in \mathcal{T}_h$,

$$(\nabla_w v_h, \mathbf{q})_K = -(v_0, \nabla\cdot\mathbf{q})_K + \langle v_b, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K}, \quad \text{for all } \mathbf{q} \in RT_j(K), \tag{2.3}$$

where $\mathbf{n}$ is the unit outward normal on ∂K. Clearly, such a discrete weak gradient is always well-defined. Also, the discrete weak gradient is a good approximation to the classical gradient, as demonstrated in [52]:

LEMMA 2.1. *For any $v_h = \{v_0,\, v_b\} \in V_h$ and $K \in \mathcal{T}_h$, $\nabla_w v_h|_K = 0$ if and only if $v_0 = v_b =$ constant on K. Furthermore, for any $v \in H^{m+1}(\Omega)$, where $0 \le m \le j+1$, we have*

$$\|\nabla_w(Q_h v) - \nabla v\| \lesssim h^m \|v\|_{m+1}.$$

We are now in a position to present the weak Galerkin finite element formulation for the biharmonic problem (1.2) in the mixed form: *Find $u_h = \{u_0,\, u_b\} \in V_{0,h}$ and $w_h = \{w_0,\, w_b\} \in V_h$ such that*

$$\begin{cases} (\!(w_h,\, \phi_h)\!) - (\nabla_w u_h,\, \nabla_w \phi_h) = 0, & \text{for all } \phi_h = \{\phi_0,\, \phi_b\} \in V_h, \\ (\nabla_w w_h,\, \nabla_w \psi_h) = (f,\, \psi_0), & \text{for all } \psi_h = \{\psi_0,\, \psi_b\} \in V_{0,h}. \end{cases} \tag{2.4}$$

THEOREM 2.2. *The weak Galerkin finite element formulation (2.4) has one and only one solution $[u_h; w_h]$ in the corresponding finite element spaces.*

Proof. For the discrete problem arising from (2.4), it suffices to show that the solution to (2.4) is trivial if $f = 0$; the existence of solution stems from its uniqueness.

Assume that $f = 0$ in (2.4). By taking $\phi_h = w_h$ and $\psi_h = u_h$ in (2.4) and adding the two resulting equations together, we immediately have $(\!(w_h, w_h)\!) = 0$, which implies $w_h \equiv 0$. Next, by setting $\phi_h = u_h$ in the first equation of (2.4), we arrive at $(\nabla_w u_h, \nabla_w u_h) = 0$. By using Lemma 2.1, we see that u_h must be a constant in Ω, which together with the fact that $u_h = 0$ on $\partial\Omega$ implies $u_h \equiv 0$ in Ω. This completes the proof of the theorem. □

One important observation of (2.4) is that the solution w_h has mean value zero over the domain Ω, which is a property that the exact solution $w = -\Delta u$ must possess. This can be seen by setting $\phi_h = 1$ in the first equation of (2.4), yielding

$$(w_h, 1) = (\!(w_h, 1)\!) = (\nabla_w u_h, \nabla_w 1) = 0,$$

where we have used the definition of $(\!(\cdot,\cdot)\!)$ and Lemma 2.1. For convenience, we introduce a space $\bar{V}_h \subset V_h$ defined as follows

$$\bar{V}_h = \{v_h : \ v_h = \{v_0, v_b\} \in V_h, \int_\Omega v_0 \, dx = 0\}.$$

3. Technical Tools: Projections, Mesh-dependent Norms and Some Estimates. The goal of this section is to establish some technical results useful for deriving an error estimate for the weak Galerkin finite element method (2.4).

3.1. Some Projection Operators and Their Properties. Let $\mathbf{P}_h$ be the L^2 projection from $(L^2(\Omega))^2$ to Σ_h, and $\mathbf{\Pi}_h$ be the classical interpolation [16] from $(H^\gamma(\Omega))^2, \gamma > \frac{1}{2}$, to Σ_h defined by using the degrees of freedom of Σ_h in the usual mixed finite element method. It follows from the definition of $\mathbf{\Pi}_h$ that $\mathbf{\Pi}_h \mathbf{q} \in H(div, \Omega) \cap \Sigma_h$ for all $\mathbf{q} \in (H^\gamma(\Omega))^2$. In other words, $\mathbf{\Pi}_h \mathbf{q}$ has continuous normal components across internal edges. It is also well-known that $\mathbf{\Pi}_h$ preserves the boundary condition $\mathbf{q}\cdot\mathbf{n}|_{\partial\Omega} = 0$, if it were imposed on $\mathbf{q}$. The properties of $\mathbf{\Pi}_h$ has been well-developed in the context of mixed finite element methods [16, 30]. For example, for all $\mathbf{q} \in (W^{m,p}(\Omega))^2$ where $\frac{1}{2} < m \le j+1$ and $2 \le p \le \infty$, we have

$$Q_0(\nabla\cdot\mathbf{q}) = \nabla\cdot\mathbf{\Pi}_h\mathbf{q}, \qquad \text{if in addition } \mathbf{q} \in H(div, \Omega), \tag{3.1}$$

$$\|\mathbf{q} - \mathbf{\Pi}_h \mathbf{q}\|_{L^p(\Omega)} \lesssim h^m \|\mathbf{q}\|_{W^{m,p}(\Omega)}. \tag{3.2}$$

It is also well-known that for all $0 \le m \le j+1$,

$$\|\mathbf{q} - \mathbf{P}_h\mathbf{q}\| \lesssim h^m \|\mathbf{q}\|_m. \tag{3.3}$$

Using the above estimates and the triangle inequality, one can easily derive the following estimate

$$\|\mathbf{\Pi}_h \nabla v - \mathbf{P}_h \nabla v\| \lesssim h^m \|v\|_{m+1} \tag{3.4}$$

for all $v \in H^{m+1}(\Omega)$ where $\frac{1}{2} < m \le j+1$.

Next, we shall present some useful relations for the discrete weak gradient ∇_w, the projection operator $\mathbf{P}_h$, and the interpolation $\mathbf{\Pi}_h$. The results can be summarized as follows.

LEMMA 3.1. *Let $\gamma > \frac{1}{2}$ be any real number. The following results hold true.*

(i) For any $v \in H^1(\Omega)$, we have

$$\nabla_w(Q_h v) = \mathbf{P}_h(\nabla v). \tag{3.5}$$

(ii) For any $\mathbf{q} \in (H^\gamma(\Omega))^2 \cap H(div, \Omega)$ and $v_h = \{v_0, v_b\} \in V_h$, we have

$$(\nabla \cdot \mathbf{q},\ v_0) = -(\mathbf{\Pi}_h \mathbf{q},\ \nabla_w v_h) + \sum_{e \in \mathcal{E}_h \cap \partial\Omega} \langle (\mathbf{\Pi}_h \mathbf{q}) \cdot \mathbf{n}, v_b \rangle_e. \tag{3.6}$$

In particular, if either $v_h \in V_{0,h}$ or $\mathbf{q} \cdot \mathbf{n} = 0$ on $\partial\Omega$, then

$$(\nabla \cdot \mathbf{q},\ v_0) = -(\mathbf{\Pi}_h \mathbf{q},\ \nabla_w v_h). \tag{3.7}$$

Proof. To prove (3.5), we first recall the following well-known relation [16]

$$\nabla \cdot RT_j(K) = P_j(K_0), \qquad RT_j(K) \cdot \mathbf{n}|_e = P_j(e).$$

Thus, for any $\mathbf{w} \in \Sigma_h$ and $K \in \mathcal{T}_h$, by the definition of ∇_w and properties of the L^2 projection, we have

$$\begin{aligned}
(\nabla_w Q_h v, \mathbf{w})_K &= -(Q_0 v, \nabla \cdot \mathbf{w})_K + \langle Q_b v, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial K} \\
&= -(v, \nabla \cdot \mathbf{w})_K + \langle v, \mathbf{w} \cdot \mathbf{n} \rangle_{\partial K} \\
&= (\nabla v, \mathbf{w})_K \\
&= (\mathbf{P}_h \nabla v, \mathbf{w})_K,
\end{aligned}$$

which implies (3.5). As to (3.6), using the fact that $\nabla \cdot RT_j(K) = P_j(K_0)$, the property (3.1), and the definition of ∇_w we obtain

$$\begin{aligned}
(\nabla \cdot \mathbf{q},\ v_0) &= (Q_0(\nabla \cdot \mathbf{q}),\ v_0) = (\nabla \cdot \mathbf{\Pi}_h \mathbf{q},\ v_0) \\
&= -\sum_{K \in \mathcal{T}_h} (\mathbf{\Pi}_h \mathbf{q}, \nabla_w v_h)_K + \sum_{K \in \mathcal{T}_h} \langle v_b, \mathbf{\Pi}_h \mathbf{q} \cdot \mathbf{n} \rangle_{\partial K} \\
&= -\sum_{K \in \mathcal{T}_h} (\mathbf{\Pi}_h \mathbf{q}, \nabla_w v_h)_K + \sum_{e \in \mathcal{T}_h \cap \partial\Omega} \langle (\mathbf{\Pi}_h \mathbf{q}) \cdot \mathbf{n}, v_b \rangle_e.
\end{aligned}$$

This completes the proof of (3.6). The equality (3.7) is a direct consequence of (3.6) since the boundary integrals vanish under the given condition. □

3.2. Discrete Norms and Inequalities. Let $v_h = \{v_0, v_b\} \in V_h$. Define on each $K \in \mathcal{T}_h$

$$\begin{aligned}
\|v_h\|_{0,h,K}^2 &= \|v_0\|_{0,K}^2 + h\|v_0 - v_b\|_{\partial K}^2, \\
\|v_h\|_{1,h,K}^2 &= \|v_0\|_{1,K}^2 + h^{-1}\|v_0 - v_b\|_{\partial K}^2, \\
|v_h|_{1,h,K}^2 &= |v_0|_{1,K}^2 + h^{-1}\|v_0 - v_b\|_{\partial K}^2.
\end{aligned}$$

Using the above quantities, we define the following discrete norms and semi-norms for the finite element space V_h

$$\begin{aligned}
\|v_h\|_{0,h} &:= \left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{0,h,K}^2 \right)^{1/2}, \\
\|v_h\|_{1,h} &:= \left(\sum_{K \in \mathcal{T}_h} \|v_h\|_{1,h,K}^2 \right)^{1/2}, \\
|v_h|_{1,h} &:= \left(\sum_{K \in \mathcal{T}_h} |v_h|_{1,h,K}^2 \right)^{1/2}.
\end{aligned}$$

It is clear that $\|v_h\|_{0,h}^2 = (\!(v_h, v_h)\!)$. Hence, $\|\cdot\|_{0,h}$ provides a discrete L^2 norm for V_h. It is not hard to see that $|\cdot|_{1,h}$ and $\|\cdot\|_{1,h}$ define a discrete H^1 semi-norm and a norm for V_h, respectively. Observe that $|v_h|_{1,h} = 0$ if and only if $v_h \equiv$ *constant*. Thus, $|\cdot|_{1,h}$ is a norm in $V_{0,h}$ and $\bar{V}_h$.

For any $K \in \mathcal{T}_h$ and e being an edge of K, the following trace inequality is well-known

$$\|g\|_e^2 \lesssim h^{-1}\|g\|_K^2 + h^{2s-1}|g|_{s,K}^2, \quad \frac{1}{2} < s \le 1, \tag{3.8}$$

for all $g \in H^1(K)$. Here $|g|_{s,K}$ is the semi-norm in the Sobolev space $H^s(K)$. The inequality (3.8) can be verified through a scaling argument for the standard Sobolev trace inequality in H^s with $s \in (\frac{1}{2}, 1]$. If g is a polynomial in K, then we have from (3.8) and the standard inverse inequality that

$$\|g\|_e^2 \lesssim h^{-1}\|g\|_K^2. \tag{3.9}$$

¿From (3.9) and the triangle inequality, it is not hard to see that for any $v_h \in V_h$ one has

$$\left(\sum_{K\in\mathcal{T}_h} (\|v_0\|_{0,K}^2 + h\|v_b\|_{\partial K}^2)\right)^{1/2} \lesssim \|v_h\|_{0,h} \lesssim \left(\sum_{K\in\mathcal{T}_h} (\|v_0\|_{0,K}^2 + h\|v_b\|_{\partial K}^2)\right)^{1/2}.$$

In the rest of this paper, we shall use the above equivalence without particular mentioning or referencing.

The following Lemma establishes an equivalence between the two semi-norms $|\cdot|_{1,h}$ and $\|\nabla_w \cdot\|$.

LEMMA 3.2. *For any $v_h = \{v_0, v_b\} \in V_h$, we have*

$$|v_h|_{1,h} \lesssim \|\nabla_w v_h\| \lesssim |v_h|_{1,h}. \tag{3.10}$$

Proof. Using the definition of ∇_w, integration by parts, the Schwarz inequality, the inequality (3.9), and the Young's inequality, we have

$$\begin{aligned}
\|\nabla_w v_h\|_K^2 &= -(v_0, \nabla\cdot\nabla_w v_h)_K + \langle v_b, \nabla_w v_h \cdot \mathbf{n}\rangle_{\partial K} \\
&= \langle v_b - v_0, \nabla_w v_h \cdot \mathbf{n}\rangle_{\partial K} + (\nabla v_0, \nabla_w v_h)_K \\
&\le \|v_0 - v_b\|_{\partial K}\|\nabla_w v_h \cdot \mathbf{n}\|_{\partial K} + \|\nabla v_0\|_K\|\nabla_w v_h\|_K \\
&\lesssim \|v_0 - v_b\|_{\partial K} h^{-\frac{1}{2}}\|\nabla_w v_h\|_K + \|\nabla v_0\|_K\|\nabla_w v_h\|_K \\
&\lesssim \|\nabla_w v_h\|_K \left(\|\nabla v_0\|_K + h^{-\frac{1}{2}}\|v_0 - v_b\|_{\partial K}\right).
\end{aligned}$$

This completes the proof of $\|\nabla_w v_h\| \lesssim |v_h|_{1,h}$.

To prove $|v_h|_{1,h} \lesssim \|\nabla_w v_h\|$, let $K \in \mathcal{T}_h$ be any element and consider the following subspace of $RT_j(K)$

$$D(j,K) := \{\mathbf{q} \in RT_j(K) : \ \mathbf{q}\cdot\mathbf{n} = 0 \text{ on } \partial K\}.$$

Note that $D(j,K)$ forms a dual of $(P_{j-1}(K))^2$. Thus, for any $\nabla v_0 \in (P_{j-1}(K))^2$, one has

$$\|\nabla v_0\|_K = \sup_{\mathbf{q}\in D(j,K)} \frac{(\nabla v_0, \mathbf{q})_K}{\|\mathbf{q}\|_K}. \tag{3.11}$$

It follows from the integration by parts and the definition of ∇_w that

$$(\nabla v_0, \mathbf{q})_K = -(v_0, \nabla \cdot \mathbf{q})_K = (\nabla_w v_h, \mathbf{q})_K,$$

which, together with (3.11) and the Cauchy-Schwarz inequality, gives

$$\|\nabla v_0\|_K \le \|\nabla_w v_h\|_K. \tag{3.12}$$

Note that for $j = 0$, we have $\nabla v_0 = 0$ and the above inequality is satisfied trivially.

Analogously, let e be an edge of K and denote by $D_e(j, K)$ the collection of all $\mathbf{q} \in RT_j(K)$ such that all degrees of freedom, except those for $\mathbf{q} \cdot \mathbf{n}|_e$, vanish. It is well-known that $D_e(j, K)$ forms a dual of $P_j(e)$. Thus, we have

$$\|v_0 - v_b\|_e = \sup_{\mathbf{q} \in D_e(j,K)} \frac{\langle v_0 - v_b, \mathbf{q} \cdot \mathbf{n}\rangle_e}{\|\mathbf{q} \cdot \mathbf{n}\|_e}. \tag{3.13}$$

It follows from (2.3) and the integration by parts on $(v_0, \nabla \cdot \mathbf{q})_K$ that

$$(\nabla_w v_h, \mathbf{q})_K = (\nabla v_0, \mathbf{q})_K + \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial K}, \qquad \forall\ \mathbf{q} \in RT_j(K). \tag{3.14}$$

In particular, for $\mathbf{q} \in D_e(j, K)$, we have

$$(\nabla v_0, \mathbf{q})_K = 0, \qquad \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n}\rangle_{\partial K} = \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n}\rangle_e.$$

Substituting the above into (3.14) yields

$$(\nabla_w v_h, \mathbf{q})_K = \langle v_b - v_0, \mathbf{q} \cdot \mathbf{n}\rangle_e, \qquad \forall\ \mathbf{q} \in D_e(j, K). \tag{3.15}$$

Using the Cauchy-Schwarz inequality we arrive at

$$|\langle v_b - v_0, \mathbf{q} \cdot \mathbf{n}\rangle_e| \le \|\nabla_w v_h\|_K\ \|\mathbf{q}\|_K,$$

for all $\mathbf{q} \in D_e(j, K)$. By the scaling argument, for such $\mathbf{q} \in D_e(j, K)$, we have $\|\mathbf{q}\|_K \lesssim h^{\frac{1}{2}}\|\mathbf{q} \cdot \mathbf{n}\|_e$. Thus, we obtain

$$|\langle v_b - v_0, \mathbf{q} \cdot \mathbf{n}\rangle_e| \lesssim h^{\frac{1}{2}}\|\nabla_w v_h\|_K\ \|\mathbf{q} \cdot \mathbf{n}\|_e, \qquad \forall \mathbf{q} \in D_e(j, K),$$

which, together with (3.13), implies the following estimate

$$\|v_0 - v_b\|_e \lesssim h^{\frac{1}{2}}\|\nabla_w v_h\|_K.$$

Combining the above estimate with (3.12) gives a proof of $|v_h|_{1,h} \lesssim \|\nabla_w v_h\|$. This completes the proof of (3.10). □

The discrete semi-norms satisfy the usual inverse inequality, as stated in the following Lemma.

LEMMA 3.3. *For any $v_h = \{v_0, v_b\} \in V_h$, we have*

$$|v_h|_{1,h} \lesssim h^{-1}\|v_h\|_{0,h}. \tag{3.16}$$

Consequently, by combining (3.10) and (3.16), we have

$$\|\nabla_w v_h\| \lesssim h^{-1}\|v_h\|_{0,h}. \tag{3.17}$$

Proof. The proof follows from the standard inverse inequality and the definition of $\|\cdot\|_{0,h}$ and $|\cdot|_{1,h}$; details are thus omitted. □

Next, let us show that the discrete semi-norm $\|\nabla_w(\cdot)\|$, which is equivalent to $|\cdot|_{1,h}$ as proved in Lemma 3.2, satisfies a Poincaré-type inequality.

LEMMA 3.4. *The Poincaré-type inequality holds true for functions in $V_{0,h}$ and $\bar{V}_h$. In other words, we have the following estimates:*

$$\|v_h\|_{0,h} \lesssim \|\nabla_w v_h\| \qquad \forall\ v_h \in V_{0,h}, \tag{3.18}$$

$$\|v_h\|_{0,h} \lesssim \|\nabla_w v_h\| \qquad \forall\ v_h \in \bar{V}_h. \tag{3.19}$$

Proof. For any $v_h \in V_{0,h}$, let $\mathbf{q} \in (H^1(\Omega))^2$ be such that $\nabla \cdot \mathbf{q} = v_0$ and $\|\mathbf{q}\|_1 \lesssim \|v_0\|$. Such a vector-valued function $\mathbf{q}$ exists on any polygonal domain [3]. One way to prove the existence of $\mathbf{q}$ is as follows. First, one extends v_h by zero to a convex domain which contains Ω. Secondly, one considers the Poisson equation on the enlarged domain and set $\mathbf{q}$ to be the flux. The required properties of $\mathbf{q}$ follow immediately from the full regularity of the Poisson equation on convex domains. By (3.1), we have

$$\|\mathbf{\Pi}_h \mathbf{q}\| \lesssim \|\mathbf{q}\|_1 \lesssim \|v_0\|.$$

Consequently, by (3.7) and the Schwarz inequality,

$$\|v_0\|^2 = (v_0, \nabla \cdot \mathbf{q}) = -(\mathbf{\Pi}_h \mathbf{q}, \nabla_w v_h) \lesssim \|v_0\| \|\nabla_w v_h\|.$$

It follows from Lemma 3.2 that

$$\sum_{K\in\mathcal{T}_h} h\|v_0 - v_b\|_{\partial K}^2 \lesssim \sum_{K\in\mathcal{T}_h} h^{-1}\|v_0 - v_b\|_{\partial K}^2 \le |v_h|_{1,h}^2 \lesssim \|\nabla_w v_h\|^2.$$

Combining the above two estimates gives a proof of the inequality (3.18).

As to (3.19), since $v_h \in \bar{V}_h$ has mean value zero, one may find a vector-valued function $\mathbf{q}$ satisfying $\nabla \cdot \mathbf{q} = v_0$ and $\mathbf{q}\cdot\mathbf{n} = 0$ on $\partial\Omega$ (see [3] for details). In addition, we have $\|\mathbf{q}\|_1 \lesssim \|v_0\|$. The rest of the proof follows the same avenue as the proof of (3.18). □

Next, we shall introduce a discrete norm in the finite element space $V_{0,h}$ that plays the role of the standard H^2 norm. To this end, for any internal edge $e \in \mathcal{E}_h$, denote by K_1 and K_2 the two triangles sharing e, and by $\mathbf{n}_1$, $\mathbf{n}_2$ the outward normals with respect to K_1 and K_2. Define the jump on e by

$$[\![\nabla_w \psi_h \cdot \mathbf{n}]\!] = (\nabla_w \psi_h)|_{K_1} \cdot \mathbf{n}_1 + (\nabla_w \psi_h)|_{K_2} \cdot \mathbf{n}_2.$$

If the edge e is on the boundary $\partial\Omega$, then there is only one triangle K which admits e as an edge. The jump is then modified as

$$[\![\nabla_w \psi_h \cdot \mathbf{n}]\!] = (\nabla_w \psi_h)|_K \cdot \mathbf{n}.$$

For $\psi_h \in V_{0,h}$, define

$$|\!|\!|\psi_h|\!|\!| = \left(\sum_{K\in\mathcal{T}_h} \|\nabla\cdot\nabla_w \psi_h\|_K^2 + \sum_{e\in\mathcal{E}_h} h^{-1}\|[\![\nabla_w \psi_h \cdot \mathbf{n}]\!]\|_e^2)\right)^{1/2}. \tag{3.20}$$

Lemma 3.5. *The map* $\|\!|\cdot|\!\|:\ V_{0,h}\to\mathbb{R}$*, as given in (3.20), defines a norm in the finite element space* $V_{0,h}$*. Moreover, one has*

$$(\nabla_w v_h,\nabla_w\psi_h)\lesssim\|v_h\|_{0,h}|\!\|\psi_h|\!\|\qquad\forall\ v_h\in V_h,\ \psi_h\in V_{0,h},\tag{3.21}$$

$$\sup_{v_h\in V_h}\frac{(\nabla_w v_h,\nabla_w\psi_h)}{\|v_h\|_{0,h}}\gtrsim|\!\|\psi_h|\!\|\qquad\forall\ \psi_h\in V_{0,h}.\tag{3.22}$$

Proof. To verify that $|\!\|\cdot|\!\|$ defines a norm, it is sufficient to show that $|\!\|\psi_h|\!\|=0$ implies $\psi_h\equiv 0$. To this end, let $|\!\|\psi_h|\!\|=0$. It follows that $\nabla\cdot\nabla_w\psi_h=0$ on each element and $[\![\nabla_w\psi_h\cdot\mathbf{n}]\!]=0$ on each edge. The definition of the discrete weak gradient ∇_w then implies the following

$$(\nabla_w\psi_h,\nabla_w\psi_h)=\sum_{K\in\mathcal{T}_h}\left(-(\psi_0,\nabla\cdot\nabla_w\psi_h)_K+\langle\psi_b,\nabla_w\psi_h\cdot\mathbf{n}\rangle_{\partial K}\right)=0.$$

Thus, we have $\nabla_w\psi_h=0$. Since $\psi_h\in V_{0,h}$, then $\nabla_w\psi_h=0$ implies $\psi_h\equiv 0$. This shows that $|\!\|\cdot|\!\|$ defines a norm in $V_{0,h}$. The inequality (3.21) follows immediately from the following identity

$$(\nabla_w v_h,\nabla_w\psi_h)=\sum_{K\in\mathcal{T}_h}\left(-(v_0,\nabla\cdot\nabla_w\psi_h)_K+\langle v_b,\nabla_w\psi_h\cdot\mathbf{n}\rangle_{\partial K}\right)$$

and the Schwarz inequality.

To verify (3.22), we chose a particular $v_h^*\in V_h$ such that

$$\begin{aligned}v_0^*&=-\nabla\cdot\nabla_w\psi_h&&\text{in } K_0,\\ v_b^*&=h^{-1}[\![\nabla_w\psi_h\cdot\mathbf{n}]\!]&&\text{on edge } e.\end{aligned}$$

It is not hard to see that $\|v_h^*\|_{0,h}\lesssim|\!\|\psi_h|\!\|$. Thus, we have

$$\begin{aligned}\sup_{v_h\in V_h}\frac{(\nabla_w v_h,\nabla_w\psi_h)}{\|v_h\|_{0,h}}&\ge\frac{(\nabla_w v_h^*,\nabla_w\psi_h)}{\|v_h^*\|_{0,h}}\\ &=\frac{\sum_{K\in\mathcal{T}_h}\left(-(v_0^*,\nabla\cdot\nabla_w\psi_h)_K+\langle v_b^*,\nabla_w\psi_h\cdot\mathbf{n}\rangle_{\partial K}\right)}{\|v_h^*\|_{0,h}}\\ &=\frac{|\!\|\psi_h|\!\|^2}{\|v_h^*\|_{0,h}}\gtrsim|\!\|\psi_h|\!\|.\end{aligned}$$

This completes the proof of the lemma. □

Remark 3.1. *Using the boundedness (3.21) and the discrete Poincare inequality (3.18) we have the following estimate for all* $\psi_h\in V_{0,h}$

$$\|\nabla_w\psi_h\|^2=(\nabla_w\psi_h,\nabla_w\psi_h)\lesssim\|\psi_h\|_{0,h}|\!\|\psi_h|\!\|\lesssim\|\nabla_w\psi_h\||\!\|\psi_h|\!\|.$$

This implies that $\|\nabla_w\psi_h\|\lesssim|\!\|\psi_h|\!\|$*. In other words,* $|\!\|\cdot|\!\|$ *is a norm that is stronger than* $\|\cdot\|_{1,h}$*. In fact, the norm* $|\!\|\cdot|\!\|$ *can be viewed as a discrete equivalence of the standard* H^2 *norm for smooth functions with proper boundary conditions.*

Next, we shall establish an estimate for the L^2 projection operator Q_h in the discrete norm $\|\cdot\|_{0,h}$.

Lemma 3.6. *Let* Q_h *be the* L^2 *projection operator into the finite element space* V_h*. Then, for any* $v\in H^m(\Omega)$ *with* $\frac{1}{2}<m\le j+1$*, we have*

$$\|v-Q_hv\|_{0,h}\lesssim h^m\|v\|_m.\tag{3.23}$$

Proof. For the L^2 projection on each element K, it is known that the following estimate holds true

$$\|v - Q_0 v\|_K \lesssim h^m \|v\|_{m,K}. \tag{3.24}$$

Thus, it suffices to deal with the terms associated with the edges/faces given by

$$\sum_K h\|(v - Q_0 v) - (v - Q_b v)\|^2_{\partial K} = \sum_K h\|Q_0 v - Q_b v\|^2_{\partial K}. \tag{3.25}$$

Since Q_b is the L^2 projection on edges, then we have

$$\|Q_0 v - Q_b v\|^2_{\partial K} \leq \|v - Q_0 v\|^2_{\partial K}.$$

Let $s \in (\frac{1}{2}, 1]$ be any real number satisfying $s \leq m$. It follows from the above inequality and the trace inequality (3.8) that

$$\|Q_0 v - Q_b v\|^2_{\partial K} \lesssim h^{-1}\|v - Q_0 v\|^2_K + h^{2s-1}|v - Q_0 v|^2_{s,K}.$$

Substituting the above into (3.25) yields

$$\begin{aligned}\sum_K h\|(v - Q_0 v) - (v - Q_b v)\|^2_{\partial K} &\lesssim \sum_K \left(\|v - Q_0 v\|^2_K + h^{2s}|v - Q_0 v|^2_{s,K}\right)\\ &\lesssim h^{2m}\|v\|^2_m,\end{aligned}$$

which, together with (3.24), completes the proof of the lemma. □

3.3. Ritz and Neumann Projections. To establish an error analysis in the forthcoming section, we shall introduce and analyze two additional projection operators, the Ritz projection R_h and the Neumann projection N_h, by applying the weak Galerkin method to the Poisson equation with various boundary conditions.

For any $v \in H^1_0(\Omega) \cap H^{1+\gamma}(\Omega)$ with $\gamma > \frac{1}{2}$, the Ritz projection $R_h v \in V_{0,h}$ is defined as the unique solution of the following problem:

$$(\nabla_w(R_h v), \nabla_w \psi_h) = (\mathbf{\Pi}_h \nabla v, \nabla_w \psi_h), \qquad \forall\ \psi_h \in V_{0,h}. \tag{3.26}$$

Here $\gamma > \frac{1}{2}$ in the definition of R_h is imposed to ensure that $\mathbf{\Pi}_h \nabla v$ is well-defined. ¿From the identity (3.7), clearly if $\Delta v \in L^2(\Omega)$, then $R_h v$ is identical to the weak Galerkin finite element solution [52] to the Poisson equation with homogeneous Dirichlet boundary condition for which v is the exact solution. Analogously, for any $v \in \bar{H}^1(\Omega) \cap H^{1+\gamma}(\Omega)$ with $\gamma > \frac{1}{2}$, we define the Neumann projection $N_h v \in \bar{V}_h$ as the solution to the following problem

$$(\nabla_w(N_h v), \nabla_w \psi_h) = (\mathbf{\Pi}_h \nabla v, \nabla_w \psi_h), \qquad \forall\ \psi_h \in \bar{V}_h. \tag{3.27}$$

It is useful to note that the above equation holds true for all $\psi_h \in V_h$ as $\nabla_w 1 = 0$. Similarly, if $\Delta v \in L^2(\Omega)$ and in addition $\partial v/\partial \mathbf{n} = 0$ on $\partial\Omega$, then $N_h v$ is identical to the weak Galerkin finite element solution to the Poisson equation with homogeneous Neumann boundary condition, for which v is the exact solution. The well-posedness of R_h and N_h follows immediately from the Poincaré-type inequalities (3.18) and (3.19).

Using (3.5), it is easy to see that for all $\psi_h \in V_{0,h}$ we have

$$(\nabla_w(Q_h v - R_h v), \nabla_w \psi_h) = ((\mathbf{P}_h - \mathbf{\Pi}_h)\nabla v, \nabla_w \psi_h). \tag{3.28}$$

And similarly, for all $\psi_h \in \bar{V}_h$,

$$(\nabla_w(Q_h v - N_h v), \nabla_w \psi_h) = ((\mathbf{P}_h - \mathbf{\Pi}_h)\nabla v, \nabla_w \psi_h). \tag{3.29}$$

¿From the definitions of $\bar{V}_h$ and Q_h, clearly Q_h maps $\bar{H}^1(\Omega)$ into $\bar{V}_h$.

For convenience, let us adopt the following notation

$$\{R_0 v, R_b v\} := R_h v, \qquad \{N_0 v, N_b v\} := N_h v,$$

where again the subscript "0" denotes the function value in the interior of triangles, while "b" denotes the trace on $\mathcal{E}_h$. For Ritz and Neumann projections, the following approximation error estimates hold true.

LEMMA 3.7. *For $v \in H_0^1(\Omega) \cap H^{m+1}(\Omega)$ or $\bar{H}^1(\Omega) \cap H^{m+1}(\Omega)$, where $\frac{1}{2} < m \leq j+1$, we have*

$$\|\nabla_w(Q_h v - R_h v)\| \lesssim h^m \|v\|_{m+1}, \tag{3.30}$$

$$\|\nabla_w(Q_h v - N_h v)\| \lesssim h^m \|v\|_{m+1}. \tag{3.31}$$

Moreover, assume $\Delta v \in L^2(\Omega)$ and that the Poisson problem in Ω with either the homogeneous Dirichlet boundary condition or the homogeneous Neumann boundary condition has H^{1+s} regularity, where $\frac{1}{2} < s \leq 1$, then

$$\|Q_0 v - R_0 v\| \lesssim h^{m+s}\|v\|_{m+1} + h^{1+s}\|(I - Q_0)\Delta v\|, \tag{3.32}$$

$$\|Q_0 v - N_0 v\| \lesssim h^{m+\min(s, j+\frac{1}{2})}\|v\|_{m+1} + h^{1+s}\|(I - Q_0)\Delta v\|. \tag{3.33}$$

Proof. The estimates (3.30)-(3.31) follow immediately from (3.28)-(3.29), (3.4), and the Schwarz inequality. Next, we prove (3.33) by using the standard duality argument. Let $\phi \in \bar{H}^1(\Omega)$ be the solution of $-\Delta\phi = Q_0 v - N_0 v$ with boundary condition $\frac{\partial \phi}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0$. Note that ϕ is well-defined since $Q_h v - N_h v \in \bar{V}_h$. According to the regularity assumption, we have $\phi \in H^{1+s}(\Omega)$ and $\|\phi\|_{1+s} \lesssim \|Q_0 v - N_0 v\|$. Then, by (3.7), (3.29), the Schwarz inequality and (3.4), we arrive at

$$\begin{aligned}
\|Q_0 v - N_0 v\|^2 &= (Q_0 v - N_0 v, -\Delta\phi) = (\mathbf{\Pi}_h \nabla\phi, \nabla_w(Q_h v - N_h v)) \\
&= (\mathbf{\Pi}_h \nabla\phi - \nabla_w(N_h\phi), \nabla_w(Q_h v - N_h v)) + ((\mathbf{P}_h - \mathbf{\Pi}_h)\nabla v, \nabla_w(N_h\phi)) \\
&\leq \Big(\|\mathbf{\Pi}_h\nabla\phi - \mathbf{P}_h\nabla\phi\| + \|\nabla_w(Q_h\phi - N_h\phi)\|\Big)\|\nabla_w(Q_h v - N_h v)\| \\
&\quad + ((\mathbf{P}_h - \mathbf{\Pi}_h)\nabla v, \nabla_w(N_h\phi - Q_h\phi)) + ((\mathbf{P}_h - \mathbf{\Pi}_h)\nabla v, \mathbf{P}_h\nabla\phi) \\
&\lesssim h^{m+s}\|\phi\|_{1+s}\|v\|_{m+1} + ((I - \mathbf{\Pi}_h)\nabla v, \mathbf{P}_h\nabla\phi).
\end{aligned}$$

Using integration by parts, the triangular inequality and the definition of $\mathbf{\Pi}_h$, we have

$$\begin{aligned}
&((I - \mathbf{\Pi}_h)\nabla v, \mathbf{P}_h\nabla\phi) \\
=&((I - \mathbf{\Pi}_h)\nabla v, (\mathbf{P}_h - I)\nabla\phi) + ((I - \mathbf{\Pi}_h)\nabla v, \nabla\phi) \\
\lesssim& h^{m+s}\|\phi\|_{1+s}\|v\|_{m+1} + ((I - \mathbf{\Pi}_h)\nabla v \cdot \mathbf{n}, \phi)_{\partial\Omega} - (\nabla \cdot (I - \mathbf{\Pi}_h)\nabla v, \phi) \\
=& h^{m+s}\|\phi\|_{1+s}\|v\|_{m+1} + ((I - \mathbf{\Pi}_h)\nabla v \cdot \mathbf{n}, \phi - Q_b\phi)_{\partial\Omega} - ((I - Q_0)\Delta v, \phi) \\
\lesssim& h^{m+s}\|\phi\|_{1+s}\|v\|_{m+1} + (h^{m-\frac{1}{2}}\|v\|_{m+\frac{1}{2},\partial\Omega})(h^{\min(s+\frac{1}{2}, j+1)}\|\phi\|_{s+\frac{1}{2},\partial\Omega}) \\
&\quad - ((I - Q_0)\Delta v, (I - Q_0)\phi) \\
\lesssim& h^{m+\min(s, j+\frac{1}{2})}\|\phi\|_{1+s}\|v\|_{m+1} + h^{1+s}\|\phi\|_{1+s}\|(I - Q_0)\Delta v\|.
\end{aligned} \tag{3.34}$$

In the proof of (3.34), we have used the fact that $\Pi_h(\nabla v\cdot\mathbf{n})$ is exactly the L^2 projection of $\nabla v \cdot \mathbf{n}$ on $\partial\Omega$. Combining the above gives

$$\begin{aligned}\|Q_0v - N_0v\|^2 &\lesssim \left(h^{m+\min(s,j+\frac{1}{2})}\|v\|_{m+1} + h^{1+s}\|(I-Q_0)\Delta v\|\right)\|\phi\|_{1+s}\\ &\lesssim \left(h^{m+\min(s,j+\frac{1}{2})}\|v\|_{m+1} + h^{1+s}\|(I-Q_0)\Delta v\|\right)\|Q_0v - N_0v\|.\end{aligned}$$

This completes the proof of the estimate (3.33). The inequality (3.32) can be verified in a similar way by considering a function $\phi \in H_0^1(\Omega)$ satisfying a Poisson equation with homogeneous Dirichlet boundary condition. Observe that in this case, the boundary integral $((I - \mathbf{\Pi}_h)\nabla v \cdot \mathbf{n}, \phi)_{\partial\Omega}$ in inequality (3.34) shall vanish due to the vanishing value of ϕ. □

Remark 3.2. *It is not hard to see from (3.34) that for the Neumann projection, if in addition we have $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$, then the term $((I - \mathbf{\Pi}_h)\nabla v \cdot \mathbf{n}, \phi)_{\partial\Omega}$ vanishes and one obtains the optimal order estimate of h^{m+s} instead of $h^{m+\min(s,j+\frac{1}{2})}$ for the Neumann projection operator.*

Remark 3.3. *If the Poisson equation has the full H^2 regularity in Ω, then for v satisfying the assumptions of Lemma 3.7, we have*

$$\|Q_0v - R_0v\| \lesssim h^{m+1}\|v\|_{m+1} + h^2\|(I-Q_0)\Delta v\| \qquad \text{for } \frac{1}{2} < m \le j+1,$$

$$\|Q_0v - N_0v\| \lesssim \begin{cases} h^{m+\frac{1}{2}}\|v\|_{m+1} + h^2\|(I-Q_0)\Delta v\| & \text{for } j=0,\ \frac{1}{2} < m \le 1,\\ h^{m+1}\|v\|_{m+1} + h^2\|(I-Q_0)\Delta v\| & \text{for } j \ge 1,\ \frac{1}{2} < m \le j+1.\end{cases}$$

Again, if in addition, $\frac{\partial v}{\partial n} = 0$ on $\partial\Omega$, then the Neumann projection has optimal order of error estimates, even for $j = 0$.

Remark 3.4. *The duality argument used in Lemma 3.7 works only for $\|Q_0v - R_0v\|$ and $\|Q_0v - N_0v\|$. For $\|Q_hv - R_hv\|_{0,h}$ and $\|Q_hv - N_hv\|_{0,h}$ involving element boundary information, we currently have only sub-optimal estimates. More precisely, for v satisfying the assumptions in Lemma 3.7, the following estimates hold true.*

$$\begin{aligned}&\|Q_hv - R_hv\|_{0,h} \lesssim \|\nabla_w(Q_hv - R_hv)\| \lesssim h^m\|v\|_{m+1} \qquad \text{for } \frac{1}{2} < m \le j+1,\\ &\|Q_hv - N_hv\|_{0,h} \lesssim \|\nabla_w(Q_hv - N_hv)\| \lesssim h^m\|v\|_{m+1} \qquad \text{for } \frac{1}{2} < m \le j+1.\end{aligned} \tag{3.35}$$

Although numerical experiments in [43] suggest an optimal order of convergence in the $\|\cdot\|_{0,h}$ norm, it remains to see if optimal order error estimates hold true or not theoretically.

Another important observation is that, for sufficiently smooth v, $\nabla_w R_h v$ is identical to the mixed finite element approximation of ∇v, discretized by using RT_j and discrete P_j elements. Indeed, we have the following lemma:

Lemma 3.8. *For any $v \in H_0^1 \cap H^{1+\gamma}(\Omega)$ with $\gamma > \frac{1}{2}$ and $\Delta v \in L^2(\Omega)$, let $\mathbf{q}_h \in \Sigma_h \cap H(div,\Omega)$ and $v_0 \in L^2(\Omega)$ be piecewise P_j polynomials solving*

$$\begin{cases}(\mathbf{q}_h, \boldsymbol{\chi}_h) - (\nabla\cdot\boldsymbol{\chi}_h, v_0) = 0 & \forall\, \boldsymbol{\chi}_h \in \Sigma_h \cap H(div,\Omega),\\ (\nabla\cdot\mathbf{q}_h, \psi_0) = (\Delta v, \psi_0) & \forall\, \psi_0 \in L^2(\Omega) \text{ piecewise } P_j \text{ polynomials}.\end{cases} \tag{3.36}$$

In other words, $\mathbf{q}_h$ and v_0 are the mixed finite element solution, discretized using the RT_j element, to the Poisson equation with homogeneous Dirichlet boundary condition for which v is the exact solution. Then, one has $\nabla_w R_h v = \mathbf{q}_h$.

Proof. We first show that $\nabla_w R_h v \in \Sigma_h \cap H(div,\Omega)$ by verifying that $(\nabla_w R_h v)\cdot \mathbf{n}$ is continuous across internal edges. Let $e \in \mathcal{E}_h \backslash \partial\Omega$ be an internal edge and K_1, K_2 be two triangles sharing e. Denote $\mathbf{n}_1$ and $\mathbf{n}_2$ the outward normal vectors on e, with respect to K_1 and K_2, respectively. Let $\psi_h \in V_{0,h}$ satisfy $\psi_b|_e \neq 0$ and ψ_0, ψ_b vanish elsewhere. By the definition of R_h, ∇_w and the fact that $\mathbf{\Pi}_h \nabla v \in H(div,\Omega)$, we have

$$
\begin{aligned}
0 &= (\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \psi_h)\\
&= (\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \psi_h)_{K_1} + (\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \psi_h)_{K_2}\\
&= ((\mathbf{\Pi}_h \nabla v - \nabla_w R_h v)|_{K_1}\cdot \mathbf{n}_1 + (\mathbf{\Pi}_h \nabla v - \nabla_w R_h v)|_{K_2}\cdot \mathbf{n}_2, \psi_b)_e\\
&= -(\nabla_w R_h v|_{K_1}\cdot \mathbf{n}_1 + \nabla_w R_h v|_{K_2}\cdot \mathbf{n}_2, \psi_b)_e.
\end{aligned}
$$

The above equation holds true for all $\psi_b|_e \in P_j(e)$. Since $\nabla_w R_h v|_{K_1}\cdot \mathbf{n}_1 + \nabla_w R_h v|_{K_2}\cdot \mathbf{n}_2$ is also in $P_j(e)$, therefore it must be 0. This completes the proof of $\nabla_w R_h v \in H(div,\Omega)$.

Next, we prove that $\nabla_w R_h v$ is identical to the solution $\mathbf{q}_h$ of (3.36). Since the solution to (3.36) is unique, we only need to show that $\nabla_w R_h v$, together with a certain v_0, satisfies both equations in (3.36). Consider the test function $\psi_h \in V_{0,h}$ with the form $\psi_h = \{\psi_0, 0\}$. By the definition of ∇_w, equations (3.26) and (3.7), we have

$$
(\nabla \cdot \nabla_w R_h v, \psi_0) = -(\nabla_w R_h v, \nabla_w \psi_h) = -(\mathbf{\Pi}_h \nabla v, \nabla_w \psi_h) = (\Delta v, \psi_0).
$$

Hence $\nabla_w R_h v$ satisfies the second equation of (3.36). Now, note that $\nabla\cdot$ is an onto operator from $\Sigma_h \cap H(div,\Omega)$ to the space of piecewise P_j polynomials, which allows us to define a v_0 that satisfies the first equation in (3.36) with $\mathbf{q}_h$ set to be $\nabla_w R_h v$. This completes the proof the the lemma. □

REMARK 3.5. *Using the same argument and noticing that (3.27) holds for all $\psi_h \in V_h$, one can analogously prove that for $v \in \bar{H}^1(\Omega)\cap H^{1+\gamma}(\Omega)$ with $\gamma > \frac{1}{2}$ and $\Delta v \in L^2(\Omega)$,*

$$
\nabla_w N_h v \in \Sigma_h \cap H(div,\Omega),
$$

and

$$
\nabla \cdot \nabla_w N_h v = Q_0 \Delta v.
$$

Because $\nabla_w R_h v$ is identical to the mixed finite element solution to the Poisson equation, by [50, 30], we have the following quasi-optimal order L^∞ estimate:

$$
\|\nabla v - \nabla_w R_h v\|_{L^\infty(\Omega)} \lesssim h^{n+1}|\ln h|\|\Delta v\|_{W^{n,\infty}(\Omega)}, \tag{3.37}
$$

for $0 \le n \le j$. Furthermore, for $j \ge 1$ and $v \in W^{j+2,\infty}(\Omega)$, we have the following optimal order error estimate

$$
\|\nabla v - \nabla_w R_h v\|_{L^\infty(\Omega)} \lesssim h^{n+1}\|v\|_{W^{n+2,\infty}(\Omega)}, \tag{3.38}
$$

for $1 \le n \le j$.

Inspired by [48], using the above L^∞ estimates we obtain the following lemma, which will play an essential role in the error analysis to be given in the next section.

LEMMA 3.9. *The following quasi-optimal and optimal order error estimates hold true:*

(i) *Let* $0 \le n \le j$ *and* $v \in H_0^1(\Omega) \cap W^{n+2,\infty}(\Omega)$. *Then for all* $\phi_h = \{v_0, v_b\} \in V_h$, *we have*

$$|(\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \phi_h)| \lesssim h^{n+\frac{1}{2}} |\ln h| \|v\|_{W^{n+2,\infty}(\Omega)} \|\phi_h\|_{0,h}. \tag{3.39}$$

(ii) *Let* $j \ge 1$, $1 \le n \le j$, *and* $v \in H_0^1(\Omega) \cap W^{n+2,\infty}(\Omega)$. *Then, for all* $\phi_h = \{v_0, v_b\} \in V_h$ *we have*

$$|(\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \phi_h)| \lesssim h^{n+\frac{1}{2}} \|v\|_{W^{n+2,\infty}(\Omega)} \|\phi_h\|_{0,h}. \tag{3.40}$$

Proof. We first prove part (i). Denote by $\mathcal{E}_{\partial\Omega}$ the set of all edges in $\mathcal{E}_h \cap \partial\Omega$. For any $e \in \mathcal{E}_{\partial\Omega}$, let K_e be the only triangle in $\mathcal{T}_h$ that has e as an edge. Denote by $\mathcal{T}_{\partial\Omega}$ the set of all K_e, for $e \in \mathcal{E}_{\partial\Omega}$. For simplicity of notation, denote $\mathbf{q}_h = \mathbf{\Pi}_h \nabla v - \nabla_w R_h v$. Since $(\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \psi_h) = 0$ for all $\psi_h \in V_{0,h}$, without loss of generality, we only need to consider ϕ_h that vanishes on the interior of all triangles and all internal edges. Then by the definition of ϕ_h and ∇_w, the scaling argument, and the Schwarz inequality,

$$\begin{aligned}
|(\mathbf{\Pi}_h \nabla v - \nabla_w R_h v, \nabla_w \phi_h)| &= \left| \sum_{K_e \in \mathcal{T}_{\partial\Omega}} (\mathbf{q}_h, \nabla_w (\phi_b|_e))_{K_e} \right| \\
&= \left| \sum_{e \in \mathcal{E}_{\partial\Omega}} (\phi_b, \mathbf{q}_h \cdot \mathbf{n})_e \right| \\
&\lesssim \sum_{e \in \mathcal{E}_{\partial\Omega}} h \|\phi_b\|_{L^\infty(e)} \|\mathbf{q}_h\|_{L^\infty(e)} \\
&\lesssim \|\mathbf{q}_h\|_{L^\infty(\Omega)} \sum_{e \in \mathcal{E}_{\partial\Omega}} h \left(\|\phi_0\|_{L^\infty(K_e)} + \|\phi_0 - \phi_b\|_{L^\infty(e)} \right) \\
&\lesssim \|\mathbf{q}_h\|_{L^\infty(\Omega)} \sum_{K_e \in \mathcal{T}_{\partial\Omega}} \|\phi_h\|_{0,h,K_e} \\
&\lesssim \|\mathbf{q}_h\|_{L^\infty(\Omega)} \left(\sum_{K_e \in \mathcal{T}_{\partial\Omega}} \|\phi_h\|_{0,h,K_e}^2 \right)^{\frac{1}{2}} \left(\sum_{K_e \in \mathcal{T}_{\partial\Omega}} 1 \right)^{\frac{1}{2}} \\
&\lesssim h^{-\frac{1}{2}} \|\mathbf{q}_h\|_{L^\infty(\Omega)} \|\phi_h\|_{0,h}.
\end{aligned}$$

Now, by inequalities (3.2) and (3.37), we have

$$\begin{aligned}
\|\mathbf{q}_h\|_{L^\infty(\Omega)} &\le \|\nabla v - \mathbf{\Pi}_h \nabla v\|_{L^\infty(\Omega)} + \|\nabla v - \nabla_w R_h v\|_{L^\infty(\Omega)} \\
&\lesssim h^{n+1} \|v\|_{W^{n+2,\infty}(\Omega)} + h^{n+1} |\ln h| \|\Delta v\|_{W^{n,\infty}(\Omega)},
\end{aligned}$$

for $0 \le n \le j$. This completes the proof of part (i).

The proof for part (ii) is similar. One simply needs to replace inequality (3.37) by (3.38) in the estimation of $\|\mathbf{q}_h\|_{L^\infty(\Omega)}$. □

4. Error analysis. The main purpose of this section is to analyze the approximation error of the weak Galerkin formulation (2.4). For simplicity, in this section, we assume that the solution of (2.4) satisfies $u \in H^{3+\gamma}(\Omega)$ and $w \in H^{1+\gamma}(\Omega)$, where $\gamma > \frac{1}{2}$. This is not an unreasonable assumption, as we know from (2.2), the solution u can have up to H^4 regularity as long as Ω satisfies certain conditions. However, our assumption does not include all the possible cases for the biharmonic equation.

Testing $w = -\Delta u$ with $\phi_h = \{\phi_0, \phi_b\} \in V_h$ and then by using (3.7) we have

$$(\!(w, \phi_h)\!) = (w, \phi_0) = -(\nabla \cdot \nabla u, \phi_0) = (\mathbf{\Pi}_h \nabla u, \nabla_w \phi_h). \tag{4.1}$$

Similarly, testing $-\Delta w = f$ with $\psi_h = \{\psi_0, \psi_b\} \in V_{0,h}$ gives

$$(\mathbf{\Pi}_h \nabla w, \nabla_w \psi_h) = (f, \psi_0). \tag{4.2}$$

Comparing (4.1)-(4.2) with the weak Galerkin form (2.4), one immediately sees that there is a consistency error between them. Indeed, since V_h and $V_{0,h}$ are not subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively, the weak Galerkin method is non-conforming. Therefore, we would like to first rewrite (4.1)-(4.2) into a form that is more compatible with (2.4). By using (3.26) and (3.27), equations (4.1)-(4.2) can be rewritten as

$$\begin{cases} (\!(N_h w, \phi_h)\!) - (\nabla_w R_h u, \nabla_h \phi_h) = E(w, u, \phi_h), \\ (\nabla_w N_h w, \nabla_w \psi_h) = (f, \psi_0), \end{cases} \tag{4.3}$$

where

$$E(w, u, \phi_h) = (\!(N_h w - w, \phi_h)\!) + (\mathbf{\Pi}_h \nabla u - \nabla_w R_h u, \nabla_w \phi_h).$$

Define $\varepsilon_u = R_h u - u_h \in V_{0,h}$ and $\varepsilon_w = N_h w - w_h \in V_h$. By subtracting (4.3) from (2.4), we have

$$\begin{cases} (\!(\varepsilon_w, \phi_h)\!) - (\nabla_w \varepsilon_u, \nabla_h \phi_h) = E(w, u, \phi_h) & \text{for all } \phi_h \in V_h, \\ (\nabla_w \varepsilon_w, \nabla_w \psi_h) = 0 & \text{for all } \psi_h \in V_{0,h}. \end{cases} \tag{4.4}$$

Notice here $(\nabla_w \varepsilon_w, \nabla_w \psi_h) = 0$ does not necessarily imply $\varepsilon_w = 0$, since the equation only holds for all $\psi_h \in V_{0,h}$ while ε_w is in V_h.

LEMMA 4.1. *The consistency error $E(w, u, \phi_h)$ is small in the sense that*

$$|E(w, u, \phi_h)| \lesssim h^m \|w\|_{m+1} \|\phi_h\|_{0,h} + h^{n+\frac{1}{2}} |\ln h| \|u\|_{W^{n+2,\infty}(\Omega)} \|\phi_h\|_{0,h},$$

where $\frac{1}{2} < m \leq j+1$ and $0 \leq n \leq j$. Moreover, for $j \geq 1$, we have the improved estimate

$$|E(w, u, \phi_h)| \lesssim h^m \|w\|_{m+1} \|\phi_h\|_{0,h} + h^{n+\frac{1}{2}} \|u\|_{W^{n+2,\infty}(\Omega)} \|\phi_h\|_{0,h},$$

where $\frac{1}{2} < m \leq j+1$ and $1 \leq n \leq j$.

Proof. The proof is straight forward by using the Schwarz inequality, Lemma 3.6, Remark 3.4, and Lemma 3.9. □

To derive an error estimate from (4.4), let us recall the standard theory for mixed finite element methods. Given two bounded bilinear forms $a(\cdot,\cdot)$ defined on $X \times X$ and $b(\cdot,\cdot)$ defined on $X \times M$, where X and M are finite dimensional spaces. Denote $X_0 \subset X$ by

$$X_0 = \{\phi \in X : \ b(\phi, \psi) = 0 \text{ for all } \psi \in M\}.$$

Then for all $\chi \in X$ and $\xi \in M$,

$$\sup_{\phi \in X,\, \psi \in M} \frac{a(\chi, \phi) + b(\phi, \xi) + b(\chi, \psi)}{\|\phi\|_X + \|\psi\|_M} \gtrsim \|\chi\|_X + \|\xi\|_M,$$

if and only if

$$\begin{aligned}\sup_{\phi\in X_0}\frac{a(\chi,\phi)}{\|\phi\|_X} &\gtrsim \|\chi\|_X, \qquad \text{for all } \chi\in X_0,\\ \sup_{\phi\in X}\frac{b(\phi,\xi)}{\|\phi\|_X} &\gtrsim \|\xi\|_M, \qquad \text{for all } \xi\in M.\end{aligned} \tag{4.5}$$

In our formulation, we set $X = V_h$ with norm $\|\cdot\|_{0,h}$ and $M = V_{0,h}$ with norm $|\!|\!|\cdot|\!|\!|$. Define

$$a(\chi,\phi) = (\!(\chi,\phi)\!), \qquad b(\phi,\xi) = -(\nabla_w\phi,\nabla_w\xi).$$

It is not hard to check that both of these bilinear forms are bounded under the given norms. In particular, the boundedness of $b(\cdot,\cdot)$ has been given in (3.21). It is also clear that the first inequality in (4.5) follows from the definition of $a(\cdot,\cdot)$ and $\|\cdot\|_{0,h}$, and the second inequality follows directly from (3.22). Combine the above, we have for all $\chi\in V_h$ and $\xi\in V_{0,h}$,

$$\sup_{\phi\in V_h,\,\psi\in V_{0,h}}\frac{(\!(\chi,\phi)\!)-(\nabla_w\phi,\nabla_w\xi)-(\nabla_w\chi,\nabla_w\psi)}{\|\phi\|_{0,h}+|\!|\!|\psi|\!|\!|}\gtrsim \|\chi\|_{0,h}+|\!|\!|\xi|\!|\!|. \tag{4.6}$$

Theorem 4.2. *The weak Galerkin formulation (2.4) for the biharmonic problem (1.1) has the following error estimate:*

$$\|\varepsilon_w\|_{0,h}+|\!|\!|\varepsilon_u|\!|\!| \lesssim h^m\|w\|_{m+1}+h^{n+\frac12}|\ln h|\|u\|_{W^{n+2,\infty}(\Omega)},$$

where $\frac12 < m\le j+1$ *and* $0\le n\le j$. *Moreover, for* $j\ge 1$, *we have the improved estimate*

$$\|\varepsilon_w\|_{0,h}+|\!|\!|\varepsilon_u|\!|\!| \lesssim h^m\|w\|_{m+1}+h^{n+\frac12}\|u\|_{W^{n+2,\infty}(\Omega)},$$

where $\frac12 < m\le j+1$ *and* $1\le n\le j$.

Proof. By (4.4) and (4.6),

$$\begin{aligned}\|\varepsilon_w\|_{0,h}+|\!|\!|\varepsilon_u|\!|\!| &\lesssim \sup_{\phi_h\in V_h,\,\psi_h\in V_{0,h}}\frac{(\!(\varepsilon_w,\phi_h)\!)-(\nabla_w\phi_h,\nabla_w\varepsilon_u)-(\nabla_w\varepsilon_w,\nabla_w\psi_h)}{\|\phi_h\|_{0,h}+|\!|\!|\psi_h|\!|\!|}\\ &= \sup_{\phi_h\in V_h,\,\psi_h\in V_{0,h}}\frac{E(w,u,\phi_h)}{\|\phi_h\|_{0,h}+|\!|\!|\psi_h|\!|\!|}.\end{aligned}$$

Combining this with Lemma 4.1, this completes the proof of the theorem. ∎

Remark 4.1. *Assume that the exact solution w and u are sufficiently smooth. It follows from the above theorem that the following convergence holds true*

$$\|\varepsilon_w\|_{0,h}+|\!|\!|\varepsilon_u|\!|\!| \lesssim \begin{cases} O(h^{\frac12}|\ln h|) & \text{for } j=0,\\ O(h^{j+\frac12}) & \text{for } j\ge 1.\end{cases}$$

At this stage, it is standard to use the duality argument and derive an error estimation for the L^2 norm of ε_u. However, estimating $\|\varepsilon_u\|_{0,h}$ is not an easy task, as is similar to the case of Poisson equations. For simplicity, we only consider $\|\varepsilon_{u,0}\|$, where ε_u is conveniently expressed as $\varepsilon_u = \{\varepsilon_{u,0}, \varepsilon_{u,b}\}$. Define

$$\begin{cases} \xi + \Delta\eta = 0, \\ -\Delta\xi = \varepsilon_{u,0}, \end{cases} \tag{4.7}$$

where $\eta = 0$ and $\frac{\partial\eta}{\partial\mathbf{n}} = 0$ on $\partial\Omega$. We assume that all internal angles of Ω are less than $126.283696\cdots^\circ$. Then, according to (2.2), the solution to (4.7) has H^4 regularity:

$$\|\xi\|_2 + \|\eta\|_4 \lesssim \|\varepsilon_{u,0}\|.$$

Furthermore, since such a domain Ω is convex, the Poisson equation with either the homogeneous Dirichlet boundary condition or the homogeneous Neumann boundary condition has H^2 regularity.

Clearly, Equation (4.7) can be written into the following form:

$$\begin{cases} (\!(N_h\xi,\, \phi_h)\!) - (\nabla_w R_h\eta,\, \nabla_w\phi_h) = E(\xi,\eta,\phi_h) & \text{for all } \phi_h = \{\phi_0,\, \phi_b\} \in V_h, \\ (\nabla_w N_h\xi,\, \nabla_w\psi_h) = (\varepsilon_{u,0},\, \psi_0) & \text{for all } \psi_h = \{\psi_0,\, \psi_b\} \in V_{0,h}. \end{cases} \tag{4.8}$$

For simplicity of the notation, denote

$$\Lambda(N_h\xi, R_h\eta_h;\, \phi_h, \psi_h) = (\!(N_h\xi,\, \phi_h)\!) - (\nabla_w R_h\eta,\, \nabla_w\phi_h) - (\nabla_w N_h\xi,\, \nabla_w\psi_h).$$

Note that Λ is a symmetric bilinear form. By setting $\phi_h = \varepsilon_w$ and $\psi_h = \varepsilon_u$ in (4.8) and then subtract these two equations, one get

$$\begin{aligned} \|\varepsilon_{u,0}\|^2 &= E(\xi,\eta,\varepsilon_w) - \Lambda(N_h\xi, R_h\eta;\, \varepsilon_w, \varepsilon_u) \\ &= E(\xi,\eta,\varepsilon_w) - \Lambda(\varepsilon_w, \varepsilon_u;\, N_h\xi, R_h\eta) \\ &= E(\xi,\eta,\varepsilon_w) - E(w,u,N_h\xi). \end{aligned} \tag{4.9}$$

Here we have used the symmetry of $\Lambda(\cdot,\cdot)$ and Equation (4.4).

The two terms, $E(\xi,\eta,\varepsilon_w)$ and $E(w,u,N_h\xi)$, in the right-hand side of Equation (4.9) will be estimated one by one. We start from $E(\xi,\eta,\varepsilon_w)$. By using Lemma 4.1, it follows that

(i) When $j = 0$,

$$\begin{aligned} E(\xi,\eta,\varepsilon_w) &\lesssim \left(h\|\xi\|_2 + h^{\frac{1}{2}}|\ln h|\|\eta\|_{W^{2,\infty}(\Omega)}\right)\|\varepsilon_w\|_{0,h} \\ &\lesssim h^{1/2}|\ln h|\left(\|\xi\|_2 + \|\eta\|_4\right)\|\varepsilon_w\|_{0,h}. \end{aligned} \tag{4.10}$$

(ii) When $j \geq 1$, let $\delta > 0$ be an infinitely small number which ensures the Sobolev embedding from $W^{4,2}(\Omega)$ to $W^{3-\delta,\infty}(\Omega)$. Then

$$\begin{aligned} E(\xi,\eta,\varepsilon_w) &\lesssim \left(h\|\xi\|_2 + h^{\frac{3}{2}-\delta}|\ln h|\|\eta\|_{W^{3-\delta,\infty}(\Omega)}\right)\|\varepsilon_w\|_{0,h} \\ &\lesssim h\left(\|\xi\|_2 + \|\eta\|_4\right)\|\varepsilon_w\|_{0,h}. \end{aligned} \tag{4.11}$$

Next, we give an estimate for $E(w,u,N_h\xi)$.

Lemma 4.3. *Assume all internal angles of Ω are less than $126.283696\cdots^\circ$, which means the biharmonic problem with clamped boundary condition in Ω has H^4 regularity. Then* *(i) For* $j=0$,

$$E(w,u,N_h\xi)\lesssim\left(h^{m+\frac12}\|w\|_{m+1}+h^2\|(I-Q_0)f\|+h^{n+1}\|u\|_{n+1}\right)\|\xi\|_2,$$

where $\frac12<m\le 1$ *and* $1/2<n\le 1$.

(ii) For $j\ge 1$,

$$E(w,u,N_h\xi)\lesssim\left(h^{m+1}\|w\|_{m+1}+h^2\|(I-Q_0)f\|+h^{n+1}\|u\|_{n+1}\right)\|\xi\|_2,$$

where $\frac12<m\le j+1$ *and* $1/2<n\le j+1$.

Proof. By definition,

$$E(w,u,N_h\xi)=(\!(N_hw-w,N_h\xi)\!)+(\mathbf{\Pi}_h\nabla u-\nabla_wR_hu,\nabla_wN_h\xi). \tag{4.12}$$

First, by the definition of $(\!(\cdot,\cdot)\!)$, the Schwarz inequality, Remark 3.3 and 3.4, we have

$$\begin{aligned}
&(\!(N_hw-w,N_h\xi)\!)\\
&=(N_0w-Q_0w,N_0\xi)+\sum_{K\in\mathcal{T}_h}h(N_0w-N_bw,N_0\xi-N_b\xi)_{\partial K}\\
&\lesssim\|N_0w-Q_0w\|\|N_0\xi\|+\|N_hw-w\|_{0,h}\|N_h\xi-\xi\|_{0,h}\\
&\lesssim\begin{cases}(h^{m+\frac12}\|w\|_{m+1}+h^2\|(I-Q_0)\Delta w\|)\|\xi\|_2 & \text{for } j=0,\ \frac12<m\le 1\\ (h^{m+1}\|w\|_{m+1}+h^2\|(I-Q_0)\Delta w\|)\|\xi\|_2 & \text{for } j\ge 1,\ \frac12<m\le j+1\end{cases}.
\end{aligned} \tag{4.13}$$

Next, by using inequalities (3.5), (3.27), (3.7), (3.4), (3.31) and (3.32) one after one, we get

$$\begin{aligned}
&(\mathbf{\Pi}_h\nabla u-\nabla_wR_hu,\nabla_wN_h\xi)\\
=&((\mathbf{\Pi}_h-\mathbf{P}_h)\nabla u,\nabla_wN_h\xi)+(\nabla_w(Q_hu-R_hu),\nabla_wN_h\xi)\\
=&((\mathbf{\Pi}_h-\mathbf{P}_h)\nabla u,\nabla_wN_h\xi)+(\nabla_w(Q_hu-R_hu),\mathbf{\Pi}_h\nabla\xi)\\
=&((\mathbf{\Pi}_h-\mathbf{P}_h)\nabla u,\nabla_w(N_h\xi-Q_h\xi))+((\mathbf{\Pi}_h-\mathbf{P}_h)\nabla u,\mathbf{P}_h\nabla\xi)-(Q_0u-R_0u,\Delta\xi)\\
\lesssim& h^{n+1}\|u\|_{n+1}\|\xi\|_2+((\mathbf{\Pi}_h-I)\nabla u,\mathbf{P}_h\nabla\xi)+h^2\|(I-Q_0)\Delta u\|\|\xi\|_2,
\end{aligned}$$

for $\frac12<n\le j+1$. The estimation for $((\mathbf{\Pi}_h-I)\nabla u,\mathbf{P}_h\nabla\xi)$ follows the same technique used in Inequality (3.34). By the definition of $\mathbf{\Pi}_h$ and since $\frac{\partial u}{\partial\mathbf{n}}=0$ on $\partial\Omega$, we know that $(\mathbf{\Pi}_h-I)\nabla u\cdot\mathbf{n}$ also vanishes on $\partial\Omega$. Therefore, using the same argument as in (3.34), one has

$$((\mathbf{\Pi}_h-I)\nabla u,\mathbf{P}_h\nabla\xi)\lesssim h^{n+1}\|u\|_{n+1}\|\xi\|_2+h^2\|(I-Q_0)\Delta u\|\|\xi\|_2$$

for $\frac12<n\le j+1$. Combining the above gives

$$(\mathbf{\Pi}_h\nabla u-\nabla_wR_hu,\nabla_wN_h\xi)\lesssim\left(h^{n+1}\|u\|_{n+1}+h^2\|(I-Q_0)\Delta u\|\right)\|\xi\|_2. \tag{4.14}$$

for $\frac12<n\le j+1$.

Notice that

$$\begin{aligned}
&h^2\|(I-Q_0)\Delta u\|=h^2\|(I-Q_0)w\|\lesssim h^{m+2}\|w\|_m \qquad \text{for } 0\le m\le j+1,\\
&h^2\|(I-Q_0)\Delta w\|=h^2\|(I-Q_0)f\|.
\end{aligned} \tag{4.15}$$

The lemma follows immediately from (4.12)-(4.15). □

Finally, combining Theorem 4.2, inequalities (4.9), (4.10)-(4.11), and Lemma 4.3, we get the following L^2 error estimation:

THEOREM 4.4. *Assume all internal angles of Ω are less than $126.283696\cdots^\circ$, which means the biharmonic problem with clamped boundary condition in Ω has H^4 regularity. Then*

(i) For $j = 0$,

$$\|\varepsilon_{u,0}\| \lesssim h^{m+\frac{1}{2}}|\ln h|\|w\|_{m+1} + h|\ln h|^2\|u\|_{W^{2,\infty}(\Omega)} + h^2\|(I - Q_0)f\| + h^{n+1}\|u\|_{n+1},$$

where $\frac{1}{2} < m \leq 1$ and $\frac{1}{2} < n \leq 1$.

(ii) For $j \geq 1$,

$$\|\varepsilon_{u,0}\| \lesssim h^{m+1}\|w\|_{m+1} + h^{l+\frac{3}{2}}\|u\|_{W^{l+2,\infty}(\Omega)} + h^2\|(I - Q_0)f\| + h^{n+1}\|u\|_{n+1},$$

where $\frac{1}{2} < m \leq j+1$, $\frac{1}{2} < n \leq j+1$ and $1 \leq l \leq j$.

REMARK 4.2. *If u, w and f are sufficiently smooth, then we get*

$$\|\varepsilon_{u,0}\| \lesssim \begin{cases} O(h|\ln h|^2) & \text{for } j = 0, \\ O(h^{j+\frac{3}{2}}) & \text{for } j \geq 1. \end{cases}$$

5. Numerical results. In this section, we would like to report some numerical results for the weak Galerkin finite element method proposed and analyzed in previous sections. Before doing that, let us briefly review some existing results for H^1-H^1 conforming, equal-order finite element discretization of the Ciarlet-Raviart mixed formulation. As discussed in [8, 48], theoretical error estimates for such schemes are indeed sub-optimal due to an effect of $\inf_{\chi_h}\|u - \chi_h\|_2$, where χ_h is taken from the employed H^1 conforming finite element space. For example, when H^1-H^1 conforming quadratic elements are used to approximate both u and w, the error satisfies $\|u - u_h\|_2 + \|w - w_h\| \lesssim \inf_{\chi_h}\|u - \chi_h\|_2 + \inf_{\chi_h}\|w - \chi_h\| \lesssim O(h)$, while intuitively, one may expect $\|w - w_h\|$ to have an $O(h^2)$ convergence. By using the L^∞ argument, Scholz [48] was able to improve the convergence rate of L^2 norm for w by $h^{\frac{1}{2}}$, and it is known that this theoretical result is indeed sharp. For the weak Galerkin approximation, from the discussing in the previous sections, clearly we are facing the same issue.

However, numerous numerical experiments have illustrated that H^1-H^1 conforming, equal-order Ciarlet-Raviart mixed finite element approximation often demonstrates convergence rates better than the theoretical prediction. Indeed, this has been partly explained theoretically in [49], in which the author proved that optimal order of convergence rates can be recovered in certain fixed subdomains of Ω, when equal order H^1 conforming elements are used. We point out that similar phenomena have been observed in the numerical experiments using weak Galerkin discretization. This means that numerical results are often better than theoretical predictions.

Another issue in the implementation of the weak Galerkin finite element method is the treatment of non-homogeneous boundary data

$$u = g_1 \qquad \text{on } \partial\Omega,$$
$$\frac{\partial u}{\partial \mathbf{n}} = g_2 \qquad \text{on } \partial\Omega.$$

Clearly, both boundary conditions are imposed on u, and $u = g_1$ is the essential boundary condition while $\frac{\partial u}{\partial \mathbf{n}} = g_2$ is the natural boundary condition. To impose the natural boundary condition, we shall modify the first equation of (2.4) into

$$(\!(w_h,\, \phi_h)\!) - (\nabla_w u_h,\, \nabla_w \phi_h) = -\langle g_2, \phi_b\rangle_{\partial\Omega}.$$

The essential boundary condition should be enforced by taking the L^2 projection of the corresponding boundary data.

Consider three test problems defined on $\Omega = [0,1] \times [0,1]$ with exact solutions

$$\begin{aligned} &u_1 = x^2(1-x)^2 y^2 (1-y)^2, \\ &u_2 = \sin(2\pi x)\sin(2\pi y) \qquad \text{and} \qquad u_3 = \sin(2\pi x + \frac{\pi}{2})\sin(2\pi y + \frac{\pi}{2}), \end{aligned}$$

respectively. The reason for choosing these three exact solutions is that they have the following type of boundary conditions

$$\begin{aligned} u_1|_{\partial\Omega} &= 0 \qquad & \left.\frac{\partial u_1}{\partial \mathbf{n}}\right|_{\partial\Omega} &= 0, \\ u_2|_{\partial\Omega} &= 0 \qquad & \left.\frac{\partial u_2}{\partial \mathbf{n}}\right|_{\partial\Omega} &\neq 0, \\ u_3|_{\partial\Omega} &\neq 0 \qquad & \left.\frac{\partial u_3}{\partial \mathbf{n}}\right|_{\partial\Omega} &= 0. \end{aligned}$$

This allows us to test the effect of different boundary data on convergence rates. Although the theoretical error estimates are given for $\varepsilon_u = R_h u - u_h$ and $\varepsilon_w = N_h w - w_h$, by using the triangle inequality and the approximation properties of R_h, N_h and Q_h, it is clear that they have at least the same order as $e_u = Q_h u - u_h$ and $e_w = Q_h w - w_h$, provided that the exact solution is smooth enough. Thus for convenience, we only compute different norms for e_u and e_w, instead of for ε_u and ε_w.

The tests are performed using an unstructured triangular initial mesh, with characteristic mesh size 0.1. The initial mesh is then refined by dividing every triangle into four sub-triangles, to generate a sequence of nested meshes with various mesh size h. All discretization schemes are formulated by using the lowest order weak Galerkin element, with $j = 0$. For simplicity of notation, for any $v \in V_h$, denote

$$\|v_b\| = \left(\sum_{K\in\mathcal{T}_h} h\|v_b\|^2_{\partial K}\right)^{1/2}.$$

The results for test problems with exact solutions u_1, u_2 and u_3, are reported in Table 5.1, 5.2 and 5.3, respectively. The results indicate that u always achieves an optimal order of convergence, while the convergence for w varies with different boundary conditions. It should be pointed out that both of them have outperformed the convergence as predicted by theory.

Our final example is a case where the exact solution has a low regularity in the domain $\Omega = [0,1] \times [0,1]$. More precisely, the exact solution is given by

$$u_4 = r^{3/2}\left(\sin\frac{3\theta}{2} - 3\sin\frac{\theta}{2}\right),$$

where (r,θ) are the polar coordinates. It is easy to check that $u \in H^{2.5}$. The errors for weak Galerkin finite element approximations are reported in Table 5.4. Here, u

TABLE 5.1
Numerical results for the test problem with exact solution u_1 and lowest order of WG elements.

h	$\|\nabla_w e_u\|$	$\|e_{u,0}\|$	$\|e_{u,b}\|$	$\|\nabla_w e_w\|$	$\|e_{w,0}\|$	$\|e_{w,b}\|$
0.1	1.33e-03	2.40e-04	4.59e-04	5.66e-02	2.96e-03	6.91e-03
0.05	4.69e-04	6.18e-05	1.17e-04	2.80e-02	9.14e-04	1.99e-03
0.025	2.00e-04	1.55e-05	2.97e-05	1.60e-02	2.64e-04	5.70e-04
0.0125	9.56e-05	3.90e-06	7.44e-06	1.21e-02	8.33e-05	1.89e-04
0.00625	4.72e-05	9.77e-07	1.86e-06	1.13e-02	3.26e-05	7.91e-05
Asym. Order $O(h^k),\ k=$	1.1930	1.9876	1.9877	0.5864	1.6461	1.6298

TABLE 5.2
Numerical results for the test problem with exact solution u_2 and lowest order of WG elements.

h	$\|\nabla_w e_u\|$	$\|e_{u,0}\|$	$\|e_{u,b}\|$	$\|\nabla_w e_w\|$	$\|e_{w,0}\|$	$\|e_{w,b}\|$
0.1	9.58e-01	8.66e-02	1.65e-01	4.39e+01	6.09e-01	2.01e+00
0.05	3.34e-01	2.18e-02	4.14e-02	2.32e+01	2.78e-01	7.19e-01
0.025	1.43e-01	5.47e-03	1.03e-02	1.37e+01	1.15e-01	2.81e-01
0.0125	6.81e-02	1.37e-03	2.59e-03	1.02e+01	5.12e-02	1.26e-01
0.00625	3.36e-02	3.42e-04	6.49e-04	9.33e+00	2.45e-02	6.12e-02
Asym. Order $O(h^k),\ k=$	1.1958	1.9958	1.9975	0.5649	1.1709	1.2587

still achieves an optimal order of convergence, while the convergence rates for w is restricted by the fact that $w \in H^{0.5}$. All the results are in consistency with the theory established in this article.

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Table 5.3
Numerical results for the test problem with exact solution u_3 and lowest order of WG elements.

h	$\|\nabla_w e_u\|$	$\|e_{u,0}\|$	$\|e_{u,b}\|$	$\|\nabla_w e_w\|$	$\|e_{w,0}\|$	$\|e_{w,b}\|$
0.1	8.23e-01	1.18e-01	2.27e-01	5.61e+01	4.25e+00	9.42e+00
0.05	3.07e-01	3.18e-02	6.09e-02	2.43e+01	1.24e+00	2.58e+00
0.025	1.35e-01	8.13e-03	1.55e-02	1.13e+01	3.28e-01	6.61e-01
0.0125	6.49e-02	2.04e-03	3.90e-03	5.58e+00	8.42e-02	1.67e-01
0.00625	3.21e-02	5.11e-04	9.78e-04	2.77e+00	2.14e-02	4.21e-02
Asym. Order $O(h^k),\ k=$	1.1599	1.9679	1.9682	1.0801	1.9157	1.9558

Table 5.4
Numerical results for the test problem with exact solution u_4 and lowest order of WG elements.

h	$\|\nabla_w e_u\|$	$\|e_{u,0}\|$	$\|e_{u,b}\|$	$\|\nabla_w e_w\|$	$\|e_{w,0}\|$	$\|e_{w,b}\|$
0.1	3.73e-02	9.44e-04	2.15e-03	2.88e+01	4.05e-01	1.78e+00
0.05	1.87e-02	2.55e-04	5.73e-04	4.08e+01	2.86e-01	1.26e+00
0.025	9.37e-03	6.60e-05	1.46e-04	5.77e+01	2.02e-01	8.91e-01
0.0125	4.68e-03	1.67e-05	3.69e-05	8.16e+01	1.42e-01	6.30e-01
0.00625	2.34e-03	4.19e-06	9.24e-06	1.15e+02	1.01e-01	4.45e-01
Asym. Order $O(h^k),\ k=$	0.9984	1.9567	1.9690	-0.4998	0.5008	0.5000

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