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CASCADIC MULTIGRID METHODS FOR MORTAR WILSON FINITE ELEMENT METHODS ON PLANAR LINEAR ELASTICITY*

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Abstract *Cascadic multigrid technique for mortar Wilson finite element method of homogeneous boundary value planar linear elasticity is described and analyzed. First the mortar Wilson finite element method for planar linear elasticity will be analyzed, and the error estimate under L^2 and H^1 norm is optimal. Then a cascadic multigrid method for the mortar finite element discrete problem is described. Suitable grid transfer operator and smoother are developed which lead to an optimal cascadic multigrid method. Finally, the computational results are presented.*

Key words *cascadic multigrid method, mortar Wilson finite element*

AMS(2000)subject classifications 65F10, 65N30

1 Introduction

The mortar finite element method is a non-conforming domain decomposition technique [2,3,9]. It handles discrete finite element approximations on independently partitioned sub-domains and is designed to be optimal by using the matching conditions to restrain the jump across sub-domain interfaces. The flexibility of this method is well recognized. Bernardi, Maday and Patera [3] proved the existence and uniqueness of discrete problem of Poisson equation, and also showed that the mortar finite element method is of the same accuration to the usual finite element method. In order to solve large scale problems, some preconditioning technique have been successfully adapted. For example, the "Dirichlet-Neumann" or "Neumann-Neumann" algorithms [2,20,20], substructuring preconditioner [1] and multigrid method [4,15,20,26], in which the preconditioners in [4,20,26] require a use of Lagrange multipliers under a primal hybrid formulation. Recently the cascadic multigrid method [5,6,23,25] for finite element is considered. Cascadic multigrid differs from usual multigrid method in that it requires no coarse grid corrections and it performs more iterations on coarser levels to obtain less iterations on finer levels. In this paper we will consider the cascadic multigrid method for mortar finite element approxi-

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mations of the pure displacement boundary problems in plane linear elasticity associated with a homogeneous isotropic elastic material. Next is the description of some notations and the variational form of our problem. In section 2 the mortar finite element discretion is introduced with the error estimate (the existence and uniqueness of solutions to the discrete problems are proved in appendix). In section 3 the cascadic multigrid for the discrete problem will be analyzed and an algorithm is given and it is proved to be optimal. Finally some results of numerical experiments are given in section 4.

Let Ω be a bounded convex polygon in the $x-y$ plane with a Lipschitz-continuous boundary $\partial\Omega$. We shall only consider the case of homogeneous boundary conditions in the pure displacement boundary problems because the results can be easily extended to the more general cases. For a given integer $m \geq 0$, we introduce the norm and seminorm over the Sobolev space $H^m(\Omega)$

$$\|v\|_{m,\Omega} = \left(\int_\Omega \sum_{|\alpha|\leq m} |\partial^\alpha v|^2 \,\mathrm{d}x\right)^{1/2}, \qquad |v|_{m,\Omega} = \left(\int_\Omega \sum_{|\alpha|=m} |\partial^\alpha v|^2 \,\mathrm{d}x\right)^{1/2}.$$

By interpolation theory and dual theory the m can be extended to all real number. In what follows, we shall be interested in the space

$$\vec{H}_0^1(\Omega) = (H_0^1(\Omega))^2.$$

For any $\vec{v} = (v_1, v_2) \in \vec{H}_0^1(\Omega)$, the expressions $(|v_1|_{m,\Omega}^2 + |v_2|_{m,\Omega}^2)^{1/2}$ and $(\|v_1\|_{m,\Omega}^2 + \|v_2\|_{m,\Omega}^2)^{1/2}$ will be denoted by $|\vec{v}|_{m,\Omega}$ and $\|\vec{v}\|_{m,\Omega}$. We let the stresses σ_{ij} and the strains ϵ_{ij} be

$$\epsilon_{ij}(\vec{v}) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad 1 \leq i,j \leq 2, \tag{1.1}$$

$$\sigma_{ij}(\vec{v}) = \lambda\,(\mathrm{div}\,\vec{v})\delta_{ij} + 2\mu\,\epsilon_{ij}(\vec{v}), \quad 1 \leq i,j \leq 2, \tag{1.2}$$

where the constants $\lambda \geq 0$ and $\mu > 0$ are the coefficients of Lamé of the continuum, and where $\delta_{ij} = 0\,(i \neq j)$, $\delta_{ii} = 1$.

The bilinear form $a(\cdot,\cdot)$ is defined on $\vec{H}_0^1(\Omega) \times \vec{H}_0^1(\Omega)$ by

$$a(\vec{u},\vec{v}) := \int_\Omega (2\mu\,\epsilon(\vec{u}) : \epsilon(\vec{v}) + \lambda\,\mathrm{div}\,\vec{u}\,\mathrm{div}\,\vec{v})\,\mathrm{d}x.$$

The homogeneous boundary value problem can be formulated as follows: to find the displacements $\vec{u} \in \vec{H}_0^1(\Omega)$ such that

$$a(\vec{u},\vec{v}) = (\vec{f},\vec{v}), \qquad \forall \vec{v} \in \vec{H}_0^1(\Omega). \tag{1.3}$$

And if $\vec{f} \in \vec{L}^2(\Omega)$, by [11] the problem (1.3) has a unique solution $\vec{u} \in \vec{H}^2(\Omega)$ and there exists a positive constant C_Ω such that

$$\|\vec{u}\|_{2,\Omega} + \lambda\|\mathrm{div}\vec{u}\|_{1,\Omega} \leq C_\Omega \|\vec{f}\|_{0,\Omega}. \tag{1.4}$$

Definition 1.1 Define the space of rigid motions

$$RM(\Omega) := \{\vec{v} : \vec{v} = \vec{c} + b\begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, \vec{c} \in R^2, b \in R\} \tag{1.5}$$

and the space

$$\widehat{\vec{H}}^m(\Omega) := \{\vec{v} \in \vec{H}^m(\Omega) : \int_\Omega \vec{v}\,dx = \vec{0}, \int_\Omega \operatorname{rot}\vec{v}\,dx = 0\}. \tag{1.6}$$

It is easy to see that $RM(\Omega)$ is the kernel of ϵ,

$$\epsilon(\vec{v}) = 0, \qquad \forall \vec{v} \in RM(\Omega),$$

and the space $\widehat{\vec{H}}^m(\Omega)$ is the subset of $\vec{H}^m(\Omega)$.

2 Mortar Wilson FEM

Assume that Ω is a rectangle. Divide Ω into several non-overlapping rectangles Ω_k $(k = 1 \cdots N)$. We consider here only the geometrically conforming version, which means that $\partial\Omega_i \bigcap \partial\Omega_j$, $(\forall i \neq j)$ is either an empty set, an edge or a vertex. The interface $\Gamma = \cup_{i=1}^N \partial\Omega_i$ is broken into a set of disjoint open straight line segments γ_k, each of which is the common side of $\partial\Omega_i \cap \partial\Omega_j$ for some i and j. Set mortar side and nonmortar side on each $\gamma \in \Gamma$, represent them by γ^M and γ^{NM} and assume that $\forall \gamma \in \Gamma$,

$$\gamma^M \in \Omega_{M(\gamma)}, \qquad \gamma^{NM} \in \Omega_{NM(\gamma)}.$$

We associate with each Ω_k a regular quasi-uniform triangulation $T_h(\Omega_k)$ made of elements that are rectangles. By h_k we denote the maximum diameter of the rectangles. The triangulations generally do not align at the sub-domain interfaces.

We first introduce the Q_1 element, $Q_1(K) = \operatorname{span}\{1, x, y, xy\}$, over the rectangle K, and then the Wilson element, $W(K) = \operatorname{span}\{1, x, y, xy, x^2, y^2\}$, over K. Each $v \in Q_1(K)$ is determined by its values at the vertices of K. Each $v \in W(K)$ is determined by the values at the vertices of K and by the values $\frac{1}{\operatorname{meas}(K)}\int_K \partial_{xx} v$ and $\frac{1}{\operatorname{meas}(K)}\int_K \partial_{yy} v$(see [13,19] for details). Define

$$\vec{X}_h(\Omega_k) = \{\vec{v}_{k,h} : \vec{v}_{k,h}|_K \in \vec{W}(K) \text{ and continuous at the vertices of } K\},$$

$$\vec{X}_h^Q(\Omega_k) = \{\vec{v}_{k,h}|_K \in \vec{Q}_1(K), \text{and continuous at the vertices of } K\},$$

$$\vec{X}_h(\Omega) = \prod_{k=1}^N \vec{X}_h(\Omega_k), \quad \vec{X}_h^Q(\Omega) = \prod_{k=1}^N \vec{X}_h^Q(\Omega_k).$$

Such spaces as $\vec{X}_h(\Omega)$ and $\vec{X}_h^Q(\Omega)$ generally do not align on the interface Γ. We introduce some spaces on interface segments γ. Every γ^{NM} is divided into sub-intervals by the vertices of mesh in the non-mortar domain of γ. Denote the sub-intervals by $\omega_{\gamma,s}$, $s = 1, \cdots, S$, where $\omega_{\gamma,1}$ and $\omega_{\gamma,S}$ are the sub-intervals at the ends of γ. Then we have

$$\begin{aligned}
\vec{W}^h(\gamma) &= \{\vec{\psi} = (\psi_1, \psi_2) : \psi_i\,(i = 1,2) \text{is linear on each } \omega_{\gamma,s} \text{and continuous on } \gamma\},\\
\vec{M}^h(\gamma) &= \{\vec{\psi} = (\psi_1, \psi_2) \in \vec{W}^h(\gamma) : \psi_i\,(i = 1,2) \text{vanishes at the end-points of } \gamma\},\\
\vec{S}^h(\gamma) &= \{\vec{\psi} = (\psi_1, \psi_2) \in \vec{W}^h(\gamma) : \psi_i\,(i = 1,2) \text{is constant on } \omega_{\gamma,1} \text{and } \omega_{\gamma,S}\},
\end{aligned}$$

$\forall \vec{v} \in \vec{X}_h(\Omega)$, let $\vec{v}^Q$ be the bilinear part of $\vec{v}$. Then $\vec{v}^Q$ is continuous in Ω_k, $1 \leq k \leq N$. Define the mortar finite element spaces as the following:

$$\begin{aligned}
\vec{V}_h &= \{\vec{v}_h \in \vec{X}_h(\Omega) : \forall \gamma \subset \Gamma, \forall \vec{\psi} \in \vec{S}^h(\gamma), \int_\gamma (\vec{v}_{i,h}^Q - \vec{v}_{j,h}^Q) \cdot \vec{\psi}\,d\tau = 0\},\\
\vec{V}_h^Q &= \{\vec{v}_h \in \vec{X}_h^Q(\Omega) : \forall \gamma \subset \Gamma, \forall \vec{\psi} \in \vec{S}^h(\gamma), \int_\gamma (\vec{v}_{i,h} - \vec{v}_{j,h}) \cdot \vec{\psi}\,d\tau = 0\}.
\end{aligned}$$

It is easy to see that $\vec{V}_h^Q \subset \vec{V}_h$.

Define the spaces $\vec{X}_{0,h}(\Omega) \subset \vec{X}_h(\Omega)$, $\vec{X}_{0,h}^Q(\Omega) \subset \vec{X}_h^Q(\Omega)$, $\vec{V}_{0,h} \subset \vec{V}_h$, $\vec{V}_{0,h}^Q \subset \vec{V}_h^Q$, the functions in which vanishing at the vertices on $\partial\Omega$. And the spaces $\hat{\vec{X}}_h(\Omega) \subset \vec{X}_h(\Omega)$, $\hat{\vec{X}}_h^Q(\Omega) \subset \vec{X}_h^Q(\Omega)$, $\hat{\vec{V}}_h \subset \vec{V}_h$, $\hat{\vec{V}}_h^Q \subset \vec{V}_h^Q$, the functions in which satisfying the condition that $\int_\Omega \vec{v}\,\mathrm{d}x = \vec{0}$, $\int_\Omega \mathrm{rot}\,\vec{v}\,\mathrm{d}x = 0$. Then we can write the discrete problem of (1.3) as: find $\vec{u}_h \in \vec{V}_{0,h}$ that

$$a_h(\vec{u}_h, \vec{v}_h) = (\vec{f}, \vec{v}_h), \qquad \forall \vec{v}_h \in \vec{V}_{0,h}, \tag{2.1}$$

in which

$$a_h(\vec{u}_h, \vec{v}_h) := \sum_{k=1}^N \sum_{K\in T_h(\Omega_k)} \int_K (2\mu\,\epsilon(\vec{u}_{k,h}) : \epsilon(\vec{v}_{k,h}) + \lambda \,\mathrm{div}\,\vec{u}_{k,h}\,\mathrm{div}\,\vec{v}_{k,h})\,\mathrm{d}x.$$

Define the discrete norm over Ω_k as

$$|\vec{v}_h|_{m,k,h} = (\sum_{K\in T_h(\Omega_k)} |\vec{v}_{k,h}|_{m,K}^2)^{\frac{1}{2}}, \quad |\vec{v}_h|_{m,h} = (\sum_{k=1}^N |\vec{v}_h|_{m,k,h}^2)^{\frac{1}{2}}, \quad m = 1, 2,$$

$$\|\vec{v}_h\|_{m,k,h} = (\sum_{K\in T_h(\Omega_k)} \|\vec{v}_{k,h}\|_{m,K}^2)^{\frac{1}{2}}, \quad \|\vec{v}_h\|_{m,h} = \left(\sum_{k=1}^N \|\vec{v}_h\|_{m,k,h}^2\right)^{\frac{1}{2}}, \quad m = 0, 1, 2$$

$$|||\vec{v}_h|||_h = \left(\sum_{k=1}^N \sum_{K\in T_h(\Omega_k)} \sum_{i,j=1}^2 \|\epsilon_{ij}(\vec{v}_{k,h})\|_{0,K}^2\right)^{\frac{1}{2}}.$$

Lemma 2.1 $|\vec{\cdot}|_{1,h}, |||\vec{\cdot}|||_h$ is the norm over $\vec{V}_{0,h}$, $\|\vec{\cdot}\|_{1,h}$ is equivalent to $|\vec{\cdot}|_{1,h}$ over $\vec{V}_{0,h}$.

According to the mortar condition we define the following projection. $\forall \gamma \in \Gamma$, define the projection $\vec{\Pi}_\gamma : \vec{L}^2(\gamma) \to \vec{M}^h(\gamma)$ as

$$\int_\gamma (\vec{\Pi}_\gamma \vec{v}) \cdot \vec{\psi}\, ds = \int_\gamma \vec{v} \cdot \vec{\psi}\, ds, \qquad \forall\, \vec{v} \in \vec{L}^2(\gamma),\ \vec{\psi} \in \vec{S}^h(\gamma).$$

In [3] the projection is proved proper and stable in $\vec{L}^2(\gamma)$ and $\vec{H}_{00}^{1/2}(\gamma)$, i.e.,

$$\|\vec{\Pi}_\gamma \vec{v}\|_{0,\gamma} \le C\|\vec{v}\|_{0,\gamma}, \qquad \forall\, \vec{v} \in \vec{L}^2(\gamma), \tag{2.2}$$

$$\|\vec{\Pi}_\gamma \vec{v}\|_{H_{00}^{1/2}(\gamma)} \le C\|\vec{v}\|_{H_{00}^{1/2}(\gamma)}, \qquad \forall\, \vec{v} \in \vec{H}_{00}^{1/2}(\gamma). \tag{2.3}$$

Let $\{y_k^j\}$ denote the nodes of $T_h(\Omega_K)$ and the operator $\epsilon_\gamma : \vec{L}^2(\gamma) \to \vec{X}_h^Q(\Omega)$ be defined by

$$\epsilon_\gamma \vec{v}(y_k^j) = \begin{cases} \vec{\Pi}_\gamma \vec{v}(y_k^j), & \text{if } y_k^j \in \gamma \cap \bar{\Omega}_{NM(\gamma)}, \\ \vec{0}, & \text{otherwise.} \end{cases}$$

It is easy to see that if $\vec{v} \in \vec{X}_h^Q(\Omega)$, $\vec{v} + \sum_{\gamma\in\Gamma} \epsilon_\gamma \vec{v} \in \vec{V}_h^Q$, and if $\vec{v} \in \vec{X}_h(\Omega)$, $\vec{v} + \sum_{\gamma\in\Gamma} \epsilon_\gamma \vec{v} \in \vec{V}_h$.

And for $\gamma \in \Gamma$, we introduce the orthogonal projection $\tilde{\Pi}_\gamma : L^2(\gamma) \to M^h(\gamma)$, which have [3]

$$\|v - \tilde{\Pi}_\gamma v\|_{L^2(\gamma)} + h_j^{-1/2}\|v - \tilde{\Pi}_\gamma v\|_{H^{-1/2}(\gamma)} \le ch_j^\sigma \|v\|_{H^\sigma(\gamma)}, \quad \forall v \in H^\sigma(\gamma), \tag{2.4}$$

where $0 \le \sigma \le 1$.

Before analyzing the error estimate of mortar finite element solution, we give two lemmas on interpolation error.

Lemma 2.2 Assume that $\vec{v}_{k,h} \in \vec{X}_h(\Omega_k)$, the bilinear part $\vec{v}_{k,h}^Q$ satisfies that

$$\|\vec{v}_{k,h} - \vec{v}_{k,h}^Q\|_{0,\Omega_k} + h_k|\vec{v}_{k,h} - \vec{v}_{k,h}^Q|_{1,k,h} \le ch_k|\vec{v}_{k,h}|_{1,k,h}. \tag{2.5}$$

And, if $\vec{v}_{k,h}^Q$ equals to 0 at all the nodes which are not on the side γ of sub-domain Ω_k, we have

$$|\vec{v}_{k,h}^Q|_{1,k,h} \le c|\vec{v}_{k,h}^Q|_{H_{00}^{1/2}(\gamma)}. \tag{2.6}$$

The inequality(2.5) is the Lemma 3.2 in [27], and the proof of inequality(2.6) follows [17].

Definition 2.3 Assume that $\vec{v} \in \sum_{k=1}^N \vec{H}^2(\Omega_k) \cap \vec{H}^1(\Omega)$, denote $\pi_{h,k}^*$ to be the Wilson interpolation operator over Ω_k. Define separately the interpolation operators π_h^* from $\vec{H}^1(\Omega)$ onto $\vec{X}_h(\Omega)$, and π_h onto $\vec{V}_h$ as the following:

$$(\pi_h^*\vec{v})|_{\Omega_k} = \pi_{h,k}^*\vec{v}|_{\Omega_k}, \qquad \forall\Omega_k, \tag{2.7}$$

$$\pi_h\vec{v} = \pi_h^*\vec{v} + \sum_{\gamma\in\Gamma} \epsilon_\gamma(\pi_h^*\vec{v})^Q. \tag{2.8}$$

It can be proved that the interpolation operators defined up here satisfy:

Lemma 2.4 Assume that $\vec{v} \in \sum_{k=1}^N \vec{H}^2(\Omega_k) \cap \vec{H}^1(\Omega)$, then $\forall K \in T_h(\Omega_k)$, $(1 \le k \le N)$ we have

$$|\vec{v}_k - (\pi_{k,h}^*\vec{v})^Q|_{m,K} \le ch_k^{2-m}|\vec{v}|_{2,K}, \qquad m = 0, 1, 2, \tag{2.9}$$

$$|\vec{v}_k - \pi_{k,h}^*\vec{v}|_{m,K} \le ch_k^{2-m}|\vec{v}|_{2,K}, \qquad m = 0, 1, 2, \tag{2.10}$$

and

$$|\vec{v} - (\pi_h\vec{v})^Q|_{m,h} \le c\sum_{k=1}^N h_k^{2-m}|\vec{v}|_{2,\Omega_k}, \qquad m = 1, 2, \tag{2.11}$$

$$|\vec{v} - \pi_h\vec{v}|_{m,h} \le c\sum_{k=1}^N h_k^{2-m}|\vec{v}|_{2,\Omega_k}, \qquad m = 1, 2. \tag{2.12}$$

We prove the existence and uniqueness of the solution in the appendix. Next some results of the error estimate will be given. First we formulate a generalization of the second Strang lemma. Using the same way as in [19] and the generalized Green formula one can prove easily that

Lemma 2.5 Assume that $\vec{u}_h \in \vec{V}_{0,h}$ is the solution of discrete problem (2.1), and $\vec{u} \in \vec{H}_0^1(\Omega)$ is the solution of the original problem (1.3), then

$$|\vec{u} - \vec{u}_h|_{1,h} \le c(\inf_{\vec{v}_h\in\vec{V}_{0,h}} |\vec{u} - \vec{v}_h|_{1,h} + \sup_{\vec{w}\in\vec{V}_{0,h}} \frac{|E_h(\vec{u}, \vec{w})|}{|\vec{w}|_{1,h}}), \tag{2.13}$$

$$\|\vec{u}-\vec{u}_h\|_{0,\Omega} \le c \sup_{\vec{\varphi}\in \vec{H}_0^2(\Omega)} \inf_{\vec{\varphi}_h\in \vec{V}_{0,h}} \frac{|E_h(\vec{u},\vec{u}_h,\vec{\varphi},\vec{\varphi}_h)|}{\|\vec{\varphi}\|_{2,\Omega}}, \tag{2.14}$$

in which the constants $c>0$ and independent of h, and

$$E_h(\vec{u},\vec{w}) = -\sum_{k=1}^{N}\sum_{K\in T_h(\Omega_k)}\int_{\partial K}(\sigma(\vec{u})\eta_K)\cdot\vec{w}\,d\tau,$$
$$E_h(\vec{u},\vec{u}_h,\vec{\varphi},\vec{\varphi}_h) = a_h(\vec{u}-\vec{u}_h,\vec{\varphi}-\vec{\varphi}_h) - E_h(\vec{u},\vec{\varphi}_h) + E_h(\vec{\varphi},\vec{u}_h),$$

the η_K being the outer normal on ∂K.

In order to prove our main result, we first state and prove some auxiliary lemmas.

Lemma 2.6 Assume that $\vec{u}\in \vec{H}_0^1(\Omega)\cap\prod_{k=1}^{N}\vec{H}^2(\Omega_k)$,

$$\inf_{\vec{v}_h\in\vec{V}_{0,h}} |\vec{u}-\vec{v}_h|_{1,h} \le c\sum_{k=1}^{N} h_k|\vec{u}_k|_{2,\Omega_k}.$$

The lemma above is the approximation error estimate, which can be derived from lemma 2.4.

Lemma 2.7 Assume that $\vec{u}\in \vec{H}_0^1(\Omega)\cap\prod_{k=1}^{N}\vec{H}^2(\Omega_k)$, $\forall\vec{w}\in\vec{V}_{0,h}$ we have

$$|E_h(\vec{u},\vec{w})| \le c\sum_{k=1}^{N} h_k|\vec{u}_k|_{2,\Omega_k}|\vec{w}|_{1,h}. \tag{2.15}$$

Proof Let

$$\begin{aligned} D_h(\vec{u},\vec{w}) &= -\sum_{k=1}^{N}\sum_{K\in T_h(\Omega_k)}\int_{\partial K}(\sigma(\vec{u})\eta_K)\cdot(\vec{w}-\vec{w}^Q)\,d\tau, \\ M_h(\vec{u},\vec{w}) &= \sum_{\gamma\in\Gamma}\int_\gamma(\sigma(\vec{u})\eta_K)\cdot[\vec{w}^Q]\,d\tau, \qquad ([\cdot]\text{is a jump of function across } \gamma). \end{aligned}$$

It is easy to see that $E_h(\vec{u},\vec{w}) = D_h(\vec{u},\vec{w}) + M_h(\vec{u},\vec{w})$.

The $D_h(\vec{u},\vec{w})$ satisfies inequality (2.15)(by the Bramble-Hilbert lemma [13]), so, we only have to prove that $M_h(\vec{u},\vec{w})$ satisfies (2.15) also. From

$$\begin{aligned} & \int_\gamma(\sigma(\vec{u})\eta_K)\cdot[\vec{w}^Q]\,d\tau \\ =\; & \int_\gamma(\sigma(\vec{u})\eta_K-\vec{\psi})\cdot(\vec{w}_i^Q-\vec{w}_j^Q)\,d\tau \qquad \forall\vec{\psi}\in\vec{M}^h(\gamma) \\ \le\; & \inf_{\vec{\psi}\in\vec{M}^h(\gamma)}\|\sigma(\vec{u})\eta_K-\vec{\psi}\|_{\vec{H}^{-1/2}(\gamma)}\|\vec{w}_i^Q-\vec{w}_j^Q\|_{\vec{H}^{1/2}(\gamma)} \\ \le\; & ch_j\|\sigma(\vec{u})\eta_K\|_{\vec{H}^{1/2}(\gamma)}\|\vec{w}_i^Q-\vec{w}_j^Q\|_{\vec{H}^{1/2}(\gamma)} \\ \le\; & ch_j|\vec{u}|_{2,\Omega_j}\|\vec{w}_i^Q-\vec{w}_j^Q\|_{\vec{H}^{1/2}(\gamma)}, \end{aligned}$$

$$\|\vec{w}_i^Q-\vec{w}_j^Q\|_{\vec{H}^{1/2}(\gamma)} \le c|\vec{w}_j^Q|_{1,j,h}+c|\vec{w}_i^Q|_{1,i,h} \le c|\vec{w}_j|_{1,j,h}+c|\vec{w}_i|_{1,i,h}$$

and summing on all $\gamma \in \Gamma$, we have that $M_h(\vec{u}, \vec{w})$ satisfies (2.15).

Theorem 2.8 Assume that $\vec{u}_h \in \vec{V}_{0,h}$ is the solution of discrete problem (2.1), and the solution of the original problem (1.3) is $\vec{u} \in \vec{H}_0^1(\Omega) \cap \prod_{k=1}^{N} \vec{H}^2(\Omega_k)$, then

$$|\vec{u} - \vec{u}_h|_{1,h} \le c \sum_{k=1}^{N} h_k |\vec{u}_k|_{2,\Omega_k}, \tag{2.16}$$

$$\|\vec{u} - \vec{u}_h\|_{0,\Omega} \le ch^2 |\vec{u}|_{2,\Omega}. \tag{2.17}$$

Proof By Lemma 2.5, Lemma 2.6, and Lemma 2.7 we can get the (2,16). Now we prove (2.17). According to Lemma 2.5

$$\begin{aligned} E_h(\vec{u}, \vec{u}_h, \vec{\varphi}, \vec{\varphi}_h) &= a_h(\vec{u} - \vec{u}_h, \vec{\varphi} - \vec{\varphi}_h) - E_h(\vec{u}, \vec{\varphi}_h) + E_h(\vec{\varphi}, \vec{u}_h) \\ &= J_1 + J_2 + J_3. \end{aligned}$$

Because $\vec{\varphi_h}$ is arbitrary, assume that $\vec{\varphi}_h = \pi_h \vec{\varphi}$. For J_1 there is

$$\begin{aligned} a_h(\vec{u} - \vec{u}_h, \vec{\varphi} - \pi_h \vec{\varphi}) &\le c \sum_{k=1}^{N} |\vec{u}_k - \vec{u}_{k,h}|_{1,k,h} |\vec{\varphi}_k - \pi_h \vec{\varphi}_k|_{1,k,h} \\ &\le c \sum_{k=1}^{N} h_k |\vec{u}_k|_{2,\Omega_k} \sum_{k=1}^{N} h_k |\vec{\varphi}_k|_{2,\Omega_k} \\ &\le ch \sum_{k=1}^{N} h_k |\vec{u}_k|_{2,\Omega_k} |\vec{\varphi}|_{2,\Omega}. \end{aligned}$$

For J_2 and J_3, similar to the proof of Lemma 2.7, we divide $E_h(\cdot,\cdot)$ into $D_h(\cdot,\cdot)$ and $M_h(\cdot,\cdot)$. First we give some useful inequalities:

$$\begin{aligned} |\vec{u}_h - \pi_h \vec{u}|_{1,h} &\le |\vec{u} - \vec{u}_h|_{1,h} + |\vec{u} - \pi_h \vec{u}|_{1,h} \le ch|\vec{u}|_{2,\Omega}, \\ |\vec{\varphi}_h|_{2,h} &= |\pi_h \vec{\varphi}|_{2,h} \le |\vec{\varphi} - \pi_h \vec{\varphi}|_{2,h} + |\vec{\varphi}|_{2,\Omega} \le c|\vec{\varphi}|_{2,\Omega}, \\ |\vec{u}_h|_{2.h} &\le |\vec{u}_h - \pi_h \vec{u}|_{2,h} + |\pi_h \vec{u}|_{2,h} \\ &\le ch^{-1} |\vec{u}_h - \pi_h \vec{u}|_{1,h} + (|\vec{u} - \pi_h \vec{u}|_{2,h} + |\vec{u}|_{2,\Omega}) \\ &\le c|\vec{u}|_{2,\Omega}, \\ |\vec{u} - \vec{u}_h^Q|_{1,h} &\le |\vec{u} - \vec{u}_h|_{1,h} + |\vec{u}_h - \vec{u}_h^Q|_{1,h} \le ch|\vec{u}|_{2,\Omega} + ch|\vec{u}_h|_{2,h} \le ch|\vec{u}|_{2,\Omega}, \\ |\vec{u}_h^Q - (\pi_h \vec{u})^Q|_{1,h} &\le |\vec{u} - \vec{u}_h^Q|_{1,h} + |\vec{u} - (\pi_h \vec{u})^Q|_{1,h} \le ch|\vec{u}|_{2,\Omega}. \end{aligned}$$

Then by Lesaint [19] and the above inequalities, because $\vec{\varphi}$ and $\vec{\varphi}_h$ are the solution of a original problem and a discrete problem respectively, we have

$$\begin{aligned} |D_h(\vec{u}, \vec{\varphi}_h)| &\le c \sum_{k=1}^{N} h_k^2 |\vec{u}_k|_{2,\Omega_k} |\vec{\varphi}_h|_{2,h} \le c \sum_{k=1}^{N} h_k^2 |\vec{u}_k|_{2,\Omega_k} |\vec{\varphi}|_{2,\Omega}, \\ |D_h(\vec{\varphi}, \vec{u}_h)| &\le c \sum_{k=1}^{N} h_k^2 |\vec{u}_{k,h}|_{2,k,h} |\vec{\varphi}|_{2,\Omega} \le ch^2 |\vec{u}|_{2,\Omega} |\vec{\varphi}|_{2,\Omega}. \end{aligned}$$

Now we will prove it for $M_h(\cdot,\cdot)$. Just the same to the prove of Lemma 2.7, in order to prove the final results, we have to prove that

$$\sum_{\gamma\in\Gamma}\|\vec{\varphi}^Q_{h,i}-\vec{\varphi}^Q_{h,j}\|_{H^{1/2}(\gamma)}\le ch|\vec{\varphi}|_{2,\Omega}, \tag{2.18}$$

$$\sum_{\gamma\in\Gamma}\|\vec{u}^Q_{h,i}-\vec{u}^Q_{h,j}\|_{H^{1/2}(\gamma)}\le ch|\vec{u}|_{2,\Omega}. \tag{2.19}$$

For (2.18) there is

$$\begin{aligned}\|\vec{\varphi}^Q_{h,i}-\vec{\varphi}^Q_{h,j}\|_{H^{1/2}(\gamma)} &\le \|\vec{\varphi}_i-\vec{\varphi}^Q_{h,i}\|_{H^{1/2}(\gamma)}+\|\vec{\varphi}_j-(\pi^*_{h,j}\vec{\varphi}_j)^Q\|_{H^{1/2}(\gamma)}\\ &\quad+\|\epsilon_\gamma(\pi^*_h\vec{\varphi})^Q\|_{H^{1/2}(\gamma)}\\ &\le c(h_i|\vec{\varphi}|_{2,\Omega_i}+h_j|\vec{\varphi}|_{2,\Omega_j})+c\|(\pi^*_{h,i}\vec{\varphi})^Q-(\pi^*_{h,j}\vec{\varphi})^Q\|_{H^{1/2}_{00}(\gamma)}\\ &\le c(h_i|\vec{\varphi}|_{2,\Omega_i}+h_j|\vec{\varphi}|_{2,\Omega_j})\\ &\quad+c(\|\vec{\varphi}_i-(\pi^*_{h,i}\vec{\varphi})^Q\|_{H^{1/2}_{00}(\gamma)}+\|\vec{\varphi}_j-(\pi^*_{h,j}\vec{\varphi})^Q\|_{H^{1/2}_{00}(\gamma)}\\ &\le c(h_i|\vec{\varphi}|_{2,\Omega_i}+h_j|\vec{\varphi}|_{2,\Omega_j}).\end{aligned}$$

Summing up on all $\gamma\in\Gamma$ then we prove the (2.18).

For (2.19) there is

$$\begin{aligned}\sum_{\gamma\in\Gamma}\|\vec{u}^Q_{h,i}-\vec{u}^Q_{h,j}\|_{H^{1/2}(\gamma)} &\le \sum_{\gamma\in\Gamma}\{\|\vec{u}^Q_{h,i}-(\pi_h\vec{u})^Q_i\|_{H^{1/2}(\gamma)}+\|\vec{u}^Q_{h,j}-(\pi_h\vec{u})^Q_j\|_{H^{1/2}(\gamma)}\\ &\quad+\|(\pi_h\vec{u})^Q_i-(\pi_h\vec{u})^Q_j\|_{H^{1/2}(\gamma)}\}\\ &\le c|\vec{u}^Q_h-(\pi_h\vec{u})^Q|_{1,h}+\sum_{\gamma\in\Gamma}\|(\pi_h\vec{u})^Q_i-(\pi_h\vec{u})^Q_j\|_{H^{1/2}(\gamma)}\\ &\quad\text{(using the inequalities given above and (2.18))}\\ &\le ch|\vec{u}|_{2,\Omega}.\end{aligned}$$

Estimate (2.17) is a consequence of all inequalities above and the theorem 2.5.

3 Cascadic Multigrid Method

In this section we will use cascadic multigrid method to the mortar Wilson finite element problem . Assume that $\vec{f}\in\vec{L}^2(\Omega)$, and $\vec{u}$ satisfies (1.4).

Denote $T_{k,l}$ to be the nested rectangular triangulation of the lth level over Ω_k, we just use $h_l=h_0\cdot 2^{-l}$ for simplicity. It is easy to see that spaces $\vec{X}_{0,h_l}(\Omega)$ are nested, but the Mortar Wilson spaces $\vec{V}_l=\vec{V}_{0,h_l}(\Omega)$ are not nested. Then we can write the discrete form of the equation on each level as

$$a_l(\vec{u}_l,\vec{v}_l)=(\vec{f},\vec{v}_l)_l\quad\forall\vec{v}_l\in\vec{V}_l, \tag{3.1}$$

in which $a_l(\cdot,\cdot)=a_{h_l}(\cdot,\cdot)$. We introduce the energy norm

$$|||\vec{v}|||_l=(a_l(\vec{v},\vec{v}))^{\frac{1}{2}},\qquad\forall\vec{v}\in\vec{V}_l.$$

And for constants c and C independent of h_l, the energy norm satisfies

$$c|||\vec{v}|||_{h_l}\le|||\vec{v}|||_l\le C|||\vec{v}|||_{h_l},\qquad\forall\vec{v}\in\vec{V}_l.$$

So we will use the energy norm in this section for all the characters of $|||\cdot|||_{h_l}$ are satisfied by the energy norm.

There are three assumptions:

(H1) There exists an intergrid transfer operator $I_l : \vec{V}_{l-1} \to \vec{V}_l$

(1) $\|\vec{v} - I_l\vec{v}\|_{0,\Omega} \le ch_l|\vec{v}|_{1,h_{l-1}}, \quad \forall \vec{v} \in \vec{V}_{l-1}$,

(2) $\|\vec{u}_l - I_l\vec{u}_{l-1}\|_{0,\Omega} \le ch_l^2\|\vec{f}\|_{0,\Omega}, \quad \vec{u}_l$ is the FEM solution on the lth level.

(H2) Chose the iterative operator on the lth level $C_{l,m_l} : \vec{V}_l \to \vec{V}_l$, There exists a linear operator $T_l : \vec{V}_l \to \vec{V}_l$ such that $|\vec{u}_l - C_{l,m_l}\vec{u}_l^0| \le |T_l^{m_l}(\vec{u}_l - \vec{u}_l^0)|$, in which m_l is the number of iteration steps on the level l, and

(1) $|||T_l^{m_l}\vec{v}|||_l \le c\frac{h_l^{-1}}{m_l^\nu}\|\vec{v}\|_{0,\Omega}, \quad \forall \vec{v} \in \vec{V}_l$,

(2) $|||T_l^{m_l}\vec{v}|||_l \le |||\vec{v}|||_l, \quad \forall \vec{v} \in \vec{V}_l$,

where ν is a positive number depending on the given iteration.

(H3) There exists a projection $P_l : \vec{V}_{l-1} + \vec{V}_l \to \vec{V}_l$ such that $a_l(P_l\vec{u}, \vec{v}) = a_l(\vec{u}, \vec{v})$, $\forall \vec{v} \in \vec{V}_l$, and

(1) $\|\vec{v} - P_l\vec{v}\|_{0,\Omega} \le ch_l|\vec{v}|_{1,h_{l-1}}, \quad \forall \vec{v} \in \vec{V}_{l-1}$.

Then the cascadic multigrid method can be written as follows:

(1) Set $\vec{u}_0^0 = \vec{u}_0^* = \vec{u}_0$ and let

$$\vec{u}_l^0 = I_l\vec{u}_{l-1}^*$$

(2) for $l = 1, \cdots, L$:

$$\vec{u}_l^{m_l} = C_{l,m_l}\vec{u}_l^0$$

(3) Set $\vec{u}_l^* = \vec{u}_l^{m_l}$

Following [5], we call a cascadic multigrid optimal in the energy norm on level L if we obtain that

$$\begin{cases} |||\vec{u}_L - \vec{u}_L^*|||_L \approx |\vec{u} - \vec{u}_L|_{1,h_L}, \\ \text{amount of work} = O(n_L), \qquad n_L = \dim\vec{V}_L. \end{cases} \tag{3.2}$$

Let $m_l (0 \le l \le L)$ be the smallest integer satisfying

$$m_l \ge \beta^{L-1}m_L \tag{3.3}$$

for some fixed $\beta \ge 1$, where m_L is the number of iterations on the finest level L. Shi and Xu have proven in [25] for elliptic problems that

Theorem 3.1 Under the assumptions (H1),(H2) and (H3), if m_l is given by (3.3), then the accuracy of the cascadic multigrid is

$$|||\vec{u}_L - \vec{u}_L^*|||_L \le \begin{cases} C\dfrac{1}{1-2\beta^{-\nu}}\dfrac{h_L}{m_l^\nu}\|\vec{f}\|_{0,\Omega}, & \beta > 2^{\frac{1}{\nu}}, \\ CL\dfrac{h_l}{m_l^\nu}\|\vec{f}\|_{0,\Omega}, & \beta = 2^{\frac{1}{\nu}}, \end{cases} \tag{3.4}$$

and the computation cost of the cascadic multigrid is proportional to

$$\sum_{l=0}^{L} m_l n_l \leq \begin{cases} C\dfrac{1}{1-2^{-d}\beta} m_L n_L, & \beta < 2^d, \\ C L m_L n_L, & \beta = 2^d, \end{cases} \tag{3.5}$$

where C is independent of h_L and L, and d is the dimension of the domain Ω.

According the theorem above, we will give a practical method which is optimal for the mortar Wilson finite element method for planar linear elasticity, it is to say that we will find an intergrid transfer operator, an iteration operator and a projection which satisfy (H1),(H2) and (H3) respectively.

Denote $\epsilon_{\gamma,l}$ on level l to be the ϵ_γ defined on section 2, then we have

$$\|\epsilon_{\gamma,l}\vec{v}\|_{0,\gamma} \leq c h_l |\vec{v}_{\gamma^M} - \vec{v}_{\gamma^{NM}}|_{1,\gamma}, \qquad \forall \vec{v} \in \vec{V}_l^Q. \tag{3.6}$$

The above inequality can be found in [15](inequality (5.4)). Then we define the intergrid transfer operation $I_l : \vec{V}_{l-1} \to \vec{V}_l$ as

$$I_l \vec{v}_{l-1}^Q = \vec{v}_{l-1}^Q + \sum_{\gamma} \epsilon_{\gamma,l}(\vec{v}_{l-1}^Q), \qquad \forall \vec{v}_{l-1}^Q \in \vec{V}_{l-1}^Q.$$

Lemma 3.2 For mortar Wilson finite element method, I_l satisfies assumption (H1).

Proof First prove the inequality (1) of assumption (H1). $\forall \vec{v} \in \vec{V}_{l-1}$, according to Lemma 2.2,

$$\begin{aligned} \|\vec{v} - I_l\vec{v}\|_{0,\Omega} &\leq \|\vec{v} - \vec{v}^Q\|_{0,\Omega} + \|\vec{v}^Q - I_l\vec{v}\|_{0,\Omega} \\ &\leq c h_l |\vec{v}|_{1,h_{l-1}} + \|\sum_{\gamma\in\Gamma} \epsilon_{\gamma,l}(\vec{v}^Q)\|_{0,\Omega}. \end{aligned}$$

Let y^i be the nodes of the level l on γ^{NM}, then

$$\|\epsilon_{\gamma,l}(\vec{v}^Q)\|_{0,\Omega}^2 \approx \sum_{y^i} h_l^2 (\epsilon_{\gamma,l}(\vec{v}^Q))(y^i)^2 \approx h_l \|\epsilon_{\gamma,l}(\vec{v}^Q)\|_{0,\gamma}^2.$$

According to (3.6),

$$\begin{aligned} \|\epsilon_{\gamma,l}(\vec{v}^Q)\|_{0,\gamma} &\leq c h_l |\vec{v}_i^Q - \vec{v}_j^Q|_{1,\gamma} \leq c h_l^{1/2} (\|\vec{v}_i^Q\|_{\frac{1}{2},\gamma} + \|\vec{v}_j^Q\|_{\frac{1}{2},\gamma}) \\ &\leq c h_l^{1/2} (|\vec{v}_i^Q|_{1,i,h_l} + |\vec{v}_j^Q|_{1,j,h_l}) \leq c h_l^{1/2} (|\vec{v}_i|_{1,i,h_{l-1}} + |\vec{v}_j|_{1,j,h_{l-1}}). \end{aligned}$$

So the inequality (1) of assumption (H1) has been proved. Next we will prove the inequality (2). Assume that $\vec{u}$ is the solution of the original problem (1.3), then

$$\|\vec{u}_l - I_l\vec{u}_{l-1}\|_{0,\Omega} \leq \|\vec{u} - \vec{u}_l\|_{0,\Omega} + \|\vec{u} - \vec{u}_{l-1}^Q\|_{0,\Omega} + \|\vec{u}_{l-1}^Q - I_l\vec{u}_{l-1}\|_{0,\Omega}.$$

Because

$$\begin{aligned} \|\vec{u} - \vec{u}_{l-1}^Q\|_{0,\Omega} &\leq \|\vec{u} - \vec{u}_{l-1}\|_{0,\Omega} + \|\vec{u}_{l-1} - \vec{u}_{l-1}^Q\|_{0,\Omega} \\ &\leq c_1 h_l^2 \|\vec{f}\|_{0,\Omega} + c_2 h_l^2 |\vec{u}|_{2,\Omega} \leq c h_l^2 \|\vec{f}\|_{0,\Omega}, \end{aligned}$$

we have

$$\|\vec{u}_l - I_l\vec{u}_{l-1}\|_{0,\Omega} \le ch_l^2\|\vec{f}\|_{0,\Omega} + \|\sum_{\gamma\in\Gamma}\epsilon_{\gamma,l}(\vec{u}_{l-1}^Q)\|_{0,\Omega}.$$

Similar to the proof of inequality (1), we obtain

$$\begin{aligned}\|\epsilon_{\gamma,l}(\vec{u}_{l-1}^Q)\|_{0,\Omega} &\le ch_l^{3/2}|\vec{u}_{l-1,i}^Q - \vec{u}_{l-1,j}^Q|_{1,\gamma}\\ &\le ch_l(|\vec{u} - \vec{u}_{l-1,i}^Q|_{\frac12,\gamma} + |\vec{u} - \vec{u}_{l-1,j}^Q|_{\frac12,\gamma})\\ &\le ch_l(|\vec{u} - \vec{u}_{l-1,i}^Q|_{1,i,h_{l-1}} + |\vec{u} - \vec{u}_{l-1,j}^Q|_{1,j,h_{l-1}}),\end{aligned}$$

$$\|\sum_{\gamma\in\Gamma}\epsilon_{\gamma,l}(\vec{u}_{l-1}^Q)\|_{0,\Omega} \le ch_l|\vec{u} - \vec{u}_{l-1}^Q|_{1,h_{l-1}} \le ch_l^2\|\vec{f}\|_{0,\Omega}.$$

Next we will give out a proper iteration operator which satisfies assumption (H2). First we write out the equivalent linear algebraic equations of the discrete problem on level l. Denote the basis function of mortar Wilson finite element method by ϕ_i^l, $1 \le i \le M_l$, then $\forall\vec{v} \in \vec{V}_l$ we have

$$\vec{v} = \sum_{i=1}^{M_l}\vec{v}_i\phi_i^l = \sum_{i=1}^{M_l}\begin{bmatrix} v_{i,1}\\ v_{i,2}\end{bmatrix}\phi_i^l.$$

Let A_l be the stiff matrix on level l and λ_l^* be the maximum eigenvalue of A_l. $\forall\vec{v} \in \vec{V}_l$, $\forall\vec{f} \in \vec{L}^2(\Omega)$, and $1 \le i \le M_l$, let

$$V_{l,i} = \begin{bmatrix} v_{i,1}\\ v_{i,2}\end{bmatrix},\qquad F_{l,i} = \begin{bmatrix} f_{i,1}\\ f_{i,2}\end{bmatrix} = \begin{bmatrix} (f_1,\phi_i^l)\\ (f_2,\phi_i^l)\end{bmatrix},$$

$$V_l = \begin{bmatrix} V_{l,1}\\ V_{l,2}\\ \vdots\\ V_{l,M_l}\end{bmatrix},\qquad F_l = \begin{bmatrix} F_{l,1}\\ F_{l,2}\\ \vdots\\ F_{l,M_l}\end{bmatrix}.$$

Let

$$\begin{aligned}(\vec{v},\vec{w})_E &= (V_l,W_l)_E = \sum_{i-1}^{M_l} V_{l,i}^T W_{l,i},\\ \|\vec{v}\|_E &= \|V_l\|_E = (\sum_{i=1}^{M_l} V_{l,i}^T V_{l,i})^{\frac12},\\ |||V_l|||_E &= (A_lV_l,V_l)_E^{\frac12} = (a_l(\vec{v},\vec{v}))^{\frac12} = |||\vec{v}|||_l,\end{aligned}$$

$\forall\vec{v},\vec{w} \in \vec{V}_l$.

With the notations above, the discrete problem on level l (3.1) can be rewritten as

$$A_lV_l = F_l. \tag{3.7}$$

And the cascadic algorithm is presented as follows:

Cascadic Multigrid Algorithm

1. $V_0^* = A_0^{-1} F_0$;

2. **for** $l = 1, \cdots, L$ **do**

   begin
   $V_l = I_l V_{l-1}^*$;
   put $V_l^* = S_l(A_l, V_l, F_l)$;
   end.

Here I_l is the intergrid transfer operator, $S_l(A_l, V_l, F_l)$ is the iteration operator on level l and the number of iteration steps are m_l. Here the CG(conjugate-gradient) method will be used, and be proved to satisfy the assumption (H2). The algorithm of CG method for planar linear elasticity can be found in [16].

We divide the basis functions ϕ_k on level l into three groups for convenience:

(1) those ϕ_k that is equal to 1 at the node inside one of the Ω_i or the two ends of $\gamma^{NM} \subset \Omega_i$, to 0 at all other nodes in Ω_i and equals to 0 outside Ω_i, and is bilinear on each elementary rectangle;

(2) those ϕ_k which have one of the second derivative at the center some $K \in T_{h_l}(\Omega_i)$ to be h^{-2} and the other 0, equaling to 0 at all nodes, and is of 2-order on each elementary rectangle;

(3) those ϕ_k which equal to 1 at the nodes on $\gamma^M \in \Gamma$(including the two ends), being similar to the ϕ_k in the first group in $\Omega_{M(\gamma)}$, equaling to $\pi_\gamma \phi_k$ on γ_{NM}, equaling to 0 at all other nodes, and are bilinear on each elementary rectangle.

Lemma 3.3 CG method satisfies assumption (H2), and $\nu = 1$.

Proof $\forall \vec{v} \in \vec{V}_l$, because CG method minimizes the error norm $|||\cdot|||_E$ [23], for all linear operator T_l,

$$|||T_l^{m_l}\vec{v}|||_l \geq |||S_l^{CG}(A_l, V_l, F_l)|||_E.$$

According to the conclusions of Shaidurov [23], for CG method there is

$$|||S_l^{CG}(A_l, V_l, F_l)|||_E \leq \frac{\sqrt{\lambda^*}}{2m_l + 1} ||V_l||_E = \frac{\sqrt{\lambda^*}}{2m_l + 1} ||\vec{v}||_E, \tag{3.8}$$

$$|||S_l^{CG}(A_l, V_l, F_l)|||_E \leq |||V_l|||_E = |||\vec{v}|||_l. \tag{3.9}$$

So, in order to prove our result, we only have to prove that

$$\text{(a) } \lambda^* \leq c, \quad c \text{ is independent of } h_l, \qquad \text{(b) } ||\vec{v}||_E \leq c h_l^{-1} ||\vec{v}||_{0,\Omega}.$$

We first prove (a). Split $\vec{v} = \vec{v}_0 + \vec{v}_\Gamma$, in which $\vec{v}_0$ is the linear combination of the basis functions in group (1)(2)(denote the set of these basis functions by Φ_0), and $\vec{v}_\Gamma$ is the linear combination of the basis functions in group (3)(denote the set of these basis functions as Φ_Γ), then

$$(A_l V_l, V_l)_E = a_l(\vec{v}, \vec{v}) \leq 2(a_l(\vec{v}_0, \vec{v}_0) + a_l(\vec{v}_\Gamma, \vec{v}_\Gamma)),$$

in which

$$\begin{aligned} a_l(\vec{v}_0,\vec{v}_0) &= \sum_{\phi_i,\phi_j\in\Phi_0} a_l(V_{l,i}\phi_i, V_{l,j}\phi_j) \\ &\le \sum_{\phi_i,\phi_j\in\Phi_0} (a_l(V_{l,i}\phi_i, V_{l,i}\phi_i))^{1/2}(a_l(V_{l,j}\phi_j, V_{l,j}\phi_j))^{1/2} \\ &\le c\sum_{\phi_k\in\Phi_0} a_l(V_{l,k}\phi_k, V_{l,k}\phi_k) \le c\sum_{\phi_k\in\Phi_0} |V_{l,k}\phi_k|^2_{1,h_l}. \end{aligned}$$

According to lemma (4.4) in [15], there is

$$a_l(\vec{v}_\Gamma,\vec{v}_\Gamma) \le c\sum_{\phi_k\in\Phi_\Gamma} |V_{l,k}\phi_k|^2_{1,h_l}.$$

By scaling argument, it is easy to see that for the $\phi_k\in\Phi_0$, $|V_{l,k}\phi_k|^2_{1,h_l}\le c(V_{l,k}^T\cdot V_{l,k})$. For the $\phi_k\in\Phi_\Gamma$, assume that the side on which it does not equal to 0 is Γ and that $\gamma_M\subset\Omega_i$, $\gamma_{NM}\subset\Omega_j$, then

$$|V_{l,k}\phi_k|^2_{1,i,h_l}\le c(V_{l,k}^T\cdot V_{l,k}),$$

and

$$\begin{aligned} |\epsilon_{\gamma,l}(V_{l,k}\phi_k)|^2_{1,j,h_l} &\le ch_l^{-2}|\epsilon_{\gamma,l}(V_{l,k}\phi_k)|^2_{0,\Omega_j}\le ch_l^{-1}|\epsilon_{\gamma,l}(V_{l,k}\phi_k)|^2_{0,\gamma} \\ &\le ch_l^{-1}|V_{l,k}\phi_k|^2_{0,\gamma}\le c(V_{l,k}^T\cdot V_{l,k}). \end{aligned}$$

Summing all these up we have

$$(A_lV_l,V_l)_E\le c(V_l,V_l)_E. \tag{3.10}$$

Because V_l is arbitrary, $\lambda^*\le c$.

Then we prove (b). $\forall K\in T_{h_l}(\Omega_k)$, let $C_1,C_2,C_3,C_4\in R^2$ be the values of the four vertices, and $G_K''(x),G_K''(y)\in R^2$ be the second derivatives at the center of K. Using scaling argument, let $\hat{\vec{v}}$ be the transformations of $\vec{v}$ on the reference square $\hat{K}=[-1,1]\times[-1,1]$. The values on the vertices does not change but

$$G_{\hat{K}}''(x)=h_l^2G_k''(x),\qquad G_{\hat{K}}''(y)=h_l^2G_k''(y).$$

Let ξ,η be the coordinates on the reference square. Because

$$\|\hat{\vec{v}}\|_{0,\hat{K}}=0\Longrightarrow C_i=\vec{0},\ (i=1,2,3,4)\text{ and }G_{\hat{K}}''(x)=G_{\hat{K}}''(y)=\vec{0},$$

we have

$$\begin{aligned} h_l^{-2}\|\vec{v}\|^2_{0,K} &\ge c\|\hat{\vec{v}}\|^2_{0,\hat{K}} \\ &\ge c(C_1^TC_1+C_2^TC_2+C_3^TC_3+C_4^TC_4 \\ &\quad +(h_l^2G_K''(x))^T(h_l^2G_K''(x))+(h_l^2G_K''(y))^T(h_l^2G_K''(y))). \end{aligned}$$

So, $\|\vec{v}\|_E\le ch_l^{-1}\|\vec{v}\|_{0,\Omega}$.

Finally we prove assumption (H3). It is easy to see that the projection P_l has the following characters:

$$\begin{cases} |||P_l\vec{v}|||_l \le |||\vec{v}|||_{l-1}, & \forall\vec{v}\in\vec{V}_{l-1}, \\ |P_l\vec{v}|_{1,h_l} \le c|\vec{v}|_{1,h_{l-1}}, & \forall\vec{v}\in\vec{V}_{l-1}. \end{cases} \tag{3.11}$$

Lemma 3.4 P_l satisfies (H3).

Proof Assume that $\vec{v} \in \vec{V}_{l-1}$, then there exists $\vec{\xi} \in \vec{H}_0^2(\Omega)$ such that

$$\begin{cases} -\mathrm{div}\sigma(\vec{\xi}) = \vec{v} - P_l\vec{v}, & \text{in } \Omega, \\ \vec{\xi} = 0, & \text{on } \partial\Omega. \end{cases}$$

By the Green formula,

$$\begin{aligned} \|\vec{v} - P_l\vec{v}\|_{0,\Omega}^2 &= (\vec{v} - P_l\vec{v}, \vec{v} - P_l\vec{v}) \\ &= a_l(\vec{\xi}, \vec{v} - P_l\vec{v}) - \sum_{k=1}^{N} \sum_{K \in T_{h_{l-1}}(\Omega_k)} \int_{\partial K} (\sigma(\vec{\xi})\eta_K) \cdot \vec{v}\, ds \\ &\quad + \sum_{k=1}^{N} \sum_{K \in T_{h_l}(\Omega_k)} \int_{\partial K} (\sigma(\vec{\xi})\eta_K) \cdot (P_l\vec{v})\, ds \\ &= a_l(\vec{\xi}, \vec{v} - P_l\vec{v}) - E_{h_{l-1}}(\vec{\xi}, \vec{v}) + E_{h_l}(\vec{\xi}, P_l\vec{v}), \end{aligned}$$

$$\begin{aligned} |a_l(\vec{\xi}, \vec{v} - P_l\vec{v})| &= |a_l(\vec{\xi} - \pi_{h_l}\vec{\xi}, \vec{v} - P_l\vec{v})| \\ &\le c|||\vec{\xi} - \pi_{h_l}\vec{\xi}|||_l |||\vec{v} - P_l\vec{v}|||_l \\ &\le ch_l\|\vec{\xi}\|_{2,\Omega}|||\vec{v}|||_{l-1} \\ &\le ch_l\|\vec{\xi}\|_{2,\Omega}|\vec{v}|_{1,h_{l-1}} \\ &\le ch_l\|\vec{v} - P_l\vec{v}\|_{0,\Omega}|\vec{v}|_{1,h_{l-1}}. \end{aligned}$$

Similar to the proof of the error estimate in section 2, we have

$$\begin{aligned} |E_{h_{l-1}}(\vec{\xi}, \vec{v})| &\le ch_l\|\vec{v} - P_l\vec{v}\|_{0,\Omega}|\vec{v}|_{1,h_{l-1}}, \\ |E_{h_l}(\vec{\xi}, P_l\vec{v})| &\le ch_l\|\vec{v} - P_l\vec{v}\|_{0,\Omega}|P_l\vec{v}|_{1,h_l} \\ &\le ch_l\|\vec{v} - P_l\vec{v}\|_{0,\Omega}|\vec{v}|_{1,h_{l-1}}. \end{aligned}$$

So, $\|\vec{v} - P_l\vec{v}\|_{0,\Omega} \le ch_l|\vec{v}|_{1,h_{l-1}}$.

Now we can see that under the intergrid transfer operator I_l and CG method, the cascadic multigrid method is optimal for mortar Wilson finite element method of planar linear elasticity.

4 Numerical Experiments

In this section we describe some numerical results. The program is designed for the domain $\Omega = [-1,1] \times [-1,1]$, which is decomposed into two sub-domains: $\Omega_1 = [-1,1] \times [0,1]$ and $\Omega_2 = [-1,1] \times [-1,0]$. Assume that $\gamma^M \subset \Omega_1$ and $\gamma^{NM} \subset \Omega_2$. And $\vec{f}$ is chosen so that the solution $\vec{u} = (u_1, u_2)$ is:

$$u_1 = 1 \times 10^{-4} \times (1 - x^2)(1 - y^2)$$
$$u_2 = 1 \times 10^{-4} \times (1 - x^2)(1 - y^2)$$

We use the Cascadic algorithm given in Section 3 and the CG method to be the smoother. Set $\beta = 2$, we can get the following results. In the table down here, L is the number of total levels, $m1$ is the iteration number on the coarsest grid and $m5$ is the iteration number on the finest grid. E is the Young modulus and ν is the Possion coefficient, and the Lamé constants can be derived like this

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}.$$

Let h_i denote the step of triangulation over sub-domain Ω_i, d.o.f be the "degree of freedom" of all, $\vec{u}_L$ is the cascadic solution on level L.

Table 1 $L = 5, E = 7 \times 10^{10}\text{Pa}, \nu = \frac{1}{3}, m1 = 128, m5 = 8$

h_1^{-1}	h_2^{-1}	d.o.f	$\|\|\|\vec{u}_L - \vec{u}\|\|\|_L$	$\|\vec{u}_L - \vec{u}\|_{1,h_L}$	$\|\|\vec{u}_L - \vec{u}\|\|_{0,\Omega}$
32	48	39586	1.44204	5.99997×10^{-6}	4.1662×10^{-7}
64	96	159042	0.73483	3.09612×10^{-6}	2.3733×10^{-7}
128	192	637570	0.373412	1.5927×10^{-6}	1.24249×10^{-7}
256	384	2553090	0.189749	8.20557×10^{-7}	6.69434×10^{-8}

We can conclude from the table above that, for the given number of level and number of iteration steps on each level, it is to say that the whole amount of work is $O(n_l)$, the energy norm and the H^1 semi-norm is $O(h)$, which means that for mortar Wilson finite element method of homogeneous boundary value problem the cascadic multigrid method given in section 3 is optimal. As for the L^2 norm, it is known that for second order elliptic value problems using Wilson element, the cascadic multigrid method can not get optimal approximation with respect to L^2 norm. And here, through the results of numerical experiments it can be concluded that it is not optimal to L^2 norm.

5 Appendix

The Korn inequality due to the variational formulation of planar linear elasticity is well known. In order to prove the discrete Korn inequality for mortar Wilson finite element method of planar linear elasticity, we will following the steps like this: first prove the Korn inequality over the space

$$\vec{X}_0^s = \left\{ \begin{array}{l} \vec{v} \in \prod_{k=1}^{N} \vec{H}^1(\Omega_k),\ \vec{v} = \vec{0},\ \text{on}\ \partial\Omega, \\ \int_\gamma (\vec{v}_i - \vec{v}_j)\, d\tau = \vec{0} \end{array} \right\}.$$

Because $\vec{V}_{0,h}^Q \subset \vec{X}_0^s$, so the Korn inequality stands over the mortar Q_1 element space $\vec{V}_{0,h}^Q$. Then we will extend it to the mortar Wilson element space $\vec{V}_{0,h}$.

Lemma 5.1 If $\vec{v} \in \vec{V}_h$, then $\forall K \in T_h(\Omega_k)$, $(1 \le k \le N)$ there exist c and C independent of h that

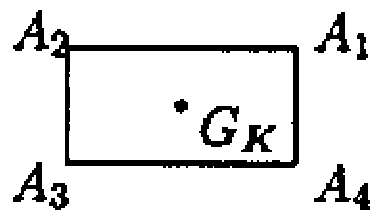
$$c\sum_{i,j=1}^{2} \|\epsilon_{ij}(\vec{v}_k)\|_{0,K}^2 \le D_K(\vec{v}_k) + \sum_{i=1}^{2}((v_{i,x}(G_K))^2 + (v_{i,y}(G_K))^2) \le C\sum_{i,j=1}^{2} \|\epsilon_{ij}(\vec{v}_k)\|_{0,K}^2, \quad (5.1)$$

$$c|\vec{v}_k|_{1,K}^2 \le B_K(\vec{v}_k) + \sum_{i=1}^{2}((v_{i,x}(G_K))^2 + (v_{i,y}(G_K))^2) \le C|\vec{v}_k|_{1,K}^2. \quad (5.2)$$

in which

$$
\begin{aligned}
B_K(\vec{v}_k) &= \sum_{i=1}^{2}[(v_i(A_2)-v_i(A_1))^2+(v_i(A_3)-v_i(A_1))^2\\
&\quad +(v_i(A_4)-v_i(A_3))^2+(v_i(A_4)-v_i(A_2))^2],\\
D_K(\vec{v}_k) &= (v_2(A_3)-v_2(A_1))^2+(v_2(A_4)-v_2(A_2))^2\\
&\quad +(v_1(A_2)-v_1(A_1))^2+(v_1(A_4)-v_1(A_3))^2\\
&\quad +[(v_2(A_2)+v_2(A_4)-v_2(A_1)-v_2(A_3))\\
&\quad +(v_1(A_3)+v_1(A_4)-v_1(A_2)-v_1(A_1))]^2,
\end{aligned}
$$

$$
v_{i,x}(G_K) = h^2\left(\frac{\partial^2 v_{h,i}}{\partial x^2}\right)(G_K),\qquad v_{i,y}(G_K) = h^2\left(\frac{\partial^2 v_{h,i}}{\partial y^2}\right)(G_K),
$$



A_1, A_2, A_3, A_4 are the four vertices of K, and G_K is the center of K (see the picture).

The prove can be found in P.Lesaint's [19].

Lemma 5.2 $\vec{H}^1(\Omega) = \hat{\vec{H}}^1(\Omega) \oplus RM(\Omega)$, it is to say that $\forall \vec{v} \in \vec{H}^1(\Omega)$, there exists $(\vec{z}, \vec{w}) \in \hat{\vec{H}}^1(\Omega) \times RM(\Omega)$ such that $\vec{v} = \vec{z} + \vec{w}$, and $\|\vec{z}\|_1 + \|\vec{w}\|_1 \le c\|\vec{v}\|_1$.

Lemma 5.3 There exists a positive constant C that

$$
\|\epsilon(\vec{v})\|_0 \ge C\|\vec{v}\|_{H_1(\Omega)}, \qquad \forall \vec{v} \in \hat{\vec{H}}^1(\Omega). \tag{5.3}
$$

Lemma 5.2 and 5.3 can be found in S.Brenner's [12]. Lemma 5.3 is the second Korn lemma. The following lemma is easy to be checked.

Lemma 5.4 Let $RM_k = RM(\Omega_k)$,

$$
\prod_{k=1}^{N}\{\vec{v}_k = \begin{pmatrix} b_k x_2 + c_{k,1} \\ -b_k x_1 + c_{k,2} \end{pmatrix} \in RM_k, \quad \text{in } \Omega_k\} \cap \vec{X}_0^s = \{0\}, \tag{5.4}
$$

in which, b_k and $c_{k,1}, c_{k,2}$ are constants.

By contradiction, the following theorem can be proven.

Theorem 5.5 Assume that constant c is independent of h, then

$$
\sum_{k=1}^{N}\|\vec{v}_k\|_{1,\Omega_k} \le c|||\vec{v}|||_h, \qquad \forall \vec{v} \in \vec{X}_0^s. \tag{5.5}
$$

Corollary 5.6

$$
|\vec{v}_n|_{1,h} \le c|||\vec{v}_n|||_h, \qquad \forall \vec{v}_h \in \vec{V}_{0,h}^Q. \tag{5.6}
$$

Next we will prove that the Korn inequality stands over the mortar Wilson element space.

Theorem 5.7 Assume that c, C are constants independent of h,

$$
c|\vec{v}_h|_{1,h} \le |||\vec{v}_h|||_h \le C|\vec{v}_h|_{1,h}, \quad \forall \vec{v}_h \in \vec{V}_{0,h}. \tag{5.7}
$$

Proof It is easy to see that $|||\vec{v}_h|||_h \le C|\vec{v}_h|_{1,h}$, so we just prove that $c|\vec{v}_h|_{1,h} \le |||\vec{v}_h|||_h$.

Because $\vec{v}_h^Q \in \vec{V}_{0,h}^Q$, according to Theorem 5.5, $|\vec{v}_h^Q|_{1,h} \le c|||\vec{v}_h^Q|||_h$, and by Lemma 5.1 and $v_{i,x}^Q(G_K) = v_{i,y}^Q(G_K) = 0$, $\forall K \in T_h(\Omega_k), (1 \le k \le N)$, we have

$$\begin{aligned} B_K(\vec{v}_{k,h}) &= B_K(\vec{v}_{k,h}^Q) \le c|\vec{v}_{k,h}^Q|_{1,K}^2, \qquad \forall k, K \in T_h(\Omega_k), \\ c\sum_{i,j=1}^{2} \|\epsilon_{ij}(\vec{v}_k^Q)\|_{0,K}^2 &\le D_K(\vec{v}_k^Q) = D_K(\vec{v}_k), \qquad \forall k, K \in T_h(\Omega_k). \end{aligned}$$

Summing it for all $K \in T_h(\Omega_k)$, $1 \le k \le N$, we get

$$\begin{aligned} \sum_{k=1}^{N} \sum_{K \in T_h(\Omega_k)} B_K(\vec{v}_{k,h}) &\le c\sum_{k=1}^{N} \sum_{K \in T_h(\Omega_k)} |\vec{v}_k^Q|_{1,K}^2 \\ &\le c\sum_{k=1}^{N} \sum_{K \in T_h(\Omega_k)} \sum_{i,j=1}^{2} \|\epsilon_{ij}(\vec{v}_k^Q)\|_{0,K}^2 \\ &\le c\sum_{k=1}^{N} \sum_{K \in T_h(\Omega_k)} D_K(\vec{v}_{k,h}). \end{aligned}$$

Using Lemma 5.1 again we have $|\vec{v}_h|_{1,h} \le c|||\vec{v}_h|||_h$.

Theorem 5.8 There exists one and only one solution for the discrete problem (2.1).

The proof follows from the Lax-Milgram lemma.

References

1 Achdou, Y., Kuznetsov, Y. A. and Pironneau, O.. Substructuring preconditioners for the Q_1 mortar element methods. Numer. Math., 1995,71: 419-449

2 Bernardi, C., Maday, Y. and Sacchi-Landriani, G.. Non conforming matching conditions for coupling spectral and finite element methods. Appl. Numer. Math., 1989, 54: 64-84

3 Bernardi, C., Maday, Y. and Patera, A. T.. A new nonconforming approach to domain decomposition: the mortar element methods. In "Nonlinear Partial Differential Equations and Their Applications", H. Brezis, J.L. Lions, eds. Number 299 in Pitman Research Notes in Mathematics series. Longman Scientific & Technical, 1990

4 Braess, D., Dahmen, W. and Wieners, C.. A multigrid algorithm for the mortar finite element method. SIAM J. Numer. Anal., 2000, 37: 48-69

5 Bornemann, F. A. and Deuflhard, P.. The cascadic multigrid method for elliptic problems. Numer. Math., 1996, 74: 135-152

6 Bornemann, F. A. and Deuflhard, P.. The cascadic multigrid method. In "The Eighth International Conference on Domain Decomposition Methods for Partial Differential Equations", R. Glowinski, J. Periaux, Z. Shi, O. Widlund, eds. Wiley and Sons, 1997

7 Bornemann, F. A. and Rolf Klause. Classical and cascadic multigrid - a methodological comparison. In "The Ninth International Conference on Domain Decomposition Methods for Partial Differential Equations", E. Bjorstad, M.S. Espedal, D.E. Keyes, eds. 1998

8 Brezzi, F. and Fortin, M.. Mixed and hybrid finite element methods. Springer-Verlag, 1991

9 Ben, F. Belgacem. The mortar finite element method with lagrange multipliers. (to appear) Numer. Math.

10 Bramble, J.. Multigrid methods. Number 294 in Pitman Research Notes in Mathematics series. Longman Scientific & Technical, 1993

11 Brenner, S. C. and Sung, L.. Linear finite element methods for planar linear elasticity. Math. Comp., 1992, 59: 321-338

12 Brenner, S C.. The mathematical theory of finite element methods. Springer-Verlag, 1994

13 Ciarlet, P. G.. The finite element method for elliptic problems. North-Holland, Amsterdam, New York, Oxford, 1978

14 Falk, R. S.. Nonconforming finite element methods for the equations of linear elasticity. Math. Comp., 1991, 57: 529-550

15 Gopalakrishnan, J. and Pasciak, J. E.. Multigrid for the mortar finite element method. (to appear) SIAM J. Numer. Anal.

16 Gilyova, L. V. and Shaidurov, V. V.. A cascadic multigrid algorithm in the finite element method for the plane elasticity problem. East-West J. Numer. Math., 1997, 5: 23-34

17 Gao, Q.. On some researches of discretization methods and algorithms for plate problems. Ph.D. Thesis, Fudan University, 1998

18 Li, L.. Introduction to sobolev spaces(Chinese). Shanghai Sci&Tech., 1981

19 Lesaint, P.. On the convergence of wilson's nonconforming element for solving the elastic problems. Comput. Meths. Appl. Mech. Engrg., 1976, 7: 1-16

20 Le, P., Tallec, and Sassi, T.. Domain decomposition with nonmatching grids: augmented lagrangian approach. Math. Comp., 1995, 64: 1367-1396

21 Le, P. Tallec "Neumann-neumann" domain decomposition algorithms for solving 2D elliptic problem with nonmatching grids. East-West J. Numer. Math., 1993, 1: 129-146

22 Liu, Q.. Cascadic method for elasticity problems(in Chinese). Master's degree Thesis, Institute of Computation Science, China Academy of Science, 1999

23 Shaidurov, V. V.. Some estimates of the rate of convergence for the cascadic conjugate gradient method. Comp. Math. Applic., 1996, 4-5: 161-171

24 Shi, Z.. Convergence properties of two nonconforming finite elements. Comput. Methods. Appl. Mech. Engrg., 1985, 48: 123-137

25 Shi, Z. and Xu, X.. Cascadic multigrid for elliptic problems. preprint

26 Wieners, C. and Wohlmuth, B. L.. A general framework for multigrid methods for mortar finite elements. preprint

27 Xu, X.. Multilevel methods for wilson element approximation of elasticity problem. Comput. Methods. Appl. Mech. Engrg., 1999, 174: 191-201

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