

A POSTERIORI ERROR ESTIMATE FOR THE $H(\text{div})$ CONFORMING MIXED FINITE ELEMENT FOR THE COUPLED DARCY-STOKES SYSTEM

WENBIN CHEN AND YANQIU WANG

ABSTRACT. An $H(\text{div})$ conforming mixed finite element method has been proposed for the coupled Darcy-Stokes flow in [30], which imposes normal continuity on the velocity field strongly across the Darcy-Stokes interface. Here, we develop an a posteriori error estimator for this $H(\text{div})$ conforming mixed method, and prove its global reliability and efficiency. Due to the strong coupling on the interface, special techniques need to be employed in the proof. This is the main difference between this paper and Babuška and Gatica's work [5], in which they analyzed an a posteriori error estimator for the mixed formulation using weakly coupled interface conditions.

1. INTRODUCTION

The coupled Darcy-Stokes problem is a well-known and well-studied problem, which has many important applications. We refer to the nice overview [19] and references therein for its physical background, modeling, and common numerical methods. One important issue in the modeling of the coupled Darcy-Stokes flow is the treatment of the interface condition, where the Stokes fluid meets the porous medium. In this paper, we only consider the so called Beavers-Joseph-Saffman condition, which was experimentally derived by Beavers and Joseph in [7], modified by Saffman in [40], and later mathematically verified in [27, 28, 29, 36].

Depending on whether to use the primal formulation or the mixed formulation in the Darcy region, there are two popular ways to formulate the weak problem of the coupled Darcy-Stokes flow. Here we concentrate on the mixed formulation, which has been studied in [3, 21, 22, 23, 30, 31, 32, 38, 39]. In [32], rigorous analysis of the mixed formulation and its weak existence have been presented. The authors studied two different mixed formulations. The first one imposes the normal continuity of the velocity field on the interface weakly, by using a Lagrange multiplier; while the second one imposes the normal continuity strongly in the functional space. Later we shall call these two mixed formulations, respectively, the weakly coupled formulation and the strongly coupled formulation.

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The weakly coupled formulation gives more freedom in choosing the discretizations for the Stokes side and the Darcy side separately. The work in [21, 22, 23, 32] are based on the weakly coupled formulation. Research on the strongly coupled formulation has been focused on developing unified discretization for the coupled problem. That is, the Stokes side and the Darcy side are discretized using the same finite element. This approach can simplify the numerical implementation, of course only if the unified discretization is not significantly more complicated than commonly used discretizations for the Darcy and the Stokes problems. One essential difficulty in choosing the unified discretization is that, the Stokes side velocity is in H^1 while the Darcy side velocity is only in $H(\text{div})$. Commonly used stable finite elements for the Stokes equation do not work for the Darcy equation, and vice versa. Special techniques usually need to be employed. In [3], a conforming, unified finite element has been proposed for the strongly coupled mixed formulation. However, it is constructed only on rectangular grids, and requires special treatment of the nodal degrees of freedom along the interface. According to the authors knowledge, the element proposed in [3] is probably the only existing conforming and unified element. Other researchers have resorted to less restrictive discretizations such as the non-conforming unified approach [31] or the discontinuous Galerkin (DG) approach [30, 38, 39]. Due to its discontinuous nature, some DG discretizations for the coupled Darcy-Stokes problem may break the strong coupling in the discrete level [38, 39], as they impose the normal continuity across the interface via interior penalties. We are interested in the $H(\text{div})$ conforming DG approach [30] which preserves the strong coupling even in the discrete level. The idea of the $H(\text{div})$ conforming DG approach is to use $H(\text{div})$ conforming elements, such as the Raviart-Thomas [37] elements and the Brezzi-Douglas-Marini [9] elements, to discretize the entire coupled problem. Such elements have normal continuity but not tangential continuity on mesh edges/faces, and thus is not conforming for the Stokes side. The solution is to use the interior penalty methods and impose the tangential continuity on the Stokes side weakly, via edge/face integrals. We point out that the $H(\text{div})$ conforming interior penalty method for the Stokes equation has been well-studied in [15, 16, 47]. In [30], the authors used this approach to develop a discretization for the coupled Darcy-Stokes problem, which strongly satisfies the normal continuity condition on the interface. Energy norm a priori error estimates is also proved. Later, an L^2 a priori error estimation for this approach is given [24].

The purpose of this paper is to develop an a posteriori error estimator for the $H(\text{div})$ conforming method proposed in [30]. A posteriori error estimations have been well-established for both the mixed formulation of the Darcy flow [2, 8, 11, 34], etc., and the Stokes flow [1, 6, 13, 18, 20, 26, 35, 42, 43, 45, 46], etc, among which [26, 45] covers a posteriori error estimation for $H(\text{div})$ conforming interior penalty methods for Stokes equations. However, there are only a few works existing for the coupled Darcy-Stokes problem [5, 17], where [5] concerns the weakly coupled mixed formulation while [17] uses the primal formulation on the Darcy side. To our knowledge, there is no a posteriori error estimation for the strongly coupled mixed formulation yet for the coupled Darcy-Stokes flow.

One immediately wants to ask, how different can the a posteriori error estimation for the strongly coupled mixed formulation be, comparing with estimations for the weakly coupled formulation or even for the pure Darcy or pure Stokes equations?

Here, the technical difficulty lies in the combination of a posteriori error estimations and the strongly imposed interface condition. For the mixed formulation of pure Darcy problem, the easiest way of performing a posteriori error estimation is to use a Helmholtz decomposition [11], while for the interior penalty method for the pure Stokes equation, one may want to define a continuous approximation to the discontinuous velocity [26, 45]. When coupling these two completely different techniques together, the normal continuity condition across the interface needs to be satisfied all the time. Special interpolation operators need to be constructed to fulfill this requirement, and there are many technical complexities that need to be clarified.

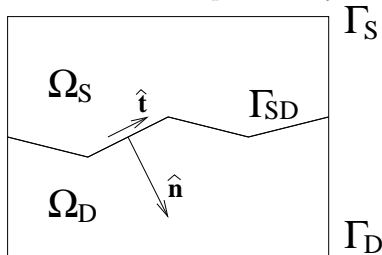
The paper is organized as follows. For simplicity, only two-dimensional problems are considered. In Section 2, the model problem for the coupled Darcy-Stokes flow and its strongly coupled mixed formulation are introduced, together with several notations. The $H(\text{div})$ conforming discretization for the strongly coupled formulation will be presented in Section 3. Then, in Section 4, an a posteriori error estimator is derived. The process of deriving actually also serves as the proof for the global reliability of the estimator. The global efficiency of the estimator is verified in Section 5. Finally in Appendix A, we construct an important interpolation operator which preserves the normal continuity on the interface while satisfying certain properties.

2. MODEL PROBLEM AND NOTATION

We follow the model developed in [24, 30]. Contents of sections 2 and 3 can be found in [24, 30] and other references as will be stated. For reader's convenience, we present some details here. The notations used in this paper are slightly different from those in [24, 30], hence sections 2 and 3 also serve the purpose of introducing the notations.

Consider the coupled Darcy-Stokes system in a polygon Ω divided into two non-overlapping subdomains Ω_S and Ω_D , which are occupied by the Stokes fluid and porous medium, respectively. For simplicity, assume both Ω_S and Ω_D are polygonal. Denote the interface between these two subdomains by Γ_{SD} . Define $\Gamma_S = \partial\Omega_S \setminus \Gamma_{SD}$ and $\Gamma_D = \partial\Omega_D \setminus \Gamma_{SD}$. When the associated domain is clear from the context, use \mathbf{n} to represent the unit outward normal vector and \mathbf{t} the unit tangential vector such that (\mathbf{n}, \mathbf{t}) forms a right-hand coordinate system. On Γ_{SD} , denote $\hat{\mathbf{n}}$ to be the unit normal vector pointing from Ω_S towards Ω_D , and $\hat{\mathbf{t}}$ accordingly, as shown in Figure 1.

FIGURE 1. Domain of the coupled Darcy-Stokes problem.



Consider the following coupled Darcy-Stokes problem, where the flow is governed by the Stokes equation in Ω_S and the Darcy's law in Ω_D :

$$(2.1) \quad \begin{aligned} -\nabla \cdot \mathbb{T}(\mathbf{u}, p) &= \mathbf{f} && \text{in } \Omega_S, \\ \mathbb{K}^{-1} \mathbf{u} + \nabla p &= \mathbf{f} && \text{in } \Omega_D, \\ \nabla \cdot \mathbf{u} &= g && \text{in } \Omega. \end{aligned}$$

Here \mathbf{u} is the velocity, p is the pressure, \mathbf{f} and g are given vector-valued and scalar-valued functions, respectively, in Ω . The stress tensor is defined by $\mathbb{T}(\mathbf{u}, p) = 2\nu D(\mathbf{u}) - pI$, where $\nu > 0$ is the fluid viscosity, $D(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the strain tensor, and I is the identity matrix. Finally, \mathbb{K} is a symmetric and uniformly positive tensor denoting the permeability tensor divided by the fluid viscosity. For simplicity, assume \mathbb{K} has smooth components and is also uniformly bounded from above.

To distinguish between the Darcy and the Stokes sides when necessary, we sometimes denote $\mathbf{u}_S = \mathbf{u}|_{\Omega_S}$, $\mathbf{u}_D = \mathbf{u}|_{\Omega_D}$, and p_S, p_D in the same fashion. The boundary condition is set to be:

$$(2.2) \quad \begin{aligned} \mathbf{u}_S &= \mathbf{0} && \text{on } \Gamma_S, \\ \mathbf{u}_D \cdot \mathbf{n} &= 0 && \text{on } \Gamma_D. \end{aligned}$$

For simplicity, we assume that $\Gamma_S \neq \emptyset$. On the interface Γ_{SD} , we impose the conservation of mass, the balance of normal forces, and the Beavers-Joseph-Saffman condition [7, 40]

$$(2.3) \quad \mathbf{u}_S \cdot \hat{\mathbf{n}} = \mathbf{u}_D \cdot \hat{\mathbf{n}},$$

$$(2.4) \quad -\mathbb{T}(\mathbf{u}_S, p_S) \hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = p_D,$$

$$(2.5) \quad -\mathbb{T}(\mathbf{u}_S, p_S) \hat{\mathbf{n}} \cdot \hat{\mathbf{t}} = \mu \mathbb{K}^{-1/2} \mathbf{u}_S \cdot \hat{\mathbf{t}},$$

where $\mu > 0$ is a variable related to the friction and shall be determined experimentally, $\mathbb{K}^{-1/2}$ is defined using the standard eigenvalue decomposition. We assume that μ is smooth and uniformly bounded both above and away from zero. It is not hard to see that conditions (2.4) and (2.5) are equivalent to

$$(2.6) \quad \mathbb{T}(\mathbf{u}_S, p_S) \hat{\mathbf{n}} + p_D \hat{\mathbf{n}} + \mu \mathbb{K}^{-1/2} (\mathbf{u}_S \cdot \hat{\mathbf{t}}) \hat{\mathbf{t}} = 0 \quad \text{on } \Gamma_{SD}.$$

Due to the boundary condition (2.2), we clearly need to assume the compatibility condition $\int_{\Omega} g \, dx = 0$. In addition, the pressure is unique only up to a constant. Thus it is convenient to assume that

$$\int_{\Omega} p \, dx = 0.$$

The mixed weak formulation and the existence of the weak solution of problem (2.1) has been thoroughly discussed in [24, 32]. Below we briefly state these results.

First, we introduce several notations. For a one- or two-dimensional polygonal domain K , denote $H^s(K)$, where $s \in \mathbb{R}$, to be the usual Sobolev space, with the norm $\|\cdot\|_{s,K}$. When $s = 0$, it coincides with the square integrable space $L^2(K)$ and we usually suppress 0 in the subscript of the norm, that is, $\|\cdot\|_K = \|\cdot\|_{0,K}$. Denote $(\cdot, \cdot)_K$ and $\langle \cdot, \cdot \rangle_K$ to be the L^2 inner-product and duality form, respectively, in K . When K is one-dimensional, the convention is to use $\langle \cdot, \cdot \rangle_K$ for both the L^2 inner-product and duality form. When $K = \Omega$, we usually suppress the subscript K in $(\cdot, \cdot)_K$. Finally, all these notations can be easily extended to vector and tensor spaces.

We follow the convention that a bold character denotes a vector or vector-valued function. Define

$$H(\operatorname{div}, K) = \{\mathbf{v} \in (L^2(K))^2, \nabla \cdot \mathbf{v} \in L^2(K)\},$$

with the norm

$$\|\mathbf{v}\|_{H(\operatorname{div}, K)} = (\|\mathbf{v}\|_K^2 + \|\nabla \cdot \mathbf{v}\|_K^2)^{1/2}.$$

Let $\Gamma \subset \partial K$. Define

$$H_{0,\Gamma}(\operatorname{div}, K) = \{\mathbf{v} \in H(\operatorname{div}, K), \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma\},$$

and

$$H_{0,\Gamma}^1(K) = \{v \in H^1(K), v = 0 \text{ on } \Gamma\}.$$

If $\Gamma = \partial K$, simply denote $H_{0,\partial K}(\operatorname{div}, K) = H_0(\operatorname{div}, K)$ and $H_{0,\partial K}^1(K) = H_0^1(K)$. We are interested in the trace of functions in $H_{0,\Gamma}(\operatorname{div}, K)$ and $H_{0,\Gamma}^1(K)$ on $\tilde{\Gamma} = \partial K \setminus \Gamma$. It is well-known that for all $v \in H_{0,\Gamma}^1(K)$, we have $v|_{\tilde{\Gamma}} \in H_{00}^{1/2}(\tilde{\Gamma})$ and for all $\mathbf{v} \in H_{0,\Gamma}(\operatorname{div}, K)$, we have $\mathbf{v} \cdot \mathbf{n}|_{\tilde{\Gamma}} \in (H_{00}^{1/2}(\tilde{\Gamma}))^*$ [32] (readers may refer to [33] for the definition and norm of $H_{00}^{1/2}(\tilde{\Gamma})$). An important property of $H_{00}^{1/2}(\tilde{\Gamma})$, is that, for any function in $H_{00}^{1/2}(\tilde{\Gamma})$, it can be extended by zero on $\partial K \setminus \tilde{\Gamma}$ and yields a function in $H^{1/2}(\partial K)$.

Define

$$\mathbf{V} = \{\mathbf{v} \in H_0(\operatorname{div}, \Omega) \mid \mathbf{v}_S \in H^1(\Omega_S)^2 \text{ and } \mathbf{v}|_{\Gamma_S} = \mathbf{0}\}.$$

The space \mathbf{V} is a Hilbert space under the norm $(\|\mathbf{v}\|_{1,\Omega_S}^2 + \|\mathbf{v}\|_{H(\operatorname{div},\Omega_D)}^2)^{1/2}$. Later we will also introduce an equivalent energy norm on \mathbf{V} . For convenience, denote \mathbf{V}_S and \mathbf{V}_D to be the confinements of \mathbf{V} on Ω_S and Ω_D respectively. It is clear that functions in \mathbf{V} satisfy the strong coupling condition (2.3) on the interface Γ_{SD} . Furthermore, $\mathbf{v}_S|_{\Gamma_{SD}} \in H_{00}^{1/2}(\Gamma_{SD})^2$ for all $\mathbf{v} \in \mathbf{V}$.

Define a bilinear form $a(\cdot, \cdot) : \mathbf{V} \times \mathbf{V} \rightarrow \mathbb{R}$ by

$$a(\mathbf{u}, \mathbf{v}) = a_S(\mathbf{u}, \mathbf{v}) + a_D(\mathbf{u}, \mathbf{v}) + a_I(\mathbf{u}, \mathbf{v}),$$

where

$$\begin{aligned} a_S(\mathbf{u}, \mathbf{v}) &= 2\nu(D(\mathbf{u}), D(\mathbf{v}))_{\Omega_S}, \\ a_D(\mathbf{u}, \mathbf{v}) &= (\mathbb{K}^{-1}\mathbf{u}, \mathbf{v})_{\Omega_D}, \\ a_I(\mathbf{u}, \mathbf{v}) &= \langle \mu\mathbb{K}^{-1/2}\mathbf{u}_s \cdot \hat{\mathbf{t}}, \mathbf{v}_s \cdot \hat{\mathbf{t}} \rangle_{\Gamma_{SD}}. \end{aligned}$$

Denote $Q = L_0^2(\Omega)$, the mean-value free subspace of $L^2(\Omega)$, and define a bilinear form $b(\cdot, \cdot) : \mathbf{V} \times Q \rightarrow \mathbb{R}$ by

$$b(\mathbf{v}, q) = -(\nabla \cdot \mathbf{v}, q)_\Omega.$$

Now we can introduce the weak formulation of the Darcy-Stokes coupled problem: Find $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$(2.7) \quad \begin{cases} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) = -(g, q) & \text{for all } q \in Q. \end{cases}$$

Clearly, Equation (2.7) can be written into

$$\Lambda((\mathbf{u}, p), (\mathbf{v}, q)) = \mathbf{F}((\mathbf{v}, q)),$$

where

$$(2.8) \quad \Lambda((\mathbf{u}, p), (\mathbf{v}, q)) = a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + b(\mathbf{u}, q),$$

$$(2.9) \quad \mathbf{F}((\mathbf{v}, q)) = (\mathbf{f}, \mathbf{v}) - (g, q).$$

It has been shown [32] that the weak formulation (2.7) is equivalent to the boundary value problem (2.1)-(2.5). Readers may also refer to [25] for a detailed discussion on the interface conditions on Γ_{SD} . To state the weak existence results for problem (2.7), we first need to introduce several norms. Define an energy norm on \mathbf{V} by

$$\|\mathbf{v}\|_{\mathbf{V}} = \left(\|\nabla \mathbf{v}\|_{\Omega_S}^2 + \|\mu^{1/2} \mathbb{K}^{-1/4} \mathbf{v}_S \cdot \hat{\mathbf{t}}\|_{\Gamma_{SD}}^2 + \|\mathbb{K}^{-1/2} \mathbf{v}\|_{\Omega_D}^2 + \|\nabla \cdot \mathbf{v}\|_{\Omega}^2 \right)^{1/2}.$$

It is not hard to see that

$$C_1 \|\mathbf{v}\|_{\mathbf{V}} \leq \left(\|\mathbf{v}\|_{1, \Omega_S}^2 + \|\mathbf{v}\|_{H(\text{div}, \Omega_D)}^2 \right)^{1/2} \leq C_2 \|\mathbf{v}\|_{\mathbf{V}},$$

where C_1 and C_2 depend only on the shape of domain, μ , and \mathbb{K} . Denote $\|\cdot\|_Q$ to be the L^2 norm on Ω and

$$\|(\mathbf{v}, q)\| \triangleq \|(\mathbf{v}, q)\|_{\mathbf{V} \times Q} = \sqrt{\|\mathbf{v}\|_{\mathbf{V}}^2 + \|q\|_Q^2}.$$

Then, it is not hard to establish the following Ladyzhenskaya-Babuška-Brezzi condition [24, 32]:

$$(2.10) \quad a(\mathbf{v}, \mathbf{v}) \geq \alpha \|\mathbf{v}\|_{\mathbf{V}}^2 \quad \text{for all } \mathbf{v} \in \mathbf{Z},$$

$$(2.11) \quad \sup_{\mathbf{v} \in \mathbf{V}} \frac{b(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}}} \geq \beta \|q\|_Q \quad \text{for all } q \in Q,$$

where

$$\mathbf{Z} = \{\mathbf{v} \in \mathbf{V} \mid \nabla \cdot \mathbf{v} = 0\}.$$

These guarantee that Equation (2.7) admits a unique solution in $\mathbf{V} \times Q$. Furthermore, it is well-known that conditions (2.10)-(2.11) are equivalent to the following Babuška's form: [10]

$$(2.12) \quad \sup_{(\mathbf{w}, \xi) \in \mathbf{V} \times Q} \frac{\Lambda((\mathbf{w}, \xi), (\mathbf{v}, q))}{\|(\mathbf{w}, \xi)\|} \geq C(\alpha, \beta) \|(\mathbf{w}, \xi)\| \quad \text{for all } (\mathbf{w}, \xi) \in \mathbf{V} \times Q.$$

Here $C(\alpha, \beta)$ is a constant depending on α and β .

Equation (2.7) is the strongly coupled formulation studied in [32]. An alternative weak formulation for problem (2.1), the weakly coupled formulation, has also been presented in [32]. The weakly coupled formulation is defined on the space $(H_{0, \Gamma_S}^1(\Omega_S)^2 \times H_{0, \Gamma_D}(\text{div}, \Omega_D)) \times Q$, and the interface condition (2.3) is then weakly imposed by using a Lagrange multiplier. In [32], Layton, Schieweck and Yotov have proved that when the porous medium is entirely enclosed in the fluid region, the weakly coupled formulation and the strongly coupled formulation are equivalent, and both are well-posed. However, for general domains, it can only be proved that the strongly coupled formulation (2.7) is well-posed, while the well-posedness of the weakly coupled formulation is unknown due to a technical difficulty of restricting $H^{-1/2}(\Omega_D)$ on Γ_{SD} [32]. Interested readers may refer to [32] for the details. Mixed finite element methods introduced in [21, 22, 23, 32] are all based on the weakly coupled formulation. The a posteriori error estimation given in [5] is also based on the weakly coupled formulation. Different from the work in [5], here we propose

an a posteriori error estimation for the strongly coupled formulation (2.7) and its $H(\text{div})$ conforming finite element discretization introduced in [24, 30].

3. MIXED FINITE ELEMENT DISCRETIZATION

Recently, an $H(\text{div})$ conforming, unified mixed method for (2.7), in which both the Stokes part and the Darcy part are approximated by the Raviart-Thomas (RT) elements, has been proposed [30]. Later, optimal error in L^2 norm for the velocity of the $H(\text{div})$ conforming formulation has been proved in [24]. Of course, the RT elements are not H^1 conforming on the Stokes side. The idea is to use the interior penalty discontinuous Galerkin $H(\text{div})$ approach for the Stokes equation [15, 16, 47] in the Stokes region, while using a usual mixed finite element discretization in the Darcy region. There are three main advantages of this approach. First, the unified finite element space may simplify the numerical simulation. Second, the normal continuity condition (2.3) on the interface is strongly imposed on the discrete level. Third, the discretization is strongly conservative [30, 47]. For example, when $g = 0$ in Ω_S , then the discrete velocity is exactly divergence free, instead of weakly divergence free. Next, we briefly present this method.

Let \mathcal{T}_h be a geometrically conformal, shape-regular mesh on Ω . We require that \mathcal{T}_h be aligned with Γ_{SD} . For each triangle $T \in \mathcal{T}_h$, denote by h_T its diameter. Let h be the maximum of all h_T . Denote \mathcal{T}_h^S and \mathcal{T}_h^D to be the meshes in Ω_S and Ω_D , respectively.

Denote by \mathcal{E}_h the set of all edges in \mathcal{T}_h . For each edge $e \in \mathcal{E}_h$, denote by h_e its length. Let \mathcal{E}_h^{SD} be the set of all edges in $\mathcal{T}_h \cap \Gamma_{SD}$, and let \mathcal{E}_h^S , \mathcal{E}_h^D be the set of all edges in $\mathcal{T}_h \cap (\Omega_S \cup \Gamma_S)$, $\mathcal{T}_h \cap (\Omega_D \cup \Gamma_D)$, respectively. We also denote $\mathcal{E}_{0,h}^S$ and $\mathcal{E}_{0,h}^D$ to be the set of edges interior to Ω_S and Ω_D , respectively.

Let \mathcal{O} , \mathcal{P} be operators defined on each $T \in \mathcal{T}_h^S$ or \mathcal{T}_h^D , but may not be well-defined on the entire Ω . For example, the gradient operator on the space of discontinuous piecewise polynomials on \mathcal{T}_h . We introduce the notation for discrete L^2 inner-products as following

$$\begin{aligned} (\mathcal{O}(\cdot), \mathcal{P}(\cdot))_{\mathcal{T}_h^S} &= \sum_{T \in \mathcal{T}_h^S} (\mathcal{O}(\cdot), \mathcal{P}(\cdot))_T, \\ (\mathcal{O}(\cdot), \mathcal{P}(\cdot))_{\mathcal{T}_h^D} &= \sum_{T \in \mathcal{T}_h^D} (\mathcal{O}(\cdot), \mathcal{P}(\cdot))_T. \end{aligned}$$

Similarly, define $\langle \cdot, \cdot \rangle_{\mathcal{E}_h^S}$, $\langle \cdot, \cdot \rangle_{\mathcal{E}_{0,h}^S}$, $\langle \cdot, \cdot \rangle_{\mathcal{E}_h^D}$, $\langle \cdot, \cdot \rangle_{\mathcal{E}_{0,h}^D}$, and $\langle \cdot, \cdot \rangle_{\mathcal{E}_h^{SD}}$ in the same fashion. With the aid of these notations, we can also denote mesh-dependent "broken" L^2 norms in a straight-forward manner. For example, $\|\cdot\|_{\mathcal{T}_h^S} \triangleq (\cdot, \cdot)_{\mathcal{T}_h^S}^{1/2}$ and $\|\cdot\|_{\mathcal{E}_h^{SD}} \triangleq \langle \cdot, \cdot \rangle_{\mathcal{E}_h^{SD}}^{1/2}$. We especially remark that the "broken" norm may even contain h_T or h_e in it. For example,

$$\begin{aligned} \|h_T \mathbf{v}\|_{\mathcal{T}_h^S} &= (h_T \mathbf{v}, h_T \mathbf{v})_{\mathcal{T}_h^S}^{1/2} = \left(\sum_{T \in \mathcal{T}_h^S} h_T^2 \|\mathbf{v}\|_T^2 \right)^{1/2}, \\ \|h_e^{-1/2} \mathbf{v}\|_{\mathcal{E}_h^{SD}} &= \langle h_e^{-1/2} \mathbf{v}, h_e^{-1/2} \mathbf{v} \rangle_{\mathcal{E}_h^{SD}}^{1/2} = \left(\sum_{e \in \mathcal{E}_h^{SD}} h_e^{-1} \|\mathbf{v}\|_e^2 \right)^{1/2}. \end{aligned}$$

Such notations shall greatly clean up the style of this paper.

Let $\mathbf{V}_h \subset H_0(\text{div}, \Omega)$ and $Q_h \subset Q$ be a pair of the Raviart-Thomas [37] finite element spaces, except for the lowest-order one, defined on \mathcal{T}_h . That is, \mathbf{V}_h is the RT_k space, with $k \geq 1$, and Q_h is the discontinuous P_k space. Notice here we exclude the RT_0 element, since otherwise $\mathbf{V}_h \cap \mathbf{V}$ will be empty. Readers may refer to [10, 14] for more details and properties of the RT elements.

For convenience, denote $\mathbf{V}_{h,S}$ and $\mathbf{V}_{h,D}$ to be the confinement of \mathbf{V}_h in Ω_S and Ω_D , respectively. Similarly, for any $\mathbf{v} \in \mathbf{V}_h$, it can be split into $\mathbf{v}_{h,S} \in \mathbf{V}_{h,S}$ and $\mathbf{v}_{h,D} \in \mathbf{V}_{h,D}$. Denote P_h the space of H^1 conforming Lagrange finite element space on \mathcal{T}_h consisting of piecewise polynomials with degree less than or equal to k , and let $P_{h,S}$, $P_{h,D}$ be the confinements of P_h on Ω_S and Ω_D , respectively. By the definition, we have

$$(P_{h,S})^2 \cap \mathbf{V}_S \subset \mathbf{V}_{h,S}.$$

Finally, we point out that all functions in \mathbf{V}_h satisfy the strong coupling condition (2.3) on the interface Γ_{SD} .

Define a discrete bilinear form

$$a_{S,h}(\mathbf{u}, \mathbf{v}) = 2\nu \left((D(\mathbf{u}), D(\mathbf{v}))_{\mathcal{T}_h^S} - \langle \{D(\mathbf{u})\}\mathbf{n}, [\mathbf{v}] \rangle_{\mathcal{E}_h^S} - \langle [\mathbf{u}], \{D(\mathbf{v})\}\mathbf{n} \rangle_{\mathcal{E}_h^S} + \langle \frac{\sigma}{h_e} [\mathbf{u}], [\mathbf{v}] \rangle_{\mathcal{E}_h^S} \right),$$

where $\{\cdot\}$ and $[\cdot]$ denote the average and the jump on edges, respectively, and $\sigma > 0$ is a parameter of $O(1)$. On the boundary edge $e \subset \Gamma_S$, $\{\cdot\}$ and $[\cdot]$ are just the one-sided values. On each edge, the direction \mathbf{n} is taken to be the same as the direction in $[\cdot]$, that is, if $[\mathbf{v}]|_e \triangleq \mathbf{v}|_{T_1} - \mathbf{v}|_{T_2}$ where T_1 and T_2 share the edge e , then \mathbf{n} points from T_1 to T_2 . The notations in the definition of bilinear form $a_{S,h}$ are standard in the discontinuous Galerkin literature and readers may refer to [15, 16, 30, 47] for more details.

Now, define

$$a_h(\mathbf{u}, \mathbf{v}) = a_{S,h}(\mathbf{u}, \mathbf{v}) + a_D(\mathbf{u}, \mathbf{v}) + a_I(\mathbf{u}, \mathbf{v}).$$

We have the discrete problem [24, 30]: Find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$(3.1) \quad \begin{cases} a_h(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) = (\mathbf{f}, \mathbf{v}) & \text{for all } \mathbf{v} \in \mathbf{V}_h, \\ b(\mathbf{u}_h, q) = -(g, q) & \text{for all } q \in Q_h. \end{cases}$$

Since $\mathbf{V}_{h,S}$ is not in $H_{0,\Gamma_S}^1(\Omega_S)^2$, the discrete space \mathbf{V}_h is not a subspace of \mathbf{V} . Therefore it does not inherit the norm of \mathbf{V} . Here we shall define a discrete norm on \mathbf{V}_h by

$$\|\mathbf{v}\|_{\mathbf{V}_h} = \left(\|\nabla \mathbf{v}\|_{\mathcal{T}_h^S}^2 + \|h_e^{-1/2}[\mathbf{v}]\|_{\mathcal{E}_h^S}^2 + \|\mu^{1/2}\mathbb{K}^{-1/4}\mathbf{v}_S \cdot \hat{\mathbf{t}}\|_{\mathcal{E}_h^S}^2 + \|\mathbb{K}^{-1/2}\mathbf{v}\|_{\Omega_D}^2 + \|\nabla \cdot \mathbf{v}\|_{\Omega}^2 \right)^{1/2}.$$

Although the norm $\|\cdot\|_{\mathbf{V}}$ is not well-defined on \mathbf{V}_h , we point out that the discrete norm $\|\cdot\|_{\mathbf{V}_h}$ is well-defined on \mathbf{V} . Indeed, it is clear that $\|\mathbf{v}\|_{\mathbf{V}_h} = \|\mathbf{v}\|_{\mathbf{V}}$ for all $\mathbf{v} \in \mathbf{V}$, since the jump term $\|h_e^{-1/2}[\mathbf{v}]\|_{\mathcal{E}_h^S}$ vanishes. Later we shall use this property and build certain discrete functions in \mathbf{V} . Then for these functions, one can easily shift from the discrete $\|\cdot\|_{\mathbf{V}_h}$ norm to the continuous $\|\cdot\|_{\mathbf{V}}$ norm.

The space Q_h is a subspace of Q and inherits its norm, which is just the L^2 norm. Define the norm in $(\mathbf{V} + \mathbf{V}_h) \times Q$ by

$$\|(\mathbf{v}, q)\|_h = \left(\|\mathbf{v}\|_{\mathbf{V}_h}^2 + \|q\|_Q^2 \right)^{1/2}.$$

Again, we have

$$\|(\mathbf{v}, q)\|_h = \|(\mathbf{v}, q)\| \quad \text{for all } \mathbf{v} \in \mathbf{V} \text{ and } q \in Q.$$

Well-posedness and a priori error estimates for (3.1) have been given in [24, 30]:

Theorem 3.1. *Equation (3.1) has a unique solution (\mathbf{u}_h, p_h) for σ large enough, but not depending on h . Assume the solution (\mathbf{u}, p) of (2.7) is in $H^s(\Omega)^2 \times H^s(\Omega)$ with $3/2 < s \leq k + 1$, then*

$$\|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_h \leq Ch^{s-1}(|\mathbf{u}|_{s,\Omega} + |p|_{s,\Omega}),$$

where C is a positive constant independent of the mesh size.

4. RESIDUAL BASED A POSTERIORI ESTIMATION

The goal of this section is to derive an a posteriori error estimator for the problem (3.1). For simplicity of notation, we shall use “ \lesssim ” to denote “less than or equal to up to a constant independent of the mesh size, variables, or other parameters appearing in the inequality”. In this section, we will also frequently use the following well-known inequality: for any function $\xi \in H^1(T)$ where T is a triangle with an edge e , the following estimate holds:

$$(4.1) \quad h_e \|\xi\|_e^2 \lesssim \|\xi\|_T^2 + h_T^2 \|\nabla \xi\|_T^2.$$

To derive an a posteriori error estimator, we first denote

$$\boldsymbol{\varepsilon}_\mathbf{u} = \mathbf{u} - \mathbf{u}_h, \quad \boldsymbol{\varepsilon}_p = p - p_h,$$

where (\mathbf{u}, p) is the solution to (2.7) and (\mathbf{u}_h, p_h) is the solution to (3.1). The idea of deriving a reliable a posteriori error estimator is to find an upper bound for $\|(\boldsymbol{\varepsilon}_\mathbf{u}, \boldsymbol{\varepsilon}_p)\|_h$.

As pointed out in the previous section, the norms $\|\cdot\|_h$ and $\|\cdot\|$ are identical on $\mathbf{V} \times Q$. Thus we introduce a new discrete function $\tilde{\mathbf{u}}_h \in \mathbf{V}$, which is defined from the discrete solution \mathbf{u}_h and satisfy

$$(4.2) \quad \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h} \lesssim \|h_e^{-1/2}[\mathbf{u}_h]\|_{\mathcal{E}_h^S}.$$

The definition of $\tilde{\mathbf{u}}_h$ and the proof of Equation (4.2) will be given in Appendix A. Note that $\tilde{\mathbf{u}}_h$ is not necessarily in \mathbf{V}_h . The term $\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h}$ is usually called the nonconformity estimator in the a posteriori error estimation literature.

Denote

$$\tilde{\boldsymbol{\varepsilon}}_\mathbf{u} = \mathbf{u} - \tilde{\mathbf{u}}_h \in \mathbf{V},$$

then

$$\|(\boldsymbol{\varepsilon}_\mathbf{u}, \boldsymbol{\varepsilon}_p)\|_h \leq \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h} + \|(\tilde{\boldsymbol{\varepsilon}}_\mathbf{u}, \boldsymbol{\varepsilon}_p)\|_h = \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h} + \|(\tilde{\boldsymbol{\varepsilon}}_\mathbf{u}, \boldsymbol{\varepsilon}_p)\|.$$

Since $\|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h}$ is bounded in Equation (4.2), we only need to estimate $\|(\tilde{\boldsymbol{\varepsilon}}_\mathbf{u}, \boldsymbol{\varepsilon}_p)\|$.

By the Babuška’s condition (2.12),

$$\|(\tilde{\boldsymbol{\varepsilon}}_\mathbf{u}, \boldsymbol{\varepsilon}_p)\| \lesssim \sup_{(\mathbf{v}, q) \in \mathbf{V} \times Q} \frac{\Lambda((\tilde{\boldsymbol{\varepsilon}}_\mathbf{u}, \boldsymbol{\varepsilon}_p), (\mathbf{v}, q))}{\|(\mathbf{v}, q)\|}.$$

Note that

$$\begin{aligned} \Lambda((\tilde{\boldsymbol{\varepsilon}}_{\mathbf{u}}, \boldsymbol{\varepsilon}_p), (\mathbf{v}, q)) &= \left(a(\tilde{\boldsymbol{\varepsilon}}_{\mathbf{u}}, \mathbf{v}) + b(\mathbf{v}, \boldsymbol{\varepsilon}_p) \right) + b(\tilde{\boldsymbol{\varepsilon}}_{\mathbf{u}}, q) \\ &= \left((\mathbf{f}, \mathbf{v}) - a(\tilde{\mathbf{u}}_h, \mathbf{v}) - b(\mathbf{v}, p_h) \right) + \left(-(g, q) + (\nabla \cdot \tilde{\mathbf{u}}_h, q) \right) \\ &= Res_1(\mathbf{v}) + Res_2(q). \end{aligned}$$

Here

$$(4.3) \quad \begin{aligned} Res_2(q) &\leq \|g - \nabla \cdot \tilde{\mathbf{u}}_h\| \|q\| \leq \left(\|g - \nabla \cdot \mathbf{u}_h\| + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h} \right) \|q\| \\ &\lesssim \left(\|g - \nabla \cdot \mathbf{u}_h\| + \|h_e^{-1/2}[\mathbf{u}_h]\|_{\mathcal{E}_h^S} \right) \|q\|. \end{aligned}$$

Next, we concentrate on estimating $Res_1(\mathbf{v})$. Let $\mathbf{v}_h \in \mathbf{V}_h$ be any function, by (3.1), we have

$$Res_{1,h}(\mathbf{v}_h) \triangleq (\mathbf{f}, \mathbf{v}_h) - a_h(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = 0.$$

Thus

$$\begin{aligned} Res_1(\mathbf{v}) &= Res_1(\mathbf{v}) - Res_{1,h}(\mathbf{v}_h) \\ &= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h) - \left(a(\tilde{\mathbf{u}}_h, \mathbf{v}) - a_h(\mathbf{u}_h, \mathbf{v}_h) \right) - b(\mathbf{v} - \mathbf{v}_h, p_h) \\ &= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} - \left(a_D(\tilde{\mathbf{u}}_h, \mathbf{v}) - a_D(\mathbf{u}_h, \mathbf{v}_h) \right) + (\nabla \cdot (\mathbf{v} - \mathbf{v}_h), p_h)_{\Omega_D} \\ &\quad + (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_S} - \left(a_S(\tilde{\mathbf{u}}_h, \mathbf{v}) - a_{S,h}(\mathbf{u}_h, \mathbf{v}_h) \right) + (\nabla \cdot (\mathbf{v} - \mathbf{v}_h), p_h)_{\Omega_S} \\ &\quad - \left(a_I(\tilde{\mathbf{u}}_h, \mathbf{v}) - a_I(\mathbf{u}_h, \mathbf{v}_h) \right) \\ &= R_D + R_S + R_I, \end{aligned}$$

Next, we shall derive upper bounds for R_D , R_S and R_I one by one. Clearly, the choice of \mathbf{v}_h is essential to the estimation. We would like \mathbf{v}_h to satisfy the following conditions:

- (1) \mathbf{v}_h is in $\mathbf{V}_h \cap \mathbf{V}$;
- (2) \mathbf{v}_h is a good approximation of \mathbf{v} ;
- (3) \mathbf{v}_h should lead to a straight-forward a posteriori error estimation, which allows us to follow well-known techniques from the a posteriori error estimations of pure Darcy and pure Stokes problems.

To satisfy the third condition, we must first investigate a posteriori error estimators for the pure Darcy and the pure Stokes equations. For the Stokes equations, we follow the proof in [26] where \mathbf{v}_h is chosen to be the Clément interpolation of \mathbf{v} . For the Darcy equation, we follow the proof in [11] where \mathbf{v}_h needs to be defined using a Helmholtz decomposition. Now the difficulty is, how to couple these two different type of definitions while ensuring that $\mathbf{v}_h \in \mathbf{V}_h \cap \mathbf{V}$? Note that \mathbf{v}_h must satisfy the strong coupling condition (2.3) across the interface Γ_{SD} .

4.1. **Defining \mathbf{v}_h .** Given $\mathbf{v} \in \mathbf{V}$, we define the Stokes side approximation $\mathbf{v}_{h,S}$ and the Darcy side approximation $\mathbf{v}_{h,D}$ separately. Then \mathbf{v}_h will be the combination of $\mathbf{v}_{h,S}$ and $\mathbf{v}_{h,D}$ as long as they satisfy

$$(4.4) \quad \mathbf{v}_{h,S} \cdot \hat{\mathbf{n}} = \mathbf{v}_{h,D} \cdot \hat{\mathbf{n}} \quad \text{on } \Gamma_{SD}.$$

On the Stokes side, the approximation can be done directly by a Clément type interpolation onto $(P_{h,S})^2 \cap \mathbf{V}_S \subset \mathbf{V}_{h,S}$. Here we pick the Scott-Zhang interpolation [41], since it also preserves the non-homogeneous boundary condition. The idea of the Scott-Zhang interpolation is to first assign, for each Lagrange interpolation point used as degrees of freedom for $P_{h,S}$, an associated integration region (see Figure 2). Then, define the interpolated value at each Lagrange point by testing the function with the dual basis in the associated integration region. For Lagrange points interior to a triangle $T \in \mathcal{T}_h^S$, the associate integration region is the triangle T . For Lagrange points lying on edges, the associate integration region is chosen to be an edge. Note for points where several edges meet, the choice may not be unique. In order to preserve the boundary condition, the associated integration region for Lagrange points lying on $\partial\Omega_S$ need to be chosen as a boundary edge on $\partial\Omega_S$. We especially note that, at the end points $\Gamma_S \cap \Gamma_{SD}$, the associated integration region needs to be chosen on Γ_S , in order to ensure the interpolated value at these points are equal to zero (see Figure 2). Denote I_h to be the Scott-Zhang interpolation mentioned above that maps $H^1(\Omega_S)$ to $P_{h,S}$, preserving the homogeneous boundary condition on Γ_S . On Γ_{SD} , I_h maps the value at the end points of Γ_{SD} into zero, while produces the interpolated value on Γ_{SD} using only the function value on Γ_{SD} . Indeed, $I_h|_{\Gamma_{SD}}$ can be viewed as a well-defined interpolation from $H_{00}^{1/2}(\Gamma_{SD})$ to $P_h|_{\Gamma_{SD}}$, by simply setting the interpolation at end-points of Γ_{SD} to be zero. Notice that $H_{00}^{1/2}(\Gamma_{SD})$ can be extended by zero on either Γ_S or Γ_D , we are able to easily make transition from Ω_S to Ω_D . That is, for $\xi \in H_{0,\Gamma_D}^1(\Omega_D)$ and consequently $\xi|_{\Gamma_{SD}} \in H_{00}^{1/2}(\Gamma_{SD})$, the interpolation $I_h|_{\Gamma_{SD}}\xi$ is also well-defined on Γ_{SD} . Furthermore, similar to the proof in [41], one can show that for any $e \in \mathcal{E}_h^{SD}$,

$$(4.5) \quad h_e \|(I - I_h|_{\Gamma_{SD}})\xi\|_e^2 \lesssim \sum_{T \in S_D(e)} h_T^2 \|\nabla \xi\|_T^2,$$

where $S_D(e)$ is the set of triangles in \mathcal{T}_h^D that have a non-empty intersection with e . In the rest of the paper, when there is no ambiguity, we will just denote $I_h|_{\Gamma_{SD}}$ by I_h .

Clearly $\mathbf{I}_h \triangleq (I_h)^2$ will map \mathbf{V}_S into $(P_{h,S})^2 \cap \mathbf{V}_S \subset \mathbf{V}_{h,S}$. It is also known [41] that \mathbf{I}_h is a projection. In other words, $\mathbf{I}_h \boldsymbol{\varphi}_S = \boldsymbol{\varphi}_S$ for all $\boldsymbol{\varphi}_S$ in $(P_{h,S})^2 \cap \mathbf{V}_S$. Define $\mathbf{v}_{h,S} = \mathbf{I}_h \mathbf{v}_S$. Since I_h is a linear operator, we have $\mathbf{v}_{h,S} \cdot \hat{\mathbf{n}} = (\mathbf{I}_h \mathbf{v}_S) \cdot \hat{\mathbf{n}} = I_h(\mathbf{v}_S \cdot \hat{\mathbf{n}})$ on Γ_{SD} .

On the Darcy side, the interpolation will be defined using a Helmholtz decomposition. That is, we first split

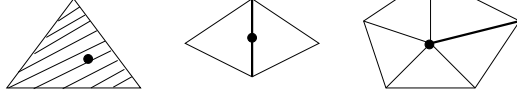
$$\mathbf{v}_D = \mathbf{w} + \mathbf{curl} \eta,$$

where

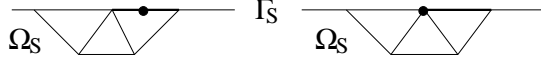
$$\mathbf{curl} \eta = \begin{pmatrix} -\frac{\partial \eta}{\partial x_2} \\ \frac{\partial \eta}{\partial x_1} \end{pmatrix},$$

FIGURE 2. Associated integration region for different type of Lagrange points. Each Lagrange point is denoted by a black dot, and the associated integration region is denoted by either a (shaded) triangle or a (bold) edge. For Lagrange points on Γ_S , Γ_{SD} , or the intersection of Γ_S and Γ_{SD} , there are special rules for choosing the associated integration region.

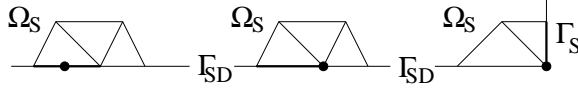
Lagrange points in Ω_S



Lagrange points on Γ_S



Lagrange points on Γ_{SD}



and \mathbf{w} satisfies

$$(4.6) \quad \nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v}_D,$$

$$(4.7) \quad \mathbf{w}|_{\Gamma_D} = \mathbf{0}, \quad \mathbf{w}|_{\Gamma_{SD}} = \mathbf{v}_S|_{\Gamma_{SD}}.$$

Here, condition (4.7) is imposed to ensure $\mathbf{w} \cdot \mathbf{n}|_{\Gamma_D} = 0$ and $\mathbf{w} \cdot \hat{\mathbf{n}}|_{\Gamma_{SD}} = \mathbf{v}_S \cdot \hat{\mathbf{n}}$. Then $\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega_D} = \mathbf{v}_D \cdot \mathbf{n}|_{\partial\Omega_D}$ and consequently the compatibility condition $\int_{\Omega_D} \nabla \cdot \mathbf{w} \, dx = \int_{\Omega_D} \nabla \cdot \mathbf{v}_D \, dx = \int_{\partial\Omega_D} \mathbf{v}_D \cdot \mathbf{n} \, ds = \int_{\partial\Omega_D} \mathbf{w} \cdot \mathbf{n} \, ds$ is satisfied.

Of course one needs to make sure such a decomposition is well-defined and \mathbf{w}, η have certain regularity results. Indeed, since $\mathbf{v}_S|_{\Gamma_{SD}} \in H_{00}^{1/2}(\Gamma_{SD})^2$, according to [4], there exists such a $\mathbf{w} \in H^1(\Omega_D)^2$ satisfying (4.6) and (4.7). Furthermore, we have

$$(4.8) \quad \|\mathbf{w}\|_{1,\Omega_D} \lesssim \|\mathbf{v}\|_{\mathbf{V}}.$$

Now $(\mathbf{v}_D - \mathbf{w})|_{\Omega_D} \in H_0(\text{div}, \Omega_D)$ is divergence-free. Thus there exists a potential function $\eta \in H_0^1(\Omega_D)$ such that $\mathbf{curl} \eta = \mathbf{v}_D - \mathbf{w}$, and

$$(4.9) \quad \|\eta\|_{1,\Omega_D} \lesssim \|\mathbf{curl} \eta\|_{0,\Omega_D} = \|\mathbf{v}_D - \mathbf{w}\|_{0,\Omega_D} \lesssim \|\mathbf{v}\|_{\mathbf{V}}.$$

Now we can start to define $\mathbf{v}_{h,D}$. First, we need an interpolation operator from $H^1(\Omega_D)^2$ to $\mathbf{V}_{h,D}$, which must map $\mathbf{w} \cdot \hat{\mathbf{n}}$ to $I_h(\mathbf{w} \cdot \hat{\mathbf{n}})$ on Γ_{SD} . Recall that a usual nodal value interpolation $\Pi_h : H^1(\Omega_D)^2 \rightarrow \mathbf{V}_{h,D}$ associated with the degrees of freedom of the RT_k elements [10] will map $\mathbf{w} \cdot \hat{\mathbf{n}}$ to $\mathbb{P}_h(\mathbf{w} \cdot \hat{\mathbf{n}})$ on Γ_{SD} , where \mathbb{P}_h is the L^2 projection onto $(\mathbf{V}_{h,D} \cdot \hat{\mathbf{n}})|_{\Gamma_{SD}} = P_h|_{\Gamma_{SD}}$. Hence, we define a new interpolation $\tilde{\Pi}_h : H^1(\Omega_D)^2 \rightarrow \mathbf{V}_{h,D}$ such that it is the same as Π_h on all other degrees of freedom except for those associated with $\mathbf{w} \cdot \hat{\mathbf{n}}|_{\Gamma_{SD}}$. On these degrees of freedom, define $\tilde{\Pi}_h$ by

$$(4.10) \quad \int_e (\tilde{\Pi}_h \mathbf{w} \cdot \hat{\mathbf{n}}) s^r \, ds = \int_e I_h(\mathbf{w} \cdot \hat{\mathbf{n}}) s^r \, ds \quad \text{for all } e \in \mathcal{E}_h^{SD} \text{ and } 0 \leq r \leq k.$$

Of course, on other degrees of freedom, $\tilde{\Pi}_h$ inherits the properties of Π_h , especially the following ones

$$(4.11) \quad \begin{aligned} \int_e (\mathbf{w} - \tilde{\Pi}_h \mathbf{w}) \cdot \mathbf{n} q_h ds &= 0 && \text{for all } e \in \mathcal{E}_h^D \text{ and } q_h \in Q_{h,D}, \\ \int_T (\mathbf{w} - \tilde{\Pi}_h \mathbf{w}) \cdot \nabla q_h dx &= 0 && \text{for all } T \in \mathcal{T}_h^D \text{ and } q_h \in Q_{h,D}. \end{aligned}$$

Combine Proposition III.3.6 in [10] and approximation property of the Scott-Zhang interpolation in [41], then use the scaling argument, Inequality (4.5), and the property of the L^2 projection, we have for all $T \in \mathcal{T}_h^D$,

$$(4.12) \quad \begin{aligned} \|\mathbf{w} - \tilde{\Pi}_h \mathbf{w}\|_{0,T} &\lesssim \|\mathbf{w} - \Pi_h \mathbf{w}\|_{0,T} + \|\tilde{\Pi}_h \mathbf{w} - \Pi_h \mathbf{w}\|_{0,T} \\ &\lesssim h_T \|\nabla \mathbf{w}\|_T + h_T^{1/2} \|(I_h - \mathbb{P}_h)(\mathbf{w} \cdot \hat{\mathbf{n}})\|_{T \cap \Gamma_{SD}} \\ &\lesssim \left(\sum_{T \in S_D(T)} h_T^2 \|\nabla \mathbf{w}\|_T^2 \right)^{1/2} \end{aligned}$$

where $S_D(T)$ is the set of all triangles in \mathcal{T}_h^D that has a non-empty intersection with $T \cap \Gamma_{SD}$.

Different from Π_h , which satisfies $\nabla \cdot (\Pi_h \mathbf{w}) = \mathbb{Q}_h(\nabla \cdot \mathbf{w})$ (see [10]), where \mathbb{Q}_h is the L^2 projection onto $Q_{h,D}$, $\tilde{\Pi}_h$ does not satisfy the same relation. Instead, for all $q_h \in Q_{h,D}$, by the definition of $\tilde{\Pi}_h$ and its properties (4.10)-(4.11), we have

$$(4.13) \quad \begin{aligned} &(\nabla \cdot (\mathbf{w} - \tilde{\Pi}_h \mathbf{w}), q_h)_{\Omega_D} \\ &= \sum_{T \in \mathcal{T}_h^D} \left(\langle (\mathbf{w} - \tilde{\Pi}_h \mathbf{w}) \cdot \mathbf{n}, q_h \rangle_{\partial T} - (\mathbf{w} - \tilde{\Pi}_h \mathbf{w}, \nabla q_h)_T \right) \\ &= - \langle (\mathbf{w} - \tilde{\Pi}_h \mathbf{w}) \cdot \hat{\mathbf{n}}, q_h \rangle_{\mathcal{E}_h^{SD}} \\ &= - \langle (I - \mathbf{I}_h) \mathbf{v}_S \cdot \hat{\mathbf{n}}, q_h \rangle_{\mathcal{E}_h^{SD}}. \end{aligned}$$

Finally, define

$$\mathbf{v}_{h,D} = \tilde{\Pi}_h \mathbf{w} + \mathbf{curl} \eta_h,$$

where η_h is the Clément interpolation of η into the continuous piecewise P_{k+1} polynomials on \mathcal{T}_h^D that preserves the homogeneous boundary condition on $\partial\Omega_D$. Of course one can also chose η_h to be the Scott-Zhang interpolation. By the properties of the Raviart-Thomas elements [10], it is easy to see that $\mathbf{v}_{h,D} \in \mathbf{V}_{h,D}$. Furthermore, $\mathbf{v}_{h,S}$ and $\mathbf{v}_{h,D}$ satisfy the interface coupling condition (4.4) and hence $\mathbf{v}_h \in \mathbf{V}_h \cap \mathbf{V}$. By the approximation properties of the Scott-Zhang interpolation \mathbf{I}_h and the Clément interpolation, we have

$$(4.14) \quad \begin{aligned} \sum_{T \in \mathcal{T}_h^S} \left(\|\mathbf{v}_S - \mathbf{v}_{h,S}\|_T^2 + h_T^2 |\mathbf{v}_S - \mathbf{v}_{h,S}|_{1,T}^2 \right) &\lesssim \sum_{T \in \mathcal{T}_h^S} h_T^2 |\mathbf{v}|_{1,T}^2, \\ \sum_{T \in \mathcal{T}_h^D} \left(\|\eta - \eta_h\|_T^2 + h_T^2 |\eta - \eta_h|_{1,T}^2 \right) &\lesssim \sum_{T \in \mathcal{T}_h^D} h_T^2 |\eta|_{1,T}^2. \end{aligned}$$

Now we are ready to derive upper bounds, or equivalently the a posteriori error estimators, for R_D , R_S and R_I .

4.2. Deriving the Darcy estimator. By the definition of $a_D(\cdot, \cdot)$, Equation (4.13), and the Schwarz inequality,

$$\begin{aligned}
R_D &= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} - \left(a_D(\tilde{\mathbf{u}}_h, \mathbf{v}) - a_D(\mathbf{u}_h, \mathbf{v}_h) \right) + (\nabla \cdot (\mathbf{v} - \mathbf{v}_h), p_h)_{\Omega_D} \\
&= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} - (\mathbb{K}^{-1}(\tilde{\mathbf{u}}_h - \mathbf{u}_h), \mathbf{v})_{\Omega_D} - (\mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} \\
&\quad + (\nabla \cdot (\mathbf{v} - \mathbf{v}_h), p_h)_{\Omega_D} \\
&= (\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} - (\mathbb{K}^{-1}(\tilde{\mathbf{u}}_h - \mathbf{u}_h), \mathbf{v})_{\Omega_D} \\
&\quad - \langle (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{n}}, p_{h,D} \rangle_{\mathcal{E}_h^{SD}} \\
&\lesssim (\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\| \|\mathbf{v}\|_{\Omega_D} \\
&\quad - \langle (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{n}}, p_{h,D} \rangle_{\mathcal{E}_h^{SD}}.
\end{aligned}$$

Next, notice that

$$(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{v} - \mathbf{v}_h)_{\Omega_D} = (\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{w} - \tilde{\Pi}_h \mathbf{w})_{\Omega_D} + (\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{curl}(\eta - \eta_h))_{\Omega_D},$$

where by (4.11), (4.12) and (4.8),

$$\begin{aligned}
(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{w} - \tilde{\Pi}_h \mathbf{w})_{\Omega_D} &\lesssim \inf_{\tilde{p}_h \in Q_{h,D}} \|h_T(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h - \nabla \tilde{p}_h)\|_{\mathcal{T}_h^D} |\mathbf{w}|_{1, \Omega_D} \\
&\lesssim \inf_{\tilde{p}_h \in Q_{h,D}} \|h_T(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h - \nabla \tilde{p}_h)\|_{\mathcal{T}_h^D} \|\mathbf{v}\|_{\mathbf{V}},
\end{aligned}$$

and by (4.1), (4.9), integration by parts, the boundary condition of η , and (4.14)

$$\begin{aligned}
&(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h, \mathbf{curl}(\eta - \eta_h)) \\
&\lesssim \left(\|h_T \mathbf{curl}(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h)\|_{\mathcal{T}_h^D} + \|h_e^{1/2}[\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h] \cdot \mathbf{t}\|_{\mathcal{E}_{0,h}^D} \right) |\eta|_{1, \Omega_D} \\
&\lesssim \left(\|h_T \mathbf{curl}(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h)\|_{\mathcal{T}_h^D} + \|h_e^{1/2}[\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h] \cdot \mathbf{t}\|_{\mathcal{E}_{0,h}^D} \right) \|\mathbf{v}\|_{\mathbf{V}}.
\end{aligned}$$

Here \mathbf{curl} is defined for any vector-valued function $\boldsymbol{\xi} = (\xi_1, \xi_2)^t$ by

$$\mathbf{curl} \boldsymbol{\xi} = -\frac{\partial \xi_1}{\partial x_2} + \frac{\partial \xi_2}{\partial x_1}.$$

Combining the above and using (4.2), we have

$$\begin{aligned}
R_D &\lesssim \left(\inf_{\tilde{p}_h \in Q_{h,D}} \|h_T(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h - \nabla \tilde{p}_h)\|_{\mathcal{T}_h^D} + \|h_T \mathbf{curl}(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h)\|_{\mathcal{T}_h^D} \right. \\
&\quad \left. + \|h_e^{1/2}[\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h] \cdot \mathbf{t}\|_{\mathcal{E}_{0,h}^D} + \|h_e^{-1/2}[\mathbf{u}_h]\|_{\mathcal{E}_h^S} \right) \|\mathbf{v}\|_{\mathbf{V}} \\
&\quad - \langle (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{n}}, p_{h,D} \rangle_{\mathcal{E}_h^{SD}}.
\end{aligned}$$

4.3. Deriving the Stokes estimator. Note that $[\mathbf{v}_h] = 0$ on all $e \in \mathcal{E}_h^S$. By using the definition of $a_S(\cdot, \cdot)$, $a_{h,S}(\cdot, \cdot)$, and \mathbb{T} , we have

$$\begin{aligned}
 R_S &= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_S} - \left(a_S(\tilde{\mathbf{u}}_h, \mathbf{v}) - a_{S,h}(\mathbf{u}_h, \mathbf{v}_h) \right) + (\nabla \cdot (\mathbf{v} - \mathbf{v}_h), p_h)_{\Omega_S} \\
 &= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_S} + (2\nu D(\mathbf{u}_h - \tilde{\mathbf{u}}_h), D(\mathbf{v}))_{\mathcal{T}_h^S} \\
 &\quad - \left((2\nu D(\mathbf{u}_h), D(\mathbf{v}))_{\mathcal{T}_h^S} - a_{S,h}(\mathbf{u}_h, \mathbf{v}_h) - (p_h \mathbf{I}, \nabla(\mathbf{v} - \mathbf{v}_h))_{\mathcal{T}_h^S} \right) \\
 &= (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_S} + (2\nu D(\mathbf{u}_h - \tilde{\mathbf{u}}_h), D(\mathbf{v}))_{\mathcal{T}_h^S} \\
 &\quad - (\mathbb{T}(\mathbf{u}_h, p_h), D(\mathbf{v} - \mathbf{v}_h))_{\mathcal{T}_h^S} - 2\nu \langle [\mathbf{u}_h], \{D(\mathbf{v}_h)\} \mathbf{n} \rangle_{\mathcal{E}_h^S} \\
 &\lesssim (\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_S} - (\mathbb{T}(\mathbf{u}_h, p_h), \nabla(\mathbf{v} - \mathbf{v}_h))_{\mathcal{T}_h^S} \\
 &\quad + \|\mathbf{u}_h - \tilde{\mathbf{u}}_h\|_{\mathbf{V}_h} \|\mathbf{v}\|_{\mathbf{V}} - 2\nu \langle [\mathbf{u}_h], \{D(\mathbf{v}_h)\} \mathbf{n} \rangle_{\mathcal{E}_h^S}.
 \end{aligned}$$

In the above we have used the algebraic relation that for any symmetric tensor τ and domain K , $(\tau, \nabla(\mathbf{v} - \mathbf{v}_h))_K = (\tau, D(\mathbf{v} - \mathbf{v}_h))_K$.

Using integration by parts and (4.1), (4.14),

$$\begin{aligned}
 &(\mathbf{f}, \mathbf{v} - \mathbf{v}_h)_{\Omega_S} - (\mathbb{T}(\mathbf{u}_h, p_h), \nabla(\mathbf{v} - \mathbf{v}_h))_{\mathcal{T}_h^S} \\
 &= (\mathbf{f} + \nabla \cdot \mathbb{T}(\mathbf{u}_h, p_h), \mathbf{v} - \mathbf{v}_h)_{\mathcal{T}_h^S} - \sum_{T \in \mathcal{T}_h^S} \langle \mathbb{T}(\mathbf{u}_h, p_h) \mathbf{n}, \mathbf{v} - \mathbf{v}_h \rangle_{\partial T} \\
 &\lesssim \left(\|h_T(\mathbf{f} + \nabla \cdot \mathbb{T}(\mathbf{u}_h, p_h))\|_{\mathcal{T}_h^S} + \|h_e^{1/2}[\mathbb{T}(\mathbf{u}_h, p_h)] \mathbf{n}\|_{\mathcal{E}_{0,h}^S} \right) \|\mathbf{v}\|_{\mathbf{V}} \\
 &\quad - \langle \mathbb{T}(\mathbf{u}_h, p_h) \hat{\mathbf{n}}, (I - \mathbf{I}_h) \mathbf{v}_S \rangle_{\mathcal{E}_h^{SD}}.
 \end{aligned}$$

Combine all the above and using (4.2), we have

$$\begin{aligned}
 R_S &\lesssim \left(\|h_T(\mathbf{f} + \nabla \cdot \mathbb{T}(\mathbf{u}_h, p_h))\|_{\mathcal{T}_h^S} + \|h_e^{1/2}[\mathbb{T}(\mathbf{u}_h, p_h)] \mathbf{n}\|_{\mathcal{E}_{0,h}^S} \right. \\
 &\quad \left. + \|h_e^{-1/2}[\mathbf{u}_h]\|_{\mathcal{E}_h^S} \right) \|\mathbf{v}\|_{\mathbf{V}} - \langle \mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) \hat{\mathbf{n}}, (I - \mathbf{I}_h) \mathbf{v}_S \rangle_{\mathcal{E}_h^{SD}}.
 \end{aligned}$$

4.4. Deriving the interface estimator. By the definition of $a_I(\cdot, \cdot)$ and using the Schwarz inequality, inequalities (4.1), (4.2), we have

$$\begin{aligned}
 R_I &= -a_I(\tilde{\mathbf{u}}_h, \mathbf{v}) + a_I(\mathbf{u}_h, \mathbf{v}_h) \\
 &= - \langle \mu \mathbb{K}^{-1/2}(\tilde{\mathbf{u}}_{h,S} - \mathbf{u}_{h,S}) \cdot \hat{\mathbf{t}}, \mathbf{v}_S \cdot \hat{\mathbf{t}} \rangle_{\mathcal{E}_h^{SD}} \\
 &\quad - \langle \mu \mathbb{K}^{-1/2} \mathbf{u}_{h,S} \cdot \hat{\mathbf{t}}, (\mathbf{v}_S - \mathbf{v}_{h,S}) \cdot \hat{\mathbf{t}} \rangle_{\mathcal{E}_h^{SD}} \\
 &\leq \|\tilde{\mathbf{u}}_h - \mathbf{u}_h\|_{\mathbf{V}_h} \|\mathbf{v}\|_{\mathbf{V}_h} - \langle \mu \mathbb{K}^{-1/2} \mathbf{u}_{h,S} \cdot \hat{\mathbf{t}}, (I - \mathbf{I}_h) \mathbf{v}_S \cdot \hat{\mathbf{t}} \rangle_{\mathcal{E}_h^{SD}} \\
 &\lesssim \|h_e^{-1/2}[\mathbf{u}_h]\|_{\mathcal{E}_h^S} \|\mathbf{v}\|_{\mathbf{V}} - \langle \mu \mathbb{K}^{-1/2} \mathbf{u}_{h,S} \cdot \hat{\mathbf{t}}, (I - \mathbf{I}_h) \mathbf{v}_S \cdot \hat{\mathbf{t}} \rangle_{\mathcal{E}_h^{SD}}.
 \end{aligned}$$

4.5. Estimator for the coupled problem. Finally, by adding R_D , R_S and R_I together, we have

$$\begin{aligned}
Res_1(\mathbf{v}) &\lesssim \left(\inf_{\tilde{p}_h \in Q_{h,D}} \|h_T(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h - \nabla\tilde{p}_h)\|_{\mathcal{T}_h^D} + \|h_T \operatorname{curl}(\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h)\|_{\mathcal{T}_h^D} \right. \\
&\quad + \|h_e^{1/2}[\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h] \cdot \hat{\mathbf{t}}\|_{\mathcal{E}_{0,h}^D} + \|h_e^{-1/2}[\mathbf{u}_h]\|_{\mathcal{E}_h^S} \\
&\quad \left. + \|h_T(\mathbf{f} + \nabla \cdot \mathbb{T}(\mathbf{u}_h, p_h))\|_{\mathcal{T}_h^S} + \|h_e^{1/2}[\mathbb{T}(\mathbf{u}_h, p_h)]\mathbf{n}\|_{\mathcal{E}_{0,h}^S} \right)^{1/2} \|\mathbf{v}\|_{\mathbf{V}} \\
&\quad - \langle (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{n}}, p_{h,D} \rangle_{\mathcal{E}_h^{SD}} - \langle \mathbb{T}(\mathbf{u}_{h,S}, p_{h,S})\hat{\mathbf{n}}, (I - \mathbf{I}_h)\mathbf{v}_S \rangle_{\mathcal{E}_h^{SD}} \\
&\quad - \langle \mu\mathbb{K}^{-1/2}\mathbf{u}_{h,S} \cdot \hat{\mathbf{t}}, (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{t}} \rangle_{\mathcal{E}_h^{SD}}.
\end{aligned}$$

Then, using (4.1) and (4.14),

$$\begin{aligned}
&\langle (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{n}}, p_{h,D} \rangle_{\mathcal{E}_h^{SD}} + \langle \mathbb{T}(\mathbf{u}_{h,S}, p_{h,S})\hat{\mathbf{n}}, (I - \mathbf{I}_h)\mathbf{v}_S \rangle_{\mathcal{E}_h^{SD}} \\
&\quad + \langle \mu\mathbb{K}^{-1/2}\mathbf{u}_{h,S} \cdot \hat{\mathbf{t}}, (I - \mathbf{I}_h)\mathbf{v}_S \cdot \hat{\mathbf{t}} \rangle_{\mathcal{E}_h^{SD}} \\
&= \langle \mathbb{T}(\mathbf{u}_{h,S}, p_{h,S})\hat{\mathbf{n}} + p_{h,D}\hat{\mathbf{n}} + \mu\mathbb{K}^{-1/2}(\mathbf{u}_{h,S} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}, (I - \mathbf{I}_h)\mathbf{v}_S \rangle_{\mathcal{E}_h^{SD}} \\
&\lesssim \left\| h_e^{1/2} \left(\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S})\hat{\mathbf{n}} + p_{h,D}\hat{\mathbf{n}} + \mu\mathbb{K}^{-1/2}(\mathbf{u}_{h,S} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}} \right) \right\|_{\mathcal{E}_h^{SD}} \|\mathbf{v}\|_{\mathbf{V}}.
\end{aligned}$$

Combining the estimation for $Res_1(\mathbf{v})$ with the estimation (4.3) for $Res_2(q)$, and setting $\tilde{p}_h = p_h$. We can now construct an a posteriori error estimator for Problem (3.1). Let \mathbf{f}_T and \mathbf{f}_e be the L^2 projection of \mathbf{f} on a triangle T and an edge e , respectively onto the space of k th order polynomials. We define the a posteriori error estimator for the coupled Darcy-Stokes equation as following:

(1) for $T \in \mathcal{T}_h^S$

$$\begin{aligned}
\eta_{T,S}^2 &= h_T^2 \|\mathbf{f}_T + \nabla \cdot \mathbb{T}(\mathbf{u}_h, p_h)\|_T^2 + \frac{1}{2} \sum_{e \in \mathcal{E}_{0,h}^S \cap \partial T} h_e \|\mathbb{T}(\mathbf{u}_h, p_h)\mathbf{n}\|_e^2 \\
&\quad + \frac{1}{2} \sum_{e \in \mathcal{E}_{0,h}^S \cap \partial T} h_e^{-1} \|\mathbf{u}_{h,S}\|_e^2 + \sum_{e \in \partial T \cap \Gamma_S} h_e^{-1} \|\mathbf{u}_{h,S}\|_e^2 + \|g - \nabla \cdot \mathbf{u}_h\|_T^2,
\end{aligned}$$

(2) for $T \in \mathcal{T}_h^D$

$$\begin{aligned}
\eta_{T,D}^2 &= h_T^2 \|\mathbf{f}_T - \mathbb{K}^{-1}\mathbf{u}_h - \nabla p_h\|_T^2 + h_T^2 \|\operatorname{curl}(\mathbf{f}_T - \mathbb{K}^{-1}\mathbf{u}_h)\|_T^2 \\
&\quad + \frac{1}{2} \sum_{e \in \mathcal{E}_{0,h}^D \cap \partial T} h_e \|\mathbf{f}_e - \mathbb{K}^{-1}\mathbf{u}_h\|_e^2 + \|g - \nabla \cdot \mathbf{u}_h\|_T^2,
\end{aligned}$$

(3) for $e \in \mathcal{E}_h^{SD}$

$$\eta_{e,SD}^2 = h_e \|\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S})\hat{\mathbf{n}} + p_{h,D}\hat{\mathbf{n}} + \mu\mathbb{K}^{-1/2}(\mathbf{u}_{h,S} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}\|_e^2.$$

Then the global a posteriori error estimator is

$$\eta^2 = \sum_{T \in \mathcal{T}_h^S} \eta_{T,S}^2 + \sum_{T \in \mathcal{T}_h^D} \eta_{T,D}^2 + \sum_{e \in \mathcal{E}_h^{SD}} \eta_{e,SD}^2.$$

In practice, one may distribute the value of $\eta_{e,SD}$ by certain formula on the two triangles sharing edge e , where one triangle is in Ω_S and another in Ω_D . This shall give a functioning adaptive refinement strategy. Of course one can also design more specific refinement strategy that uses $\eta_{e,SD}$ directly. Here we do not move further

into the adaptive refinement strategies, since we are only interested in the global upper and lower bounds for η .

To conclude this section, in the above we have constructed and proved the reliability of the a posteriori estimator η , that is

Theorem 4.1. *Let $\boldsymbol{\varepsilon}_u$, $\boldsymbol{\varepsilon}_p$ and η be defined as in this section, then*

$$\|(\boldsymbol{\varepsilon}_u, \boldsymbol{\varepsilon}_p)\|_h \lesssim \eta + \mathcal{R}(\mathbf{f}),$$

where $\mathcal{R}(\mathbf{f}, g)$ is the higher order oscillation term

$$\mathcal{R}(\mathbf{f}) = \|h_T(\mathbf{f} - \mathbf{f}_T)\|_{\mathcal{T}_h} + \|h_T \operatorname{curl}(\mathbf{f} - \mathbf{f}_T)\|_{\mathcal{T}_h^D} + \|h_e^{1/2}(\mathbf{f} - \mathbf{f}_e) \cdot \mathbf{t}\|_{\mathcal{E}_{0,h}^D}.$$

5. EFFICIENCY OF THE A POSTERIORI ERROR ESTIMATOR

The a posteriori error estimator is consider efficient if it also satisfies

$$(5.1) \quad \eta \lesssim \|(\boldsymbol{\varepsilon}_u, \boldsymbol{\varepsilon}_p)\|_h + \mathcal{R}(\mathbf{f}).$$

In this section, we shall prove this.

By examining $\eta_{T,S}$ and $\eta_{T,D}$, we immediately realize that all terms are either entirely interior to the Darcy side or to the Stokes side. In other words, when using the standard technique of defining bubble functions, the support of each bubble function is contained either in Ω_S or Ω_D . Thus, to prove

$$(5.2) \quad \sum_{T \in \mathcal{T}_h^S} \eta_{T,S}^2 + \sum_{T \in \mathcal{T}_h^D} \eta_{T,D}^2 \lesssim \|(\boldsymbol{\varepsilon}_u, \boldsymbol{\varepsilon}_p)\|_h^2 + \mathcal{R}(\mathbf{f})^2,$$

it suffices to use only the Darcy equation or only the Stokes equation. The proof will be exactly the same as the proof for pure Darcy and pure Stokes equations. Reader can refer to [11, 26, 45] for details.

Now we only need to prove the upper bound for $\sum_{e \in \mathcal{E}_h^{SD}} \eta_{e,SD}^2$. The proof is very similar to the proof of Lemma 4.5 in [5]. Below are given the details.

For each $e \in \mathcal{E}_h^{SD}$, define an edge bubble function ϕ_e which has support only in the two triangles sharing e . Let T_e^S and T_e^D be the triangles in \mathcal{T}_h^S and \mathcal{T}_h^D , respectively, that contain edge e , and $L : P_k(e) \rightarrow P_k(T_e^S)$ be an extension such that $L(q)|_e = q$ for all $q \in P_k(e)$. One may refer to [44] for the definition of ϕ_e , L and the proof of the following properties:

- For any polynomial q with degree at most m , there exist positive constants d_m , D_m and E_m , depending only on m , such that

$$(5.3) \quad d_m \|q\|_e^2 \leq \int_e q^2 \phi_e ds \leq D_m \|q\|_e^2,$$

$$(5.4) \quad \|L(q)\phi_e\|_{T_e^S} \leq E_m h_e^{1/2} \|q\|_e.$$

Denote $\mathbf{L} = (L)^2$ which maps $P_k(e)^2$ to $P_k(T_e^S)^2$.

Denote $\chi_e = \mathbb{T}(\mathbf{u}_{h,S}, p_{h,S})\hat{\mathbf{n}} + p_{h,D}\hat{\mathbf{n}} + \mu\mathbb{K}^{-1/2}(\mathbf{u}_{h,S} \cdot \hat{\mathbf{t}})\hat{\mathbf{t}}$ on $e \in \mathcal{E}_h^{SD}$. Then using (2.6) and (5.3),

$$\begin{aligned} \|\chi_e\|_e^2 &\lesssim \int_e \chi_e^2 \phi_e ds \\ &= \int_e \chi_e \phi_e \left(\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) - \mathbb{T}(\mathbf{u}_S, p_S) \right) \hat{\mathbf{n}} + (p_{h,D} - p_D) \hat{\mathbf{n}} \\ &\quad + \mu\mathbb{K}^{-1/2}((\mathbf{u}_{h,S} - \mathbf{u}_S) \cdot \hat{\mathbf{t}}) \hat{\mathbf{t}} ds \\ &\lesssim \int_e \chi_e \phi_e (\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) - \mathbb{T}(\mathbf{u}_S, p_S)) \hat{\mathbf{n}} ds \\ &\quad + \|\chi_e\|_e \left(\|p_{h,D} - p_D\|_e + \|\mu\mathbb{K}^{-1/2}(\mathbf{u}_{h,S} - \mathbf{u}_S) \cdot \hat{\mathbf{t}}\|_e \right). \end{aligned}$$

Then, using the support of ϕ_e , the inverse inequality and Inequality (5.4),

$$\begin{aligned} &\int_e \chi_e \phi_e (\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) - \mathbb{T}(\mathbf{u}_S, p_S)) \hat{\mathbf{n}} ds \\ &= \int_{T_e^S} \nabla(\mathbf{L}(\chi_e)\phi_e) (\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) - \mathbb{T}(\mathbf{u}_S, p_S)) dx \\ &\quad + \int_{T_e^S} \mathbf{L}(\chi_e)\phi_e \nabla \cdot (\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) - \mathbb{T}(\mathbf{u}_S, p_S)) dx \\ &\lesssim h_e^{1/2} \|\chi_e\|_e \left(h_{T_e^S}^{-1} \|\mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) - \mathbb{T}(\mathbf{u}_S, p_S)\|_{T_e^S} + \|\nabla \cdot \mathbb{T}(\mathbf{u}_{h,S}, p_{h,S}) + \mathbf{f}\|_{T_e^S} \right) \\ &\lesssim h_e^{-1/2} \|\chi_e\|_e \left(\|\nabla \boldsymbol{\varepsilon}_u\|_{T_e^S} + \|\boldsymbol{\varepsilon}_p\|_{T_e^S} + \eta_{T_e^S, S} + h_{T_e^S} \|\mathbf{f} - \mathbf{f}_T\|_{T_e^S} \right). \end{aligned}$$

Using (4.1), we have

$$\begin{aligned} &\|p_D - p_{h,D}\|_e \\ &\lesssim h_e^{-1/2} \|p_D - p_{h,D}\|_{T_e^D} + h_e^{1/2} \|\nabla(p_D - p_{h,D})\|_{T_e^D} \\ &= h_e^{-1/2} \|\boldsymbol{\varepsilon}_p\|_{T_e^D} + h_e^{1/2} \|\mathbf{f} - \mathbb{K}^{-1}\mathbf{u} - \nabla p_{h,D}\|_{T_e^D} \\ &\lesssim h_e^{-1/2} \|\boldsymbol{\varepsilon}_p\|_{T_e^D} + h_e^{1/2} \|\mathbf{f} - \mathbb{K}^{-1}\mathbf{u}_h - \nabla p_{h,D}\|_{T_e^D} + h_e^{1/2} \|\mathbb{K}^{-1}(\mathbf{u} - \mathbf{u}_h)\|_{T_e^D} \\ &\lesssim h_e^{-1/2} \left(\|\boldsymbol{\varepsilon}_p\|_{T_e^D} + \eta_{T_e^D, D} + h_e \|\mathbf{f} - \mathbf{f}_{T_e^D}\|_{T_e^D} + h_e \|\boldsymbol{\varepsilon}_u\|_{T_e^D} \right). \end{aligned}$$

Combining the above, using (5.2), the definition of $\|\cdot\|_h$ and $\mathcal{R}(\mathbf{f})$, we have

Theorem 5.1. *The a posteriori error estimator η satisfies (5.1).*

Remark 5.2. In both Theorem 4.1 and 5.1, the constant contained in “ \lesssim ” may depend on σ , the stabilization parameter in the definition of the bilinear form $a_{S,h}(\cdot, \cdot)$. However, since σ is of $O(1)$ and does not depend on h , its effect on the stability and efficiency of the a posteriori error estimator η is restricted.

APPENDIX A. DEFINITION AND PROPERTIES OF $\tilde{\mathbf{u}}_h \in \mathbf{V}_h \cap \mathbf{V}$

Given $\mathbf{u}_h \in \mathbf{V}_h$, here we define $\tilde{\mathbf{u}}_h \in \mathbf{V}$ satisfying (4.2). Note that $\tilde{\mathbf{u}}_h$ is not necessarily in \mathbf{V}_h . It is a commonly used technique to introduce such a $\tilde{\mathbf{u}}_h$ in a posteriori error estimations for nonconforming or discontinuous Galerkin methods.

Readers may refer to [12] and reference therein for similar usage. In [12], $\tilde{\mathbf{u}}_h$ is constructed using the Helmholtz decomposition. Here we can not borrow their results directly, for two reasons. First, we need an estimation of $\mathbf{u}_h - \tilde{\mathbf{u}}_h$ in the \mathbf{V}_h norm while the construction in [12] only provides a broken H^1 semi-norm estimation. Second, special treatment needs to be taken in order to ensure that $\tilde{\mathbf{u}}_h$ satisfies the interface condition (2.3) strongly.

Noticing that in each $T \in \mathcal{T}_h$, \mathbf{u}_h is a polynomial with degree less than or equal to $k + 1$. Denote $P_{k+1}(\mathcal{T}_h, S)$ to be the H^1 conforming discrete functional space which consists of piecewise polynomials of degree up to $k + 1$ on each $T \in \mathcal{T}_{h,S}$. We define $\tilde{\mathbf{u}}_h$ as following (as partly illustrated in Figure 3):

- (1) $\tilde{\mathbf{u}}_{h,S} \in P_{k+1}(\mathcal{T}_{h,S})^2$ is defined by setting its values on all $(k + 1)$ st order Lagrange interpolation points in $T \in \mathcal{T}_h^S$. At Lagrange points interior to any $T \in \mathcal{T}_h^S$, its value is inherited from the value of $\mathbf{u}_{h,S}$. At Lagrange points on Γ_S , including $\Gamma_S \cap \Gamma_{SD}$, the value is set to be zero. At Lagrange points located on edges in $\mathcal{E}_{0,h}^S \cup \mathcal{E}_h^{SD}$ but not on Γ_S , define the value of $\tilde{\mathbf{u}}_{h,S}$ to be the value of $\mathbf{u}_{h,S}$ from a prescribed triangle among all triangles in \mathcal{T}_h^S sharing this Lagrange point. Note that by such a definition, the values at Lagrange points on Γ_{SD} are set by using only the Stokes side solution $\mathbf{u}_{h,S}$.
- (2) Now $\tilde{\mathbf{u}}_{h,S}$ has been defined. Next, define $\tilde{\mathbf{u}}_{h,D}$ in the $H(\text{div})$ conforming RT_{k+1} space on $\mathcal{T}_{h,D}$ by copying the values of $\mathbf{u}_{h,D}$ on all degrees of freedom except for those associated with $e \in \mathcal{E}_h^{SD}$, namely, the degrees of freedom defined by

$$\int_e (\tilde{\mathbf{u}}_{h,D} \cdot \mathbf{n}) s^r ds \quad \text{for all } e \in \mathcal{E}_h^{SD} \text{ and } 0 \leq r \leq k + 1.$$

At these degrees of freedom, to make sure that $\tilde{\mathbf{u}}_{h,S} \cdot \hat{\mathbf{n}} = \tilde{\mathbf{u}}_{h,D} \cdot \hat{\mathbf{n}}$ on Γ_{SD} , we define

$$\int_e (\tilde{\mathbf{u}}_{h,D} \cdot \hat{\mathbf{n}}) s^r ds = \int_e (\tilde{\mathbf{u}}_{h,S} \cdot \hat{\mathbf{n}}) s^r ds.$$

Clearly, $\tilde{\mathbf{u}}_h$ defined as above is in \mathbf{V} , but not \mathbf{V}_h . We have the following lemma:

Lemma A.1. For all $T \in \mathcal{T}_h^S$,

$$(A.1) \quad \|\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}\|_T^2 + h_T^2 \|\nabla(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})\|_T^2 \lesssim \sum_{e \in \mathcal{E}_h^S(T)} h_e \|\mathbf{u}_{h,S}\|_e^2,$$

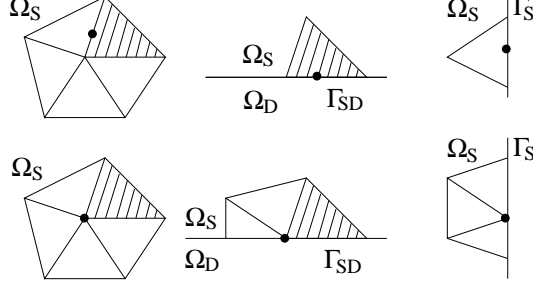
where $\mathcal{E}_h^S(T)$ denotes the set of edges in \mathcal{E}_h^S who have non-empty intersections with T . We especially point out that, benefited from the definition of $\tilde{\mathbf{u}}_{h,S}$, $\mathcal{E}_h^S(T)$ does not contain edges that lie on \mathcal{E}_h^{SD} .

Proof. The proof follows from a routine scaling argument and the fact that all norms on finite dimensional spaces are equivalent. To this end, we observe that

$$\|\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}\|_T^2 + h_T^2 \|\nabla(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})\|_T^2 \lesssim h_T^2 \sum_{x_j \in G_k(T)} |(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})(x_j)|^2,$$

where $G_k(T)$ is the set of all $(k + 1)$ st order Lagrange interpolation points in T and $|\cdot|$ denotes the Euclidean norm of a vector. It follows from the definition of $\tilde{\mathbf{u}}_h$ that $\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}$ vanishes at all internal Lagrange points in T . We only need to examine the value of $\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}$ at Lagrange points on ∂T . There are several different cases as illustrated in Figure 3.

FIGURE 3. Setting the values of $\tilde{\mathbf{u}}_{h,S}$ at different type of Lagrange points on edges. For each Lagrange point, the shaded triangle means it is the designated triangle that defines the value of $\tilde{\mathbf{u}}_{h,S}$ on this Lagrange point. On Lagrange points on Γ_S , including the intersection of Γ_S and Γ_{SD} , the value of $\tilde{\mathbf{u}}_{h,S}$ is simply set to be zero.



At Lagrange points on edges in $\mathcal{E}_{0,h}^S \cup \mathcal{E}_h^{SD}$ but not on Γ_S , there are two possibilities: (1) x_j is in the interior of an edge e ; (2) x_j is a vertex of T . In the first case, we see that $|(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})(x_j)|$ is either 0 or $||[\mathbf{u}_{h,S}]_e(x_j)|$, where $[\cdot]_e$ denotes the jump on edge e , on the two triangles sharing edge e . Furthermore, if x_j lies in the interior of an edge on Γ_{SD} , then $|(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})(x_j)| = 0$. In the second case, we can use the triangle inequality to traverse through all edges $e \in \mathcal{E}_{0,h}^S$ that has x_j as one end point, which we shall denote as $e \in \mathcal{E}_h^S(x_j)$, and to obtain

$$(A.2) \quad |(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})(x_j)| \leq \sum_{e \in \mathcal{E}_h^S(x_j)} |[\mathbf{u}_{h,S}]_e(x_j)|.$$

Notice that $\mathcal{E}_h^S(x_j)$ does not contain edges in \mathcal{E}_h^{SD} . Finally, consider Lagrange points on Γ_S . Clearly for all x_j in the interior of edge $e \subset \Gamma_S$,

$$|(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})(x_j)| = |\mathbf{u}_{h,S}(x_j)| = |[\mathbf{u}_{h,S}]_e(x_j)|.$$

For x_j at the end of edge $e \subset \Gamma_S$, again by traversing through all edges $e \in \mathcal{E}_{0,h}^S$, we have Inequality (A.2).

Combining the above analysis, we have for all $T \in \mathcal{T}_h^S$

$$\sum_{x_j \in G_k(T)} |(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})(x_j)|^2 \lesssim \sum_{e \in \mathcal{E}_h^S(T)} \sum_{x_j \in G_k(e)} |[\mathbf{u}_{h,S}]_e(x_j)|^2,$$

where $G_k(e)$ denotes the corresponding Lagrange points on edge e . Then, using the routine scaling argument on edges, inequality (A.1) follows immediately. \square

Using Lemma A.1, inequalities (4.1) and (A.1), we have

$$\begin{aligned} & \|\mu^{1/2} \mathbb{K}^{-1/4} (\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}) \cdot \hat{\mathbf{t}}\|_{\mathcal{E}_h^{SD}}^2 \\ & \lesssim \sum_{T \in \mathcal{T}_h^S} \left(h_T^{-1} \|\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}\|_T^2 + h_T \|\nabla(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S})\|_T^2 \right) \\ & \lesssim \|[\mathbf{u}_{h,S}]\|_{\mathcal{E}_h^S}^2 \lesssim \|h_e^{-1/2} [\mathbf{u}_{h,S}]\|_{\mathcal{E}_h^S}^2. \end{aligned}$$

Next, consider the Darcy side. Clearly $\mathbf{u}_{h,D} - \tilde{\mathbf{u}}_{h,D}$ is only non-zero on triangles who has at least an edge on Γ_{SD} . Using the definition of $\tilde{\mathbf{u}}_h$, the scaling argument, the normal direction continuity on Γ_{SD} , inequalities (4.1) and (A.1), we have on such triangles

$$\begin{aligned} & \|\mathbf{u}_{h,D} - \tilde{\mathbf{u}}_{h,D}\|_T^2 + h_T^2 \|\nabla \cdot (\mathbf{u}_{h,D} - \tilde{\mathbf{u}}_{h,D})\|_T^2 \\ & \lesssim \sum_{e \subset \bar{T} \cap \Gamma_{SD}} h_e \|(\mathbf{u}_{h,D} - \tilde{\mathbf{u}}_{h,D}) \cdot \hat{\mathbf{n}}\|_e^2 \\ & = \sum_{e \subset \bar{T} \cap \Gamma_{SD}} h_e \|(\mathbf{u}_{h,S} - \tilde{\mathbf{u}}_{h,S}) \cdot \hat{\mathbf{n}}\|_e^2 \\ & \lesssim \sum_{e \in \mathcal{E}_h^S(T)} h_e \|[\mathbf{u}_{h,S}]\|_e^2. \end{aligned}$$

Here since T lies on the Darcy side, $\mathcal{E}_h^S(T)$ means the set of edges in \mathcal{E}_h^S who has non-empty intersection with all triangles in \mathcal{T}_h^S that shares an edge with T .

Combining the above and using the fact that $[\tilde{\mathbf{u}}_h] = 0$ on all $e \in \mathcal{E}_h^S$, this completes the proof of (4.2).

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SCHOOL OF MATHEMATICS, FUDAN UNIVERSITY, SHANGHAI, CHINA
E-mail address: wbchen@fudan.edu.cn

DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, STILLWATER, OK, USA
E-mail address: yqwang@math.okstate.edu