A UNIFIED A POSTERIORI ERROR ESTIMATOR FOR FINITE VOLUME METHODS FOR THE STOKES EQUATIONS

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Abstract. In this paper, the authors established a unified framework for deriving and analyzing a posteriori error estimators for finite volume methods for the Stokes equations. The a posteriori error estimators are residual-based, and are applicable to various finite volume methods for the Stokes equations. In particular, the unified theoretical analysis works well for finite volume schemes arising from using trial functions of conforming, non-conforming, and discontinuous finite element functions, yielding new results that are not seen in existing literature.

Key words. A posteriori error estimate, finite volume methods, finite element methods, Stokes equations

AMS subject classifications. Primary, 65N15, 65N30, 76D07; Secondary, 35B45, 35J50

1. Introduction.

In scientific computing for science and engineering problems, finite volume methods are widely used and appreciated by users due to their local conservative properties for quantities which are of physical interest (e.g., mass or energy). Among many references for finite volume methods, we would like to cite some which addresses theoretical issues such as stability and convergence [8, 9, 15, 16, 21, 22, 25, 27, 28, 11, 13, 15, 38, 39, 34]. In [17], a unified framework has been developed for the finite volume methods for the Stokes equations. The framework of [17] covers various type of finite volume schemes including those arising from conforming, non-conforming, and discontinuous finite element functions. The goal of this paper is to establish a general theory for a posteriori error estimation for the Stokes equations based on such a framework of finite volume methods.

We shall focus our attention on residual type a posteriori error estimators, in which the computable formula for judging the efficiency and reliability of numerical schemes is given as functions of residuals. Along this avenue, many fine results have been developed for finite element methods for the Stokes equations [31, 5, 3, 29, 32, 19, 20, 6, 26, 24, 35, 36]. However, little can be seen in existing literature for the finite volume methods for Stokes equations.

This paper will first introduce a general finite volume formulation which covers conforming, non-conforming, and discontinuous Galerkin methods as examples, for the Stokes equations. Then, a general residual type a posteriori error estimator shall be presented with a unified mathematical analysis. The paper is organized as follows. In Section 2, the Stokes problem and some notations are introduced. In Section 3, a general framework of finite volume methods for the Stokes equations is presented. A priori error estimation for this framework will be stated under certain assumptions on the discrete spaces. In Section 4, a posteriori error analysis is presented and analyzed

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for this framework. Finally, in Section 5, examples of discrete spaces for conforming, non-conforming, and discontinuous Galerkin finite volume methods are given. It will be shown that the aforementioned spaces satisfy the assumptions, and hence both the a priori and the a posteriori analysis are applicable to them all.

2. Preliminaries and notations. Let Ω be a bounded open domain in $\mathbb{R}^d, d = 2, 3$. Denote by $\partial\Omega$ the boundary of Ω. Consider the Stokes equations

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega, \tag{2.2}$$

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega, \tag{2.3}$$

where the symbols Δ, ∇, and $\nabla\cdot$ denote the Laplacian, gradient, and divergence operators, respectively, and $\mathbf{f}$ is the external volumetric force.

For simplicity, the algorithm and its analysis will be presented for the model Stokes problem (2.1)-(2.3) only in two-dimensional spaces (i.e.; $d = 2$) with polygonal domains. An extension to the Stokes problem in three dimensions can be made formally for general polyhedral domains.

For any open subset D of Ω, we introduce standard definitions for the Sobolev spaces $H^s(D)$ and their associated inner-products $(\cdot,\cdot)_{s,D}$, norms $\|\cdot\|_{s,D}$, and semi-norms $|\cdot|_{s,D}$ for $s \geq 0$ (see [1, 7] for details). For example, for any integer $s \geq 0$, the semi-norm $|\cdot|_{s,D}$ is given by

$$|v|_{s,D} = \left(\sum_{|\alpha|=s} \int_D |\partial^\alpha v|^2 dD \right)^{\frac{1}{2}},$$

with the usual notation

$$\alpha = (\alpha_1, \alpha_2) \quad \text{where } \alpha_1, \alpha_2 \text{ are nonnegative integers,}$$
$$|\alpha| = \alpha_1 + \alpha_2, \qquad \partial^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2}.$$

Then the Sobolev norm $\|\cdot\|_{s,D}$ can be written by

$$\|v\|_{s,D} = \left(\sum_{j=0}^{s} |v|_{j,D}^2 \right)^{\frac{1}{2}}.$$

The space $H^0(D)$ coincides with $L^2(D)$. In this case, we suppress the subscript s in its norm and inner-product, i.e. they are are denoted by $\|\cdot\|_D$ and $(\cdot,\cdot)_D$, respectively. Moreover, when $D = \Omega$, we also suppress the subscript D in the notations of norms and inner-products. Define $L_0^2(\Omega)$ to be the subspace of $L^2(\Omega)$ consisting of functions with mean value zero.

The above definition/notation can easily be extended to vector-valued and matrix-valued functions. The norm, semi-norm, and inner-product for such functions shall follow the same naming convention. In addition, all these definitions can be transferred from a polygonal domain D to an edge e, a domain with lower dimension. Similar notation system will be employed. For example, $\|\cdot\|_{s,e}$ and $\|\cdot\|_e$ would denote the norm in $H^s(e)$ and $L^2(e)$, etc.

Throughout the paper, we follow the convention that a bold Latin letter denotes a vector. Let $\mathbf{u} = [u_i]_{1\le i\le 2}$, $\mathbf{v} = [v_i]_{1\le i\le 2}$ be two vectors, and $\sigma = [\sigma_{ij}]_{1\le i,j\le 2}$, $\tau = [\tau_{ij}]_{1\le i,j\le 2}$ be two matrices, define

$$\nabla \mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} \end{pmatrix}, \qquad \nabla \cdot \mathbf{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y},$$

$$\mathbf{u} \otimes \mathbf{v} = \begin{pmatrix} u_1 v_1 & u_1 v_2 \\ u_2 v_1 & u_2 v_2 \end{pmatrix}, \qquad \sigma : \tau = \sum_{i,j=1}^{2} \sigma_{ij}\tau_{ij},$$

$$\mathbf{v} \cdot \sigma = \begin{pmatrix} \sigma_{11} v_1 + \sigma_{12} v_2 \\ \sigma_{21} v_1 + \sigma_{22} v_2 \end{pmatrix}, \qquad \mathbf{v} \cdot \sigma \cdot \mathbf{u} = \sum_{i,j=1}^{2} \sigma_{ij} u_i v_j.$$

It is not hard to see that

$$\sigma : (\mathbf{u} \otimes \mathbf{v}) = \mathbf{v} \cdot \sigma \cdot \mathbf{u}.$$

Let $\mathcal{T}_h$ be a geometrically conformal triangulation of the domain Ω; i.e., the intersection of any two triangles in $\mathcal{T}_h$ is either empty, a common vertex, or a common edge. Denote by h_T the diameter of triangle $T \in \mathcal{T}_h$, and h the maximum of all h_T. We assume that $\mathcal{T}_h$ is shape regular in the sense that for each $T \in \mathcal{T}_h$, the ratio between h_T and the diameter of the inscribed circle is bounded from above. The shape regularity of $\mathcal{T}_h$ ensures a validity of the inverse inequality for finite element functions. In addition, shape regularity allows one to apply the routine scaling arguments in finite element analysis.

We then introduce a dual partition $\mathcal{T}_h^*$ of $\mathcal{T}_h$. Three different type of dual partitions will be considered, as shown in Figure 2.1. We call the first one a vertex-based dual partition. It is defined as the union of the convex hulls around each vertex, which are obtained by connecting the barycenters of the triangles and the midpoints of corresponding edges. The second one is an edge-based dual partition. Each triangle $T \in \mathcal{T}_h$ is further divided into three subtriangles by connecting the barycenter to the vertices. Associated with each interior edge, the two subtriangles which share this edge form a quadrilateral. Similarly, each boundary edge is associated with one subtriangle. Define the edge-based dual partition $\mathcal{T}_h^*$ to be the union of these interior quadrilaterals and the border triangles. The third on is a triangle-based dual partition. Each triangle $T \in \mathcal{T}_h$ is further divided into three subtriangles by connecting the barycenter to the vertices. Define the triangle-based dual partition $\mathcal{T}_h^*$ to be the union of all these subtriangles.

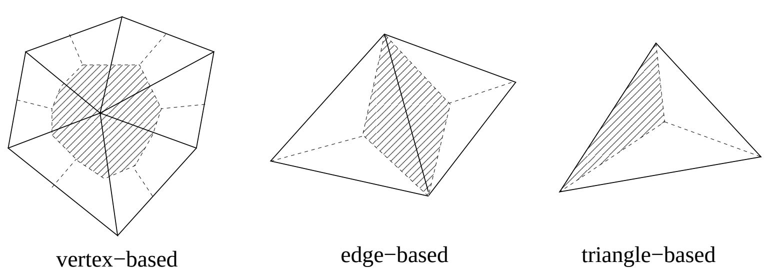

Fig. 2.1. *Three different type of dual partitions.*

Finally, we define jumps and averages on the edges of the mesh. Let $\mathcal{E}_h$ denote the union of the boundaries of all triangles T in $\mathcal{T}_h$ and $\mathcal{E}_h^0 := \mathcal{E}_h \backslash \partial\Omega$ denote the collection of all interior edges. For an interior edge e shared by two triangles T_1 and T_2, denote $\mathbf{n}_1$ and $\mathbf{n}_2$ to be the unit normal vectors on e pointing exterior to T_1 and T_2, respectively. Define the average and jump on e for scalar q, vector $\mathbf{w}$ and matrix τ, respectively, by

$$
\begin{aligned}
\{q\} &= \frac{1}{2}(q|_{T_1} + q|_{T_2}), & [\![q]\!] &= q|_{T_1}\mathbf{n}_1 + q|_{T_2}\mathbf{n}_2,\\
\{\mathbf{w}\} &= \frac{1}{2}(\mathbf{w}|_{T_1} + \mathbf{w}|_{T_2}), & [\![\mathbf{w}]\!] &= \mathbf{w}|_{T_1}\cdot\mathbf{n}_1 + \mathbf{w}|_{T_2}\cdot\mathbf{n}_2,\\
\{\tau\} &= \frac{1}{2}(\tau|_{T_1} + \tau|_{T_2}), & [\![\tau]\!] &= \mathbf{n}_1\cdot\tau|_{T_1} + \mathbf{n}_2\cdot\tau|_{T_2}.
\end{aligned}
$$

We also define a matrix-valued jump $[\cdot]$ for $\mathbf{w}$ on e by

$$
[\mathbf{w}] = \mathbf{w}|_{T_1}\otimes\mathbf{n}_1 + \mathbf{w}|_{T_2}\otimes\mathbf{n}_2.
$$

If e is a boundary edge, the above definitions need to be adjusted accordingly so that both the average and the jump are equal to the one-sided values on e. That is,

$$
\begin{aligned}
&\{q\} = q|_e, && \{\mathbf{w}\} = \mathbf{w}|_e, && \{\tau\} = \tau|_e,\\
&[\![q]\!] = q|_e\mathbf{n}, && [\![\mathbf{w}]\!] = \mathbf{w}|_e\cdot\mathbf{n}, && [\![\tau]\!] = \mathbf{n}\cdot\tau|_e,\\
&[\mathbf{w}] = \mathbf{w}|_e\otimes\mathbf{n},
\end{aligned}
$$

where $\mathbf{n}$ is the unit outward normal of Ω.

Let I be the 2×2 identity matrix. It is not hard to see that $[\![q]\!] = [\![qI]\!]$ for all scalar function q. Let q, $\mathbf{v}$ and τ be scalar-, vector-, and matrix-valued functions that are regular enough to make all involving terms well-defined, then the following identities are standard [2]:

$$
\sum_{T\in\mathcal{T}_h}\int_{\partial T} q\,\mathbf{v}\cdot\mathbf{n}\,ds = \sum_{e\in\mathcal{E}_h^0}\int_e [\![q]\!]\cdot\{\mathbf{v}\}\,ds + \sum_{e\in\mathcal{E}_h}\int_e \{q\}[\![\mathbf{v}]\!]\,ds, \tag{2.4}
$$

$$
\sum_{T\in\mathcal{T}_h}\int_{\partial T} \mathbf{n}\cdot\tau\cdot\mathbf{v}\,ds = \sum_{e\in\mathcal{E}_h^0}\int_e [\![\tau]\!]\cdot\{\mathbf{v}\}\,ds + \sum_{e\in\mathcal{E}_h}\int_e \{\tau\}:[\mathbf{v}]\,ds. \tag{2.5}
$$

We shall also need the well-know trace theorem: for any polygon K with an edge e and any function $g\in H^1(K)$,

$$
\|g\|_e^2 \lesssim h_K^{-1}\|g\|_K^2 + h_K\|\nabla g\|_K^2. \tag{2.6}
$$

3. Finite volume formulation. We start from defining the discrete spaces for a general finite volume approximation. It is a framework with the flexibility of choosing conforming, non-conforming or discontinuous Galerkin approximations to the velocity. To this end, denote V to be either $[H_0^1(\Omega)]^2$ or $[L^2(\Omega)]^2$. Let $P_l(D)$ be the space of all polynomials, with degree less than or equal to l, on a given polygon D. The finite dimensional trial function space V_h for the velocity is a subspace of piecewise linears, i.e.,

$$
V_h \subseteq \{\mathbf{v}\in V:\ \mathbf{v}|_T\in P_1(T)^2,\ \forall T\in\mathcal{T}_h\}.
$$

Certain continuity conditions may be imposed on V_h, depending on the type of methods. For example, V_h can be the continuous P_1 conforming space, the Crouzeix-Raviart P_1 nonconforming space [18] (continuous at midpoints of edges), or the totally discontinuous P_1 space to be used in conjunction with the discontinuous finite volume method [39]. More details of these spaces will be given in Section 5. To ensure that the analysis of a posteriori error estimation works, we also need the space V_h to satisfy

$$\{\mathbf{v} \in [H_0^1(\Omega)]^2 : \mathbf{v}|_T \in P_1(T)^2, \forall T \in \mathcal{T}_h\} \subseteq V_h. \tag{3.1}$$

Note that all three examples of V_h listed above have this property.

The test function space W_h for the velocity is defined on the dual mesh $\mathcal{T}_h^*$,

$$W_h = \{\mathbf{w} \in L^2(\Omega)^2 : \mathbf{w}|_K \in P_0(K)^2, \ \forall K \in \mathcal{T}_h^*\}. \tag{3.2}$$

Here, depending on the type of V_h, appropriate dual partition $\mathcal{T}_h^*$ will be chosen. Details will be presented in Section 5.

Define Q_h by

$$Q_h = \{q \in L_0^2(\Omega) : q|_T \in P_0(T), \ \forall T \in \mathcal{T}_{h'}\},$$

where $h' = h$ or $h' = 2h$. When $h' = 2h$, it actually means that $\mathcal{T}_h$ must be derived by dividing each triangle in $\mathcal{T}_{2h}$ into four subtriangles, through connecting the midpoint of its three edges. Whether to use $h' = h$ or $h' = 2h$ depends on the choice of V_h. For different V_h, different Q_h shall be chosen to guarantee the discrete formulation is well-posed. Again, details will be given in Section 5. The space Q_h serves as both the trial and the test spaces for the pressure.

We assume the existence of a transfer operator γ which maps $V(h) := V_h + [H^2(\Omega) \cap H_0^1(\Omega)]^2$ onto the test space W_h. In particular, γ connects the trial space V_h with the test space W_h. Throughout the paper, operator γ is required to satisfy the following assumption:

Assumption A1. For $T \in \mathcal{T}_h$,

$$\begin{aligned}
&\int_T (\mathbf{v} - \gamma\mathbf{v})dx = 0 && \forall\, \mathbf{v} \in V_h, \quad T \in \mathcal{T}_h,\\
&\int_e (\mathbf{v} - \gamma\mathbf{v})ds = 0 && \forall\, \mathbf{v} \in V_h, \quad e \in \mathcal{E}_h,\\
&\text{if } [\mathbf{v}] = 0, \text{ then } [\gamma\mathbf{v}] = 0, && \forall\, \mathbf{v} \in V(h), \quad e \in \mathcal{E}_h,\\
&\|\gamma\mathbf{v} - \mathbf{v}\|_T \le c h_T |\mathbf{v}|_{1,T} && \forall\, \mathbf{v} \in V(h), \quad T \in \mathcal{T}_h,\\
&\|[\gamma\mathbf{v}]\|_e \le \|[\mathbf{v}]\|_e && \forall\, \mathbf{v} \in V(h), \quad e \in \mathcal{E}_h,
\end{aligned}$$

where c is a general constant independent of the mesh size or the functions involved.

The purpose of introducing γ is to substitute the test space W_h by γV_h, and a unified framework of finite volume methods can then be defined. Such a technique has been used in [11, 12, 16] for the finite volume analysis of second order elliptic equations, and in [34] for Stokes equations. Of course, it remains a question whether such an operator γ exists or not. We will show in Section 5 that it does exist for some choices of V_h and $\mathcal{T}_h^*$. For now, we would like to skip such details, and concentrate on the a posteriori error analysis under a very general framework.

We are ready to derive the general finite volume formulation for problem (2.1)-

(2.3). Denote

$$(\mathbf{v},\ \mathbf{w})_{\mathcal{T}_h} = \sum_{T\in\mathcal{T}_h}\int_K \mathbf{v}\cdot\mathbf{w}\,dx, \qquad (\mathbf{v},\ \mathbf{w})_{\mathcal{T}_h^*} = \sum_{K\in\mathcal{T}_h^*}\int_K \mathbf{v}\cdot\mathbf{w}\,dx,$$

$$(\mathbf{v},\ \mathbf{w})_{\mathcal{E}_h} = \sum_{e\in\mathcal{E}_h}\int_e \mathbf{v}\cdot\mathbf{w}\,ds, \qquad (\mathbf{v},\ \mathbf{w})_{\mathcal{E}_h^0} = \sum_{e\in\mathcal{E}_h^0}\int_e \mathbf{v}\cdot\mathbf{w}\,ds.$$

Note that similar notations can also be defined for scalar functions. Testing the momentum equation (2.1) by $\gamma\mathbf{v}\in W_h$ gives

$$-(\Delta\mathbf{u},\ \gamma\mathbf{v})_{\mathcal{T}_h^*} + (\nabla p,\ \gamma\mathbf{v})_{\mathcal{T}_h^*} = (\mathbf{f},\ \gamma\mathbf{v}), \tag{3.3}$$

and testing the continuity equation (2.2) by $q\in Q_h$ gives

$$(\nabla\cdot\mathbf{u},\ q)_{\mathcal{T}_{h'}} = 0. \tag{3.4}$$

Define bilinear forms $a: V(h)\times V(h)\to R$ and $c: V(h)\times L_0^2(\Omega)\to R$ by

$$a(\mathbf{u},\mathbf{v}) := -\sum_{K\in\mathcal{T}_h^*}\int_{\partial K}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\gamma\mathbf{v}ds + \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\gamma\mathbf{v}ds$$

and

$$c(\mathbf{v},p) := \sum_{K\in\mathcal{T}_h^*}\int_{\partial K} p\gamma\mathbf{v}\cdot\mathbf{n}ds - \sum_{T\in\mathcal{T}_h}\int_{\partial T} p\gamma\mathbf{v}\cdot\mathbf{n}ds.$$

Since for the continuous solution $\mathbf{u}$ and p, both $[\![\nabla\mathbf{u}]\!]$ and $[\![p]\!]$ vanish on $e\in\mathcal{E}_h^0$. Thus, by using integrating by parts, equations (2.4)-(2.5), and the fact that $\gamma\mathbf{v}$ is piecewise constant, we have

$$\begin{aligned}-(\Delta\mathbf{u},\ \gamma\mathbf{v})_{\mathcal{T}_h^*} &= -\sum_{K\in\mathcal{T}_h^*}\int_{\partial K}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\gamma\mathbf{v}ds\\ &= a(\mathbf{u},\mathbf{v}) - \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\gamma\mathbf{v}ds\\ &= a(\mathbf{u},\mathbf{v}) - (\{\nabla\mathbf{u}\},\ [\gamma\mathbf{v}])_{\mathcal{E}_h}.\end{aligned}$$

Similarly,

$$\begin{aligned}(\nabla p,\ \gamma\mathbf{v}) &= \sum_{K\in\mathcal{T}_h^*}\int_{\partial K} p\gamma\mathbf{v}\cdot\mathbf{n}ds\\ &= c(\mathbf{v},p) + \sum_{T\in\mathcal{T}_h}\int_{\partial T} p\gamma\mathbf{v}\cdot\mathbf{n}ds\\ &= c(\mathbf{v},p) + ([\![\gamma\mathbf{v}]\!],\ \{p\})_{\mathcal{E}_h}.\end{aligned}$$

Combining the above, Equation (3.3) becomes

$$a(\mathbf{u},\mathbf{v}) - (\{\nabla\mathbf{u}\},\ [\gamma\mathbf{v}])_{\mathcal{E}_h} + c(\mathbf{v},p) + ([\![\gamma\mathbf{v}]\!],\ \{p\})_{\mathcal{E}_h} = (\mathbf{f},\gamma\mathbf{v}).$$

We also notice that for the exact velocity $\mathbf{u}$, by Assumption **A1**, the jump $[\gamma\mathbf{u}]$ and hence also $[\![\gamma\mathbf{u}]\!]$ vanish on all $e \in \mathcal{E}_h$. Therefore Equation (3.4) can be written as

$$(\nabla \cdot \mathbf{u},\ q)_{\mathcal{T}_{h'}} - ([\![\gamma\mathbf{u}]\!], \{q\})_{\mathcal{E}_h} = 0.$$

Now define

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) &:= a(\mathbf{u},\mathbf{v}) - (\{\nabla\mathbf{u}\},\ [\gamma\mathbf{v}])_{\mathcal{E}_h} \\ &\qquad + \delta(\{\nabla\mathbf{v}\},\ [\gamma\mathbf{u}])_{\mathcal{E}_h} + \alpha(h_e^{-1}[\mathbf{u}],\ [\mathbf{v}])_{\mathcal{E}_h}, \\ C(\mathbf{v},p) &:= c(\mathbf{v},p) + ([\![\gamma\mathbf{v}]\!], \{p\})_{\mathcal{E}_h}, \\ B(\mathbf{u},\ q) &= (\nabla\cdot\mathbf{u},\ q)_{\mathcal{T}_{h'}} - ([\![\gamma\mathbf{u}]\!], \{q\})_{\mathcal{E}_h}, \end{aligned} \tag{3.5}$$

where $\alpha \geq 0$ and $\delta = 1, -1$ or 0 are parameters.

Consider the following framework of finite volume methods: find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that

$$\begin{aligned} A(\mathbf{u}_h,\mathbf{v}) + C(\mathbf{v},p_h) &= (\mathbf{f},\gamma\mathbf{v}) \qquad && \forall\, \mathbf{v} \in V_h, \\ B(\mathbf{u}_h,\ q) &= 0 && \forall\, q \in Q_h. \end{aligned} \tag{3.6}$$

Notice that the formulation (3.6) is consistent, i.e., the true solution $(\mathbf{u}, p)$ satisfies

$$\begin{aligned} A(\mathbf{u},\mathbf{v}) + C(\mathbf{v},p) &= (\mathbf{f},\gamma\mathbf{v}) \qquad && \forall\, \mathbf{v} \in V_h, \\ B(\mathbf{u},\ q) &= 0 && \forall\, q \in Q_h. \end{aligned} \tag{3.7}$$

Subtracting (3.6) from (3.7) gives the orthogonality property of the error

$$\begin{aligned} A(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) + C(\mathbf{v},p-p_h) &= 0 \qquad && \forall\, \mathbf{v} \in V_h, \\ B(\mathbf{u}-\mathbf{u}_h,\ q) &= 0 && \forall\, q \in Q_h. \end{aligned} \tag{3.8}$$

In order to perform a priori or a posteriori error estimations, certain conditions need to be imposed on V_h, Q_h and γ. For now we only give an abstract theory built on several assumptions. Proof of these assumptions, together with suitable choices for V_h, Q_h and γ will be given in Section 5.

Define a norm $|\!|\!|\cdot|\!|\!|$ on $V(h)$ as

$$|\!|\!|\mathbf{v}|\!|\!|^2 = |\mathbf{v}|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\mathbf{v}]\|_e^2, \tag{3.9}$$

where $|\mathbf{v}|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} |\mathbf{v}|_{1,T}^2$. We will make the following assumptions:

Assumption A2. For $\mathbf{v},\mathbf{w} \in V(h)$ and $q \in L_0^2(\Omega)$,

$$\begin{aligned} a(\mathbf{v},\mathbf{w}) &= (\nabla\mathbf{v},\nabla\mathbf{w})_{\mathcal{T}_h} + \sum_{T\in\mathcal{T}_h} (\gamma\mathbf{w}-\mathbf{w},\ \nabla\mathbf{v}\cdot\mathbf{n})_{\partial T} \\ &\qquad + (\Delta\mathbf{v},\ \mathbf{w}-\gamma\mathbf{w})_{\mathcal{T}_h}, \\ c(\mathbf{v},q) &= -(\nabla\cdot\mathbf{v},\ q)_{\mathcal{T}_h} + \sum_{T\in\mathcal{T}_h} ((\mathbf{v}-\gamma\mathbf{v})\cdot\mathbf{n},\ q)_{\partial T} \\ &\qquad + (\nabla q, \gamma\mathbf{v}-\mathbf{v})_{\mathcal{T}_h}. \end{aligned}$$

Assumption A3. For $\mathbf{v}, \mathbf{w} \in V(h)$ and $q \in L_0^2(\Omega)$,

$$
\begin{aligned}
&A(\mathbf{v}, \mathbf{v}) \geq c|\!|\!|\mathbf{v}|\!|\!|^2, \qquad \text{for all } \mathbf{v} \in V_h, \\
&A(\mathbf{v}, \mathbf{w}) \leq c|\!|\!|\mathbf{v}|\!|\!| \left(|\!|\!|\mathbf{w}|\!|\!| + \left(\sum_T h_T^2 |\mathbf{w}|_{2,T}^2 \right)^{1/2} \right), \\
&C(\mathbf{v}, q) \leq c|\!|\!|\mathbf{v}|\!|\!| \left(\|q\| + \left(\sum_{T \in \mathcal{T}_h} h_T^2 |q|_{1,T}^2 \right)^{1/2} \right) \qquad \text{if } q|_T \in H^1(T) \text{ for } T \in \mathcal{T}_h, \\
&C(\mathbf{v}, q) = -B(\mathbf{v}, q), \quad \text{for all } \ q \in Q_h,
\end{aligned}
$$

where c is a general constant independent of the mesh size or the functions involved.

Assumption A4. There exists an operator $\Pi_1 : V(h) \to V_h$ such that

$$
B(\mathbf{v} - \Pi_1 \mathbf{v}, \ q) = 0, \qquad \forall\, q \in Q_h.
$$

In addition, the operator Π_1 is assumed to satisfy

$$
|\mathbf{v} - \Pi_1 \mathbf{v}|_{s,T} \leq c h^{t-s} |\mathbf{v}|_{t,T}, \quad \forall\, T \in \mathcal{T}_h, \ \ s = 0, 1, \ t = 1, 2,
$$

where the constant c depends only on the shape of T and parameters s and t.

Note that when $\mathbf{v} \in [H_0^1(\Omega) \cap H^2(\Omega)]^2$, we have $B(\mathbf{v}, \ q) = (\nabla \cdot \mathbf{v}, \ q)$. Then by Assumption **A4**, the continuous inf-sup condition, and inequality (2.6), we have the following discrete inf-sup condition, or the so-called LBB condition (see [4]):

$$
\sup_{\mathbf{v} \in V_h} \frac{B(\mathbf{v}, q)}{|\!|\!|\mathbf{v}|\!|\!|} \geq \beta \|q\|, \quad \forall\, q \in Q_h,
$$

where β is a positive constant independent of the mesh size h. Define Π_2 to be the L^2 orthogonal projection from $L_0^2(\Omega)$ to the finite dimensional space Q_h. Then we have the following a priori error estimations [17]:

THEOREM 3.1. *Let* $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ *be the solution of* (3.6) *and* $(\mathbf{u}, p) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 \times [L_0^2(\Omega) \cap H^1(\Omega)]$ *be the solution of* (2.1)–(2.3). *Under the assumptions* **A1**-**A4**, *there exists a constant* c *independent of* h *such that*

$$
\begin{aligned}
|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| + \|p - p_h\| \leq c \Bigg(&|\!|\!|\mathbf{u} - \Pi_1 \mathbf{u}|\!|\!| + \|p - \Pi_2 p\| + \left(\sum_{T \in \mathcal{T}_h} h^2 |\mathbf{u} - \Pi_1 \mathbf{u}|_{2,T}^2 \right)^{\frac{1}{2}} \\
&+ \left(\sum_{T \in \mathcal{T}_h} h^2 |p - \Pi_2 p|_{1,T}^2 \right)^{\frac{1}{2}} \Bigg).
\end{aligned}
$$

THEOREM 3.2. *Let* $(\mathbf{u}, p) \in [H^2(\Omega) \cap H_0^1(\Omega)]^2 \times [L_0^2(\Omega) \cap H^1(\Omega)]$ *and* $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ *be the solutions of (2.1)-(2.3) and (3.6) respectively with* $\delta = -1$. *Then*

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\| \leq ch \Bigg(&|\!|\!|\mathbf{u} - \Pi_1 \mathbf{u}|\!|\!| + \|p - \Pi_2 p\| + \left(\sum_{T \in \mathcal{T}_h} h^2 |\mathbf{u} - \Pi_1 \mathbf{u}|_{2,T}^2 \right)^{\frac{1}{2}} \\
&+ \left(\sum_{T \in \mathcal{T}_h} h^2 |p - \Pi_2 p|_{1,T}^2 \right)^{\frac{1}{2}} + h\|\mathbf{f}\|_1 \Bigg).
\end{aligned}
$$

4. A posteriori error estimates. In this section, we will derive an a posteriori error estimator for the finite volume formulation (3.6). Currently, the analysis only works when the bilinear form $A(\cdot,\cdot)$ is symmetric, i.e. $\delta = -1$. Hence we will set $\delta = -1$ throughout this section. For simplicity of the notation, we shall use "$\lesssim$" to denote "less than or equal to up to a constant independent of the mesh size, variables, or other parameters appearing in the inequality".

Define

$$\mathbf{J}_1(\nabla \mathbf{u}_h - p_h I) = \begin{cases} [\![\nabla \mathbf{u}_h - p_h I]\!] & \text{if } e \in \mathcal{E}_h^0 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathbf{J}_2(\mathbf{u}_h) = \begin{cases} [\mathbf{u}_h] & \text{if } e \in \mathcal{E}_h^0 \\ 2\mathbf{u}_h \otimes \mathbf{n} & \text{otherwise.} \end{cases}$$

Define a global error estimator as

$$\eta^2 = \sum_{T \in \mathcal{T}_h} \eta_T^2,$$

with

$$\eta_T^2 = h_T^2 \|\mathbf{f}\|_T^2 + \|\nabla \cdot \mathbf{u}_h\|_T^2 + \frac{1}{2} \sum_{e \in \mathcal{E}_h} \int_e \left(h_e \mathbf{J}_1(\nabla \mathbf{u}_h - p_h I)^2 + h_e^{-1} \mathbf{J}_2(\mathbf{u}_h)^2 \right) ds,$$

where h_e denotes the length of edge e. Our ultimate goal is to establish the following result:

THEOREM 4.1. *Let $(\mathbf{u}, p)$ and $(\mathbf{u}_h, p_h)$ be the solutions of (2.1)-(2.2) and (3.6), and $A(\cdot,\cdot)$ be symmetric, i.e. $\delta = -1$. Then, one has*

$$|\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| + \|p - p_h\| \lesssim \eta \tag{4.1}$$

and

$$\eta \lesssim |\!|\!|\mathbf{u} - \mathbf{u}_h|\!|\!| + \|p - p_h\| + \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\mathbf{f} - \mathbf{f}_T\|_T^2 \right)^{1/2}, \tag{4.2}$$

where $\mathbf{f}_T$ is the average of $\mathbf{f}$ on T and $(\sum_{T \in \mathcal{T}_h} h_T^2 \|\mathbf{f} - \mathbf{f}_T\|_T^2)^{1/2}$ is the data oscillation term. For convenience, the relation (4.1) is referred to as a reliability estimate and (4.2) as an efficiency estimate.

We will need a technical lemma in the proof of the above theorem. Define $H^1(\mathcal{T}_h) = \prod_{T \in \mathcal{T}_h} H^1(T)$ and $V_k = \prod_{T \in \mathcal{T}_h} P_k(T)$. For any triangle $T \in \mathcal{T}_h$, denote by $\mathcal{T}(T)$ the set of all triangles in $\mathcal{T}_h$ having a nonempty intersection with T, including T itself. Denote by $\mathcal{E}(T)$ the set of all edges in $\mathcal{E}_h$ having a nonempty intersection with T, including all three edges of T. Then following lemma has been proved in [36]:

LEMMA 4.2. *For any $v \in H^1(\mathcal{T}_h)$, there exists a $v_I \in V_k \cap H_0^1(\Omega)$, $k \geq 1$, satisfying*

$$\|v - v_I\|_T^2 + h_T^2 \|\nabla(v - v_I)\|_T^2 \lesssim \sum_{T' \in \mathcal{T}(T)} h_{T'}^2 \|\nabla v\|_{T'}^2 + \sum_{e \in \mathcal{E}(T)} h_e \|[\![v]\!]\|_e^2 \qquad \forall T \in \mathcal{T}_h. \tag{4.3}$$

Furthermore, if $v \in V_h$, then there exists a $v_I \in V_k \cap H_0^1(\Omega)$, $k \geq 1$, satisfying

$$\|v - v_I\|_T^2 + h_T^2 \|\nabla(v - v_I)\|_T^2 \lesssim \sum_{e \in \mathcal{E}(T)} h_e \|[\![v]\!]\|_e^2 \qquad \forall T \in \mathcal{T}_h, \tag{4.4}$$

4.1. Proof of reliability. In this section, we will prove the reliability estimate (4.1). Let $\mathbf{e} = \mathbf{u} - \mathbf{u}_h$ and $\epsilon = p - p_h$. To streamline the proof, we first state several technical inequalities as the following lemmas.

LEMMA 4.3. *For $\mathbf{w} \in V_h$, we have*

$$(\{\gamma\mathbf{w} - \mathbf{w}\},\ [\![\nabla\mathbf{e}]\!])_{\mathcal{E}_h^0} = 0, \tag{4.5}$$
$$(\{\gamma\mathbf{w} - \mathbf{w}\},\ [\![\epsilon]\!])_{\mathcal{E}_h^0} = 0. \tag{4.6}$$

Proof. (4.5) and (4.6) follow directly from Assumption **A1**, by noticing that $[\![\nabla\mathbf{e}]\!] = -[\![\nabla\mathbf{u}_h]\!]$ and $[\![\epsilon]\!] = -[\![p_h]\!]$ are both constants on edges. ∎

LEMMA 4.4. *For $\mathbf{w} \in V_h$ and $A(\cdot,\cdot)$ be symmetric, i.e. $\delta = -1$, we have*

$$\begin{aligned}
&(\{\nabla\mathbf{e}\},\ [\mathbf{w}])_{\mathcal{E}_h} - (\{\epsilon\},\ [\![\mathbf{w}]\!])_{\mathcal{E}_h}\\
=&(\nabla\mathbf{e},\ \nabla\mathbf{w})_{\mathcal{T}_h} + (\Delta\mathbf{u} - \nabla p,\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h}\\
&- (\nabla\cdot\mathbf{w},\ \epsilon)_{\mathcal{T}_h} - (\{\nabla\mathbf{w}\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h} + \alpha(h_e^{-1}[\mathbf{w}],\ [\mathbf{e}])_{\mathcal{E}_h}.
\end{aligned} \tag{4.7}$$

Furthermore, when $\mathbf{w}$ is also continuous,

$$(\nabla\cdot\mathbf{w},\ \epsilon)_{\mathcal{T}_h} = (\nabla\mathbf{e},\ \nabla\mathbf{w})_{\mathcal{T}_h} + (\Delta\mathbf{u} - \nabla p,\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h} - (\{\nabla\mathbf{w}\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h}. \tag{4.8}$$

Proof. Using the definition of $A(\cdot,\cdot)$, Assumption **A2**, Equation (2.5), Lemma 4.3, and the fact that $\Delta\mathbf{u}_h|_T = \mathbf{0}$, we have,

$$\begin{aligned}
A(\mathbf{e},\mathbf{w}) &= (\nabla\mathbf{e},\ \nabla\mathbf{w})_{\mathcal{T}_h} + \sum_{T\in\mathcal{T}_h}(\nabla\mathbf{e}\cdot\mathbf{n},\ \gamma\mathbf{w} - \mathbf{w})_{\partial T} + (\Delta\mathbf{u},\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h}\\
&\quad -(\{\nabla\mathbf{w}\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h} - (\{\nabla\mathbf{e}\},\ [\gamma\mathbf{w}])_{\mathcal{E}_h} + \alpha(h_e^{-1}[\mathbf{w}],\ [\mathbf{e}])_{\mathcal{E}_h}\\
&= (\nabla\mathbf{e},\ \nabla\mathbf{w})_{\mathcal{T}_h} + (\{\nabla\mathbf{e}\},\ [\gamma\mathbf{w} - \mathbf{w}])_{\mathcal{E}_h} + (\Delta\mathbf{u},\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h}\\
&\quad -(\{\nabla\mathbf{w}\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h} - (\{\nabla\mathbf{e}\},\ [\gamma\mathbf{w}])_{\mathcal{E}_h} + \alpha(h_e^{-1}[\mathbf{w}],\ [\mathbf{e}])_{\mathcal{E}_h}\\
&= (\nabla\mathbf{e},\ \nabla\mathbf{w})_{\mathcal{T}_h} + (\Delta\mathbf{u},\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h} - (\{\nabla\mathbf{e}\},\ [\mathbf{w}])_{\mathcal{E}_h}\\
&\quad -(\{\nabla\mathbf{w}\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h} + \alpha(h_e^{-1}[\mathbf{w}],\ [\mathbf{e}])_{\mathcal{E}_h}.
\end{aligned} \tag{4.9}$$

Similarly,

$$\begin{aligned}
C(\mathbf{w},\ \epsilon) &= -(\nabla\cdot\mathbf{w},\ \ \epsilon)_{\mathcal{T}_h} + \sum_{T\in\mathcal{T}_h}((\mathbf{w} - \gamma\mathbf{w})\cdot\mathbf{n},\ \epsilon)_{\partial T}\\
&\quad -(\nabla\epsilon,\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h} + (\{\epsilon\},\ [\![\gamma\mathbf{w}]\!])_{\mathcal{E}_h}\\
&= -(\nabla\cdot\mathbf{w},\ \epsilon)_{\mathcal{T}_h} + (\{\epsilon\},\ [\![\mathbf{w} - \gamma\mathbf{w}]\!])_{\mathcal{E}_h}\\
&\quad -(\nabla p,\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h} + (\{\epsilon\},\ [\![\gamma\mathbf{w}]\!])_{\mathcal{E}_h}\\
&= -(\nabla\cdot\mathbf{w},\ \epsilon)_{\mathcal{T}_h} - (\nabla p,\ \mathbf{w} - \gamma\mathbf{w})_{\mathcal{T}_h} + (\{\epsilon\},\ [\![\mathbf{w}]\!])_{\mathcal{E}_h}.
\end{aligned} \tag{4.10}$$

Combining equations (3.8), (4.9) and (4.10) gives Equation (4.7).

If $\mathbf{w} \in V_h$ is also continuous, then $[\mathbf{w}] = 0$ and $[\![\mathbf{w}]\!] = 0$. In this case, (4.7) becomes (4.8). ∎

LEMMA 4.5. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,p_h)$ be the solutions of (2.1)-(2.2) and (3.6), respectively, and $A(\cdot,\cdot)$ be symmetric, i.e. $\delta=-1$. Then*

$$\|p-p_h\|^2 \lesssim \eta^2 + \sum_{T\in\mathcal{T}_h} \|\nabla \mathbf{e}\|_T^2. \tag{4.11}$$

Proof. Let $\mathbf{v}\in[H_0^1(\Omega)]^2$ and $\mathbf{v}_I\in V_h$ be an interpolation of $\mathbf{v}$ such that both components satisfy (4.3). Observe that such an interpolation $\mathbf{v}_I$ is possible if V_h satisfies the assumption (3.1). Using the integration by parts, equations (4.8), (2.4), (2.5), (2.6), Assumption **A1**, the Schwartz inequality, the inverse inequality, and the fact that both $\mathbf{v}$ and $\mathbf{v}_I$ are continuous across each interior edge, we have

$$\begin{aligned}
(\nabla\cdot\mathbf{v},\ \epsilon) &= (\nabla\cdot(\mathbf{v}-\mathbf{v}_I),\ \epsilon)+(\nabla\cdot\mathbf{v}_I,\ \epsilon)\\
&= -(\mathbf{v}-\mathbf{v}_I,\ \nabla p)_{\mathcal{T}_h}+\sum_{T\in\mathcal{T}_h}((\mathbf{v}-\mathbf{v}_I)\cdot\mathbf{n},\ \epsilon\ )_{\partial T}\\
&\quad +(\nabla\mathbf{e},\ \nabla\mathbf{v}_I)_{\mathcal{T}_h}+(\Delta\mathbf{u}-\nabla p,\ \mathbf{v}_I-\gamma\mathbf{v}_I)_{\mathcal{T}_h}-(\{\nabla\mathbf{v}_I\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h}\\
&= -(\mathbf{v}-\mathbf{v}_I,\ \nabla p)_{\mathcal{T}_h}-(\{\mathbf{v}-\mathbf{v}_I\},\ [\![p_h]\!])_{\mathcal{E}_h^0}\\
&\quad -(\nabla\mathbf{e},\ \nabla(\mathbf{v}-\mathbf{v}_I))_{\mathcal{T}_h}+(\nabla\mathbf{e},\ \nabla\mathbf{v})_{\mathcal{T}_h}\\
&\quad +(\Delta\mathbf{u}-\nabla p,\ \mathbf{v}_I-\gamma\mathbf{v}_I)_{\mathcal{T}_h}-(\{\nabla\mathbf{v}_I\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h}\\
&= (\mathbf{v}-\mathbf{v}_I,\ \Delta\mathbf{u}-\nabla p)_{\mathcal{T}_h}+(\{\mathbf{v}-\mathbf{v}_I\},\ [\![\nabla\mathbf{u}_h-p_hI]\!])_{\mathcal{E}_h^0}\\
&\quad +(\Delta\mathbf{u}-\nabla p,\ \mathbf{v}_I-\gamma\mathbf{v}_I)_{\mathcal{T}_h}-(\{\nabla\mathbf{v}_I\},\ [\gamma\mathbf{e}])_{\mathcal{E}_h}+(\nabla\mathbf{e},\ \nabla\mathbf{v})_{\mathcal{T}_h}\\
&\lesssim \|\mathbf{v}\|_1\Bigg(h\|f\|+(\sum_{e\in\mathcal{E}_h{}^0} h_e\|[\![\nabla\mathbf{u}_h-p_hI]\!]\|_e^2)^{1/2}\\
&\qquad +(\sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\mathbf{u}_h]\|_e^2)^{1/2}+(\sum_{T\in\mathcal{T}_h}\|\nabla\mathbf{e}\|_T^2)^{1/2}\Bigg).
\end{aligned}$$

Combining the above and the inf-sup condition

$$\|p-p_h\|\lesssim \sup_{\mathbf{v}\in H_0^1(\Omega)^2}\frac{(\nabla\cdot\mathbf{v},\ p-p_h)}{\|\mathbf{v}\|_1}$$

gives Inequality (4.11). This completes the proof of the lemma. ◻

Now we are able to prove the main theorem on the reliability of the a posteriori error estimator. For simplicity of the notation, denote ∇_h to be the element-wise gradient associated with the mesh $\mathcal{T}_h$.

THEOREM 4.6. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,p_h)$ be the solutions of (2.1)-(2.2) and (3.6), and $A(\cdot,\cdot)$ be symmetric, i.e. $\delta=-1$. Then we have the following global reliability bounds:*

$$|\!|\!|\mathbf{u}-\mathbf{u}_h|\!|\!|+\|p-p_h\|\lesssim\eta. \tag{4.12}$$

Proof. Let $\mathbf{e}_I\in V_h$ be an interpolation of $\mathbf{e}$ such that both components satisfy (4.3). Using integration by parts, equation (4.8), (2.4), (2.5), and the facts that

$[\mathbf{u}] = [\mathbf{e}_I] = 0$ on all $e \in \mathcal{E}_h$, $[\nabla \mathbf{u}] = [p] = 0$ on all $e \in \mathcal{E}_h^0$, we have

$$
\begin{aligned}
&(\nabla \mathbf{e},\ \nabla \mathbf{e})_{\mathcal{T}_h}\\
=&(\nabla \mathbf{e},\ \nabla \mathbf{e} - \nabla \mathbf{e}_I)_{\mathcal{T}_h} + (\nabla \mathbf{e},\ \nabla \mathbf{e}_I)_{\mathcal{T}_h}\\
=&(\nabla \mathbf{e},\ \nabla \mathbf{e} - \nabla \mathbf{e}_I)_{\mathcal{T}_h} + (\nabla \cdot \mathbf{e}_I,\ \epsilon)\\
&- (\Delta \mathbf{u} - \nabla p,\ \mathbf{e}_I - \gamma \mathbf{e}_I)_{\mathcal{T}_h} + (\{\nabla \mathbf{e}_I\},\ [\gamma \mathbf{e}])_{\mathcal{E}_h}\\
=&(\nabla \mathbf{e},\ \nabla \mathbf{e} - \nabla \mathbf{e}_I)_{\mathcal{T}_h} - (\nabla \cdot (\mathbf{e} - \mathbf{e}_I),\ \epsilon)_{\mathcal{T}_h} + (\nabla \cdot \mathbf{e},\ \epsilon)_{\mathcal{T}_h}\\
&- (\Delta \mathbf{u} - \nabla p,\ \mathbf{e}_I - \gamma \mathbf{e}_I)_{\mathcal{T}_h} + (\{\nabla \mathbf{e}_I\},\ [\gamma \mathbf{e}])_{\mathcal{E}_h}\\
=&(-\Delta \mathbf{u} + \nabla p,\ \mathbf{e} - \mathbf{e}_I)_{\mathcal{T}_h} + \sum_{T \in \mathcal{T}_h} (\nabla \mathbf{e} \cdot \mathbf{n},\ \mathbf{e} - \mathbf{e}_I)_{\partial T} - \sum_{T \in \mathcal{T}_h} ((\mathbf{e} - \mathbf{e}_I) \cdot \mathbf{n},\ \epsilon)_{\partial T}\\
&+ (\nabla \cdot \mathbf{e},\ \epsilon)_{\mathcal{T}_h} - (\Delta \mathbf{u} - \nabla p,\ \mathbf{e}_I - \gamma \mathbf{e}_I)_{\mathcal{T}_h} + (\{\nabla \mathbf{e}_I\},\ [\gamma \mathbf{e}])_{\mathcal{E}_h}\\
=&(-\Delta \mathbf{u} + \nabla p,\ \mathbf{e} - \mathbf{e}_I)_{\mathcal{T}_h} - (\{\mathbf{e} - \mathbf{e}_I\},\ [\![\nabla \mathbf{u}_h - p_h I]\!])_{\mathcal{E}_h{}^0}\\
&- (\{\nabla \mathbf{e}\},\ [\mathbf{u}_h - \chi])_{\mathcal{E}_h} + (\{\epsilon\},\ [\![\mathbf{u}_h - \chi]\!])_{\mathcal{E}_h}\\
&+ (\nabla \cdot \mathbf{e},\ \epsilon)_{\mathcal{T}_h} - (\Delta \mathbf{u} - \nabla p,\ \mathbf{e}_I - \gamma \mathbf{e}_I)_{\mathcal{T}_h} + (\{\nabla \mathbf{e}_I\},\ [\gamma \mathbf{e}])_{\mathcal{E}_h},
\end{aligned}
\tag{4.13}
$$

where $\chi \in V_h$ is the continuous interpolation of $\mathbf{u}_h$ such that both components satisfy (4.4). Note that $[\chi] = 0$ and $[\![\chi]\!] = 0$ on all $e \in \mathcal{E}_h$. By equations (4.7), (4.4), (2.6), Assumption **A1**, and the Schwartz inequality,

$$
\begin{aligned}
&(\{\nabla \mathbf{e}\},\ [\mathbf{u}_h - \chi])_{\mathcal{E}_h} + (\{\epsilon\},\ [\![\mathbf{u}_h - \chi]\!])_{\mathcal{E}_h}\\
=&(\nabla \mathbf{e},\ \nabla(\mathbf{u}_h - \chi))_{\mathcal{T}_h} + (\Delta \mathbf{u} - \nabla p,\ (\mathbf{u}_h - \chi) - \gamma(\mathbf{u}_h - \chi))_{\mathcal{T}_h}\\
&- (\nabla \cdot (\mathbf{u}_h - \chi),\ \epsilon)_{\mathcal{T}_h} - (\{\nabla(\mathbf{u}_h - \chi)\},\ [\gamma \mathbf{e}])_{\mathcal{E}_h} + \alpha(h_e^{-1}[\mathbf{u}_h - \chi],\ [\mathbf{e}])_{\mathcal{E}_h}\\
\lesssim&\eta\,(|\!|\!|\mathbf{e}|\!|\!| + \|\epsilon\|) + \sum_{T \in \mathcal{T}_h} \|\Delta \mathbf{u} - \nabla p\|\,(h_T |\mathbf{u}_h - \chi|_{1,T})\\
&+ \sum_{e \in \mathcal{E}_h} \left(h_e^{-1} \sum_{T' \in \{T_1^e,\, T_2^e\}} \|\nabla(\mathbf{u}_h - \chi)\|_{T'}^2 \right)^{1/2} \|[\gamma \mathbf{e}]\|_e + \alpha \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\mathbf{u}_h]\|_e^2\\
\lesssim&\eta\,(|\!|\!|\mathbf{e}|\!|\!| + \|\epsilon\|) + \left(\sum_{T \in \mathcal{T}_h} h_T^2 \|\mathbf{f}\|_T^2 \right)^{1/2} \left(\sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\mathbf{u}_h]\|_e^2 \right)^{1/2}\\
&+ \sum_{e \in \mathcal{E}_h} \left(\sum_{T' \in \{T_1^e,\, T_2^e\}} \|\nabla(\mathbf{u}_h - \chi)\|_{T'}^2 \right)^{1/2} h_e^{-1/2} \|[\mathbf{u}_h]\|_e + \alpha \sum_{e \in \mathcal{E}_h} h_e^{-1} \|[\mathbf{u}_h]\|_e^2\\
\lesssim&\eta\,(|\!|\!|\mathbf{e}|\!|\!| + \|\epsilon\|) + \eta^2.
\end{aligned}
\tag{4.14}
$$

Here T_1^e, T_2^e are the two triangles in $\mathcal{T}_h$ sharing the edge e. For boundary edges, we simply set both of them be equal to the only triangle associated with that edge. Since $\nabla \cdot \mathbf{u} = 0$, we have

$$
(\nabla \cdot \mathbf{e},\ \epsilon)_{\mathcal{T}_h} = -(\nabla \cdot \mathbf{u}_h,\ \epsilon)_{\mathcal{T}_h} \lesssim \|\epsilon\|\,\eta. \tag{4.15}
$$

Other terms in Equation (4.13) can be similarly estimated. Combining (4.13)-(4.15) and using equations (2.6), (4.3), Assumption **A1**, Lemma 4.5, the Schwartz inequality, and the inverse inequality, we have

$$
\begin{aligned}
\|\!|\mathbf{u}-\mathbf{u}_h\|\!|^2 =& (\nabla \mathbf{e},\ \nabla \mathbf{e})_{\mathcal{T}_h} + \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[\mathbf{u}_h]\|_e^2 \\
\lesssim & \left(\|\!|\mathbf{e}\|\!| + \|\epsilon\|\right)\eta + \eta^2.
\end{aligned}
$$

By Lemma 4.5 and the Young's inequality, we have

$$
\|\!|\mathbf{u}-\mathbf{u}_h\|\!|^2 \lesssim \eta^2.
$$

Combining this with Lemma 4.5 gives (4.12). $\square$

4.2. Proof of efficiency. In this section, we will prove the efficiency estimate (4.2). We first define two bubble functions, which are widely used in a posteriori error estimations [33].

For each triangle $T\in\mathcal{T}_h$, denote by ϕ_T the following bubble function

$$
\phi_T = \begin{cases} 27\,\lambda_1\lambda_2\lambda_3 & \text{in } T, \\ 0 & \text{in } \Omega\backslash T, \end{cases}
$$

where $\lambda_i,\ i=1,2,3$ are barycentric coordinates on T. It is clear that $\phi_T\in H_0^1(\Omega)$ and satisfies the following properties [33]:

- For any polynomial q with degree at most m, there exist positive constants c_m and C_m, depending only on m, such that

$$
c_m\|q\|_T^2 \le \int_T q^2\phi_T\,dx \le \|q\|_T^2, \tag{4.16}
$$

$$
\|\nabla(q\phi_T)\|_T \le C_m h_T^{-1}\|q\|_T. \tag{4.17}
$$

For each $e\in\mathcal{E}_h^0$, we can analogously define an edge bubble function ϕ_e. Let T_1 and T_2 be two triangles sharing the edge e. To this end, denote by $\omega_e = T_1\cup T_2$ the union of the elements T_1 and T_2. Assume that in $T_i,\ i=1,2$, the barycentric coordinates associated with the two ends of e are $\lambda_1^{T_i}$ and $\lambda_2^{T_i}$, respectively. The edge bubble function can be defined as follows

$$
\phi_e = \begin{cases} 4\lambda_1^{T_1}\lambda_2^{T_1} & \text{in } T_1, \\ 4\lambda_1^{T_2}\lambda_2^{T_2} & \text{in } T_2, \\ 0 & \text{in } \Omega\backslash\omega_e. \end{cases}
$$

Then $\phi_e\in H_0^1(\Omega)$ and satisfies the following properties [33]:

- For any polynomial q with degree at most m, there exist positive constants d_m, D_m and E_m, depending only on m, such that

$$
d_m\|q\|_e^2 \le \int_e q^2\phi_e\,ds \le \|q\|_e^2, \tag{4.18}
$$

$$
\|\nabla(q\phi_e)\|_{\omega_e} \le D_m h_e^{-1/2}\|q\|_e, \tag{4.19}
$$

$$
\|q\phi_e\|_{\omega_e} \le E_m h_e^{1/2}\|q\|_e. \tag{4.20}
$$

Then we have the following efficiency bound.

THEOREM 4.7. *Let* $(\mathbf{u},p)$ *and* $(\mathbf{u}_h,p_h)$ *be the solutions of (2.1)-(2.2) and (3.6), and* $A(\cdot,\cdot)$ *be symmetric, i.e.* $\delta=-1$. *Then*

$$h_T\|\mathbf{f}\|_T \lesssim \|\nabla(\mathbf{u}-\mathbf{u}_h)\|_T + \|p-p_h\|_T + h_T\|\mathbf{f}-\mathbf{f}_T\|_T, \tag{4.21}$$

$$h_e^{1/2}\|[\![\nabla\mathbf{u}_h - p_hI]\!]\|_e \lesssim h_e\|\mathbf{f}-\mathbf{f}_T\|_{\omega_e} + \|\nabla_h(\mathbf{u}-\mathbf{u}_h)\|_{\omega_e} + \|p-p_h\|_{\omega_e}, \tag{4.22}$$

where $\mathbf{f}_T$ *is the average of* $\mathbf{f}$ *on* T, *and*

$$\|\nabla\cdot\mathbf{u}_h\|_T \lesssim \|\nabla\mathbf{e}\|_T. \tag{4.23}$$

Proof. Let $T\in\mathcal{T}_h$ and $\mathbf{w}_T=\mathbf{f}_T\phi_T$. Testing equation (2.1) with $\mathbf{w}_T$ gives

$$(\mathbf{f},\ \mathbf{w}_T)_T = (\nabla\mathbf{u},\ \nabla\mathbf{w}_T)_T - (\nabla\cdot\mathbf{w}_T,\ p)_T.$$

Notice that $(\nabla\mathbf{u}_h,\nabla\mathbf{w}_T)_T=0$ and $(\nabla\cdot\mathbf{w}_T,\ p_h)_T=0$, we obviously have

$$(\mathbf{f}-\mathbf{f}_T,\ \mathbf{w}_T)_T + (\mathbf{f}_T,\ \mathbf{w}_T)_T = (\nabla\mathbf{e},\ \nabla\mathbf{w}_T)_T - (\nabla\cdot\mathbf{w}_T,\ \epsilon\ )_T.$$

Then, by inequalities (4.16)-(4.17),

$$\begin{aligned}\|\mathbf{f}_T\|_T^2 \lesssim&(\mathbf{f}_T,\ \mathbf{w}_T)_T\\ =&(\mathbf{f}-\mathbf{f}_T,\ \mathbf{w}_T)_T - (\nabla\mathbf{e},\ \nabla\mathbf{w}_T)_T + (\nabla\cdot\mathbf{w}_T,\ \epsilon\ )_T\\ \lesssim& h_T^{-1}(\|\nabla\mathbf{e}\|_T + \|\epsilon\|_T)\|\mathbf{f}_T\|_T + \|\mathbf{f}-\mathbf{f}_T\|_T\|\mathbf{f}_T\|_T.\end{aligned}$$

This completes the proof of (4.21).

Next we shall prove (4.22). Let $e\in\mathcal{E}_h^0$ and $\mathbf{w}_e=[\![\nabla\mathbf{u}_h-p_hI]\!]\phi_e$. Testing (2.1) with $\mathbf{w}_e$ gives

$$(\mathbf{f},\ \mathbf{w}_e)_{\omega_e} = (\nabla\mathbf{u},\nabla\mathbf{w}_e)_{\omega_e} - (\nabla\cdot\mathbf{w}_e,\ p)_{\omega_e}. \tag{4.24}$$

Using integration by parts and the fact that $\mathbf{w}_e=0$ on $\partial\omega_e$, we have

$$(\nabla\mathbf{u}_h,\nabla\mathbf{w}_e)_{\omega_e} = \sum_{T\in\omega_e}(\nabla\mathbf{u}_h\cdot\mathbf{n},\ \mathbf{w}_e)_{\partial T} = \int_e [\![\nabla\mathbf{u}_h]\!]\cdot\mathbf{w}_e ds, \tag{4.25}$$

and

$$(\nabla\cdot\mathbf{w}_e,\ p_h)_{\omega_e} = \sum_{T\in\omega_e}(p_h\mathbf{n},\ \mathbf{w}_e)_{\partial T} = \int_e [\![p_hI]\!]\cdot\mathbf{w}_e ds. \tag{4.26}$$

Subtracting (4.25) and (4.26) from (4.24), and then using the properties of ϕ_e, we have

$$\begin{aligned}&\|[\![\nabla\mathbf{u}_h-p_hI]\!]\|_e^2\\ \lesssim&([\![\nabla\mathbf{u}_h-p_hI]\!],\ \mathbf{w}_e)_e\\ =&\sum_{T\in\omega_e}((\mathbf{f}-\mathbf{f}_T,\ \mathbf{w}_e)_T + (\mathbf{f}_T,\ \mathbf{w}_e)_T - (\nabla\mathbf{e},\nabla\mathbf{w}_e)_T + (\epsilon,\ \nabla\cdot\mathbf{w}_e)_T)\\ \lesssim&\|[\![\nabla\mathbf{u}_h-p_hI]\!]\|_e(h_e^{1/2}\|\mathbf{f}-\mathbf{f}_T\|_{\omega_e} + h_e^{1/2}\|\mathbf{f}_T\|_{\omega_\mathbf{e}}\\ &\qquad + h_e^{-1/2}\|\nabla_h\mathbf{e}\|_{\omega_e} + h_e^{-1/2}\|\epsilon\|_{\omega_e})\end{aligned}$$

Combining the above with (4.21), this completes the proof of (4.22).

Finally, the estimate (4.23) holds true as $\nabla \cdot \mathbf{u} = 0$ and clearly

$$\|\nabla \cdot \mathbf{u}_h\|_T = \|\nabla \cdot \mathbf{u}_h - \nabla \cdot \mathbf{u}\|_T = \|\nabla \cdot \mathbf{e}\|_T \leq \|\nabla \mathbf{e}\|_T.$$

This completes the proof of the theorem. ☐

Inequality (4.2) follows immediately from the above theorem.

5. Choices of $\mathcal{T}_h^*$, V_h, Q_h and γ. In this section, we will illustrate how our general theory can be applied to analyze different type of finite volume schemes. That is, we will give several choices for $\mathcal{T}_h^*$, V_h, Q_h and γ, and then prove that Assumptions **A1-A4** hold for them.

5.1. Finite volume method with conforming trial functions. For a given geometrically conformal triangular mesh $\mathcal{T}_h$, let $\mathcal{T}_h^*$ be the vertex-based dual partition as shown in Figure 2.1.

The trial function space for velocity associated with $\mathcal{T}_h$ for the traditional finite volume method is defined as

$$V_h = \{\mathbf{v} \in H_0^1(\Omega)^2 : \ \mathbf{v}|_T \in P_1(T)^2, \ \forall T \in \mathcal{T}_h\},$$

with $V = H_0^1(\Omega)^2$. The test function space W_h for velocity is defined as in (3.2), on the dual partition $\mathcal{T}_h^*$. Let Q_h be the finite dimensional space for pressure associated with the triangulation $\mathcal{T}_{h'} = \mathcal{T}_{2h}$, that is

$$Q_h = \{q \in L_0^2(\Omega) : \ q|_T \in P_0(T), \ \ \forall T \in \mathcal{T}_{2h}\}.$$

Denote $\mathcal{N}$ to be the set containing all interior nodes in $\mathcal{T}_h$. The operator $\gamma : V(h) \to W_h$ is defined by

$$\gamma \mathbf{v}(x) = \sum_{P \in \mathcal{N}} \mathbf{v}(P) \chi_P(x), \quad \forall x \in \Omega, \tag{5.1}$$

where χ_P is the characteristic function of the dual element associated with the node P. It can be easily verified that γ defined in (5.1) satisfies Assumption **A1** (see [16, 21]), while the proof of Assumption **A2** can be found in [21, 37].

By assumptions **A1**, **A2** and the facts that V_h contains piecewise linear functions, Q_h contains piecewise constant functions, it is easy to see that for $\mathbf{v}, \mathbf{w} \in V_h$ and $q \in Q_h$,

$$a(\mathbf{v}, \mathbf{w}) = (\nabla \mathbf{v}, \nabla \mathbf{w}), \quad c(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, \ q).$$

Now let us consider Assumption **A3**. For $\mathbf{v} \in V_h \subset H_0^1(\Omega)^2$, clearly $[\gamma \mathbf{v}]_e = [\mathbf{v}]_e = 0$ on all $e \in \mathcal{E}_h$. The bilinear forms $A(\mathbf{w}, \mathbf{v})$ and $C(\mathbf{v}, q)$ reduce to $a(\mathbf{w}, \mathbf{v})$ and $c(\mathbf{v}, q)$ respectively, and $C(\mathbf{v}, q) = -B(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, \ q)$. Then the conforming finite volume method can be written as: find $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ such that for any $(\mathbf{v}, q) \in V_h \times Q_h$,

$$\begin{aligned} a(\mathbf{u}_h, \ \mathbf{v}) - (\nabla \cdot \mathbf{v}, \ p_h) &= (\mathbf{f}, \ \gamma \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_h, \ q) &= 0. \end{aligned}$$

The boundedness of $A(\mathbf{v}, \mathbf{w})$ and $C(\mathbf{v}, q)$ follows directly from the above analysis, Assumption **A2**, and the Schwartz inequality. The coercivity of $A(\mathbf{v}, \mathbf{w})$ on V_h is a direct consequence of $A(\mathbf{v},\mathbf{v}) = a(\mathbf{v},\mathbf{v}) = (\nabla\mathbf{v},\ \nabla\mathbf{v})$ and $|\!|\!|\mathbf{v}|\!|\!| = |\mathbf{v}|_1$, for all $\mathbf{v} \in V_h$. This completes the proof of Assumption **A3**.

Assumption **A4** follows from the analysis of the stable $P_1 - P_0$ macro-element [23, 37]. We also note that the same conclusions hold for the conforming bilinear trial function case [14].

5.2. Finite volume method with nonconforming trial functions. For a given geometrically conformal triangulation $\mathcal{T}_h$, let $\mathcal{T}_h^*$ be its edge-based dual partition, as shown in Figure 2.1.

The nonconforming trial function space for the velocity is defined as

$$\begin{aligned} V_h = \{\mathbf{v} \in L^2(\Omega)^2 :\ & \mathbf{v}|_T \in P_1(T)^2,\ \ \forall T \in \mathcal{T}_h, \\ & \mathbf{v} \text{ is continuous at the midpoint of all } e \in \mathcal{E}_h^0 \\ & \text{and } \mathbf{v} \text{ is zero at the midpoint of all boundary edges}\}. \end{aligned}$$

The test function space W_h for velocity is defined as in (3.2), on the dual partition $\mathcal{T}_h^*$. Define $\mathcal{T}_{h'} = \mathcal{T}_h$ and hence the finite dimensional space Q_h for the pressure is

$$Q_h = \{q \in L_0^2(\Omega) :\ q|_T \in P_0(T),\ \ \forall T \in \mathcal{T}_h\}.$$

Let $\mathcal{M}$ be a set containing all the midpoints of interior edges in $\mathcal{T}_h$. The operator $\gamma : V(h) \to W_h$ is defined by

$$\gamma\mathbf{v}(x) = \sum_{P\in\mathcal{M}} \mathbf{v}(P)\chi_P(x), \quad \forall x \in \Omega, \tag{5.2}$$

where χ_P is the characteristic function of dual element associated with the node P.

Finite volume methods using the above nonconforming trial functions were considered in [10, 11]. In [16], it has been verified that the mapping γ defined in (5.2) satisfies Assumption **A1**. The proof of Assumption **A2** can be found in [37].

By assumptions **A1** and **A2** and the facts that V_h contains piecewise linear functions, Q_h contains piecewise constant functions, it is easy to see that for $\mathbf{v},\mathbf{w} \in V_h$ and $q \in Q_h$,

$$a(\mathbf{v},\mathbf{w}) = (\nabla\mathbf{v},\nabla\mathbf{w})_{\mathcal{T}_h}, \quad c(\mathbf{v},q) = -(\nabla\cdot\mathbf{v},\ q)_{\mathcal{T}_h}.$$

Now let us consider Assumption **A3**. By the definition of γ, we have $[\gamma\mathbf{v}]_e = 0$ on all $e \in \mathcal{E}_h$. The bilinear forms become

$$\begin{aligned} A(\mathbf{w},\mathbf{v}) &= a(\mathbf{w},\mathbf{v}) + \alpha(h_e^{-1}[\mathbf{w}],\ [\mathbf{v}])_{\mathcal{E}_h}, \\ C(\mathbf{v},q) &= c(\mathbf{v},q), \\ C(\mathbf{v},q) &= -B(\mathbf{v},q) = -(\nabla\cdot\mathbf{v},\ q)_{\mathcal{T}_h}. \end{aligned}$$

The nonconforming finite volume method is to find $(\mathbf{u}_h,p_h) \in V_h \times Q_h$ such that for any $(\mathbf{v},q) \in V_h \times Q_h$

$$\begin{aligned} a(\mathbf{u}_h,\ \mathbf{v}) + \alpha(h_e^{-1}[\mathbf{u}_h],\ [\mathbf{v}])_{\mathcal{E}_h} - (\nabla\cdot\mathbf{v},\ p_h)_{\mathcal{T}_h} &= (\mathbf{f},\ \gamma\mathbf{v}), \\ (\nabla\cdot\mathbf{u}_h,\ q)_{\mathcal{T}_h} &= 0. \end{aligned}$$

The boundedness of $a(\mathbf{v},\mathbf{w})$, $c(\mathbf{v},q)$ and the coercivity of $A(\mathbf{v},\mathbf{w})$ on V_h both follow directly from the above analysis, Assumption **A2**, and the Schwartz inequality. This completes the proof of Assumption **A3**.

Finally, Assumption **A4** is the well-known stability of the lowest order Crouzeix-Raviart element [18]. The same conclusions hold for the finite volume method using the rotated bilinear trial functions [30], i.e., the nonconforming Q_1 elements on rectangular grids. Details of such a finite volume method can be found in [13].

5.3. Finite volume method with totally discontinuous trial functions. The finite volume method using totally discontinuous trial functions was first proposed in [38]. For a given geometrically conformal triangulation $\mathcal{T}_h$, let $\mathcal{T}_h^*$ be its triangle-based dual partition, as shown in Figure 2.1. Define $\mathcal{T}_{h'} = \mathcal{T}_h$.

The discontinuous trial function space for the velocity is defined as

$$V_h = \{\mathbf{v} \in L^2(\Omega)^2 : \ \mathbf{v}|_T \in P_1(T)^2, \ \forall T \in \mathcal{T}_h\}.$$

The test function space W_h for velocity is defined as in (3.2), on the dual partition $\mathcal{T}_h^*$, and the finite dimensional space Q_h for the pressure is

$$Q_h = \{q \in L_0^2(\Omega) : \ q|_T \in P_0(T), \ \ \forall T \in \mathcal{T}_h\}.$$

Define $\gamma : V(h) \to W_h$ by.

$$\gamma\mathbf{v}|_K = \frac{1}{h_e}\int_e \mathbf{v}|_K ds \quad \forall K \in \mathcal{T}_h^*.$$

The operator γ defined above satisfies the first four conditions in Assumption **A1** (see [38]). The last one follows from the Schwartz inequality:

$$\begin{aligned}\|[\gamma\mathbf{v}]\|_e^2 &= \frac{1}{h_e}\left(\int_e [\mathbf{v}]ds\right)^2 \le \frac{1}{h_e}\left(\int_e [\mathbf{v}]^2 ds\right)\left(\int_e ds\right)\\ &= \int_e [\mathbf{v}]^2 ds = \|[\mathbf{v}]\|_e^2.\end{aligned}$$

Proof for assumptions **A2-A4** can be found in [39].

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