

INTERIOR PENALTY DISCONTINUOUS GALERKIN METHOD ON VERY GENERAL POLYGONAL AND POLYHEDRAL MESHES

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Abstract. This paper provides a theoretical foundation for interior penalty discontinuous Galerkin methods for second order elliptic equations on very general polygonal or polyhedral meshes. The mesh can be composed of any polygons or polyhedra which satisfies certain shape regularity conditions characterized in a recent paper by two of the authors in [18]. The usual H^1 conforming finite element methods on such meshes are either very complicated or impossible to implement in practical computation. The interior penalty discontinuous Galerkin method provides a simple and effective alternative approach which is efficient and robust. Results with such general meshes have important application in computational sciences.

Key words. discontinuous Galerkin, finite element, interior penalty, second-order elliptic equations, hybrid mesh.

AMS subject classifications. 65N15, 65N30.

1. Introduction. Most finite element methods are constructed on triangular and quadrilateral meshes, or on tetrahedral, hexahedral, prismatic, and pyramidal meshes. To extend the idea of the finite element method into meshes employing general polygonal and polyhedral elements, one immediately faces the problem of choosing suitable discrete spaces on general polygons and polyhedrons. This issue has rarely been addressed in the past, partly because it can usually be circumvented by dividing the polygon or polyhedron into sub-elements using only one or two basic shapes. However, allowing the use of general polygonal and polyhedral elements does provide more flexibility, especially for complex geometries or problems with certain physical constraints. One of such example is the modeling of composite microstructures in material sciences. A well-known solution to this problem is the Voronoi cell finite element method [9, 10, 11, 14], in which the mesh is composed of polygons or polyhedrons representing the grained microstructure of the given material. The main difficulty of constructing conforming finite element methods on Voronoi meshes is that, the finite element space has to be carefully chosen so that it is continuous along interfaces. Although the constructions on triangles, quadrilaterals, or three-dimensional simplexes are straight forward, it is not easy for general polygons and polyhedrons. Probably the only practically used solution is the rational polynomial interpolants proposed by Wachspress [17], in which rational basis functions are defined using distances from several “nodes”. An important constraint in the construction of the Wachspress basis is that, the rational basis functions need to be piecewise linear along the boundary of every element, in order to ensure H^1 conformity of the finite element space. This

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not only limits the approximation order of the entire Wachspress finite element space, but also complicates the construction. The Wachspress element has gained a renewed interest recently [7, 8, 15]. However, as we have pointed out above, its construction is complicated and usually requires the aid of computational algebraic systems such as Maple.

Another practically important issue is to define finite element methods on hybrid meshes. Hybrid meshes are frequently used nowadays. It can handle complicated geometries, and can sometimes reduce the total number of unknowns. Another possible reason for using the hybrid mesh is that, some engineers argue that in three-dimensions, a hexahedral mesh yields more accurate solution than a tetrahedral mesh for the same geometry [19, 20], as partly verified by numerical experiments. However, pure hexahedral meshes lack the ability of handling complicated geometries. Hence a hybrid mesh becomes a welcomed compromise between accuracy and flexibility. For conforming finite element methods based on hybrid meshes, continuity requirements on interfaces must be satisfied. Such a coupling is straight-forward for the H^1 -conforming finite elements on a triangular-quadrilateral hybrid mesh. However, for three-dimensional meshes, high order finite elements, or other complicated finite element spaces, it usually requires special treatments.

An alternative solution, that can address both issues mentioned above, is to use the weak Galerkin method proposed in [18]. The weak Galerkin method uses discontinuous piecewise polynomials inside each element and on the interfaces to approximate the variational solution. In [18], the authors have proved optimal order of convergence for the weak Galerkin method for the mixed formulation of second order elliptic equations on very general polygonal and polyhedral meshes. Most of the existing error analysis of finite element methods assume triangular, quadrilateral, or some commonly-seen three-dimensional meshes. To our knowledge, [18] is the first one where optimal order of convergence for the finite element solutions has been rigorously proved for general meshes of arbitrary polygons and polyhedra.

The discontinuous Galerkin method imposes the interface continuity weakly, and is known to be able to handle non-conformal, hybrid meshes as well as a variety of basis functions. There have been many research works in this direction, for example, nodal discontinuous Galerkin methods [3, 6, 13] for hyperbolic conservation laws. However, we would like to point out that so far there has been no theoretical analysis on the convergence rate of discontinuous Galerkin method on very general polygonal or polyhedral meshes. Motivated by the work of [18], this paper shall fill the gap by using the technical results developed over there. More precisely, the objective of this paper is to establish a theoretical analysis for the interior penalty discontinuous Galerkin method [2] for elliptic equations on very general meshes and discrete spaces.

The paper is organized as follows. In Section 2, we briefly describe the interior penalty discontinuous Galerkin method in an abstract setting. In Section 3, several assumptions on the discrete spaces are listed, which form a minimum requirement for the well-posedness and the approximation property of the discrete formulation. An abstract theory of error estimates is given in this section too. In Section 4, we discuss choices of meshes and discrete spaces that satisfy the assumptions given in Section 3. Finally, we present some numerical results in Section 5.

2. The model problem and the interior penalty method. Consider the model problem

$$(2.1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \in \mathbb{R}^d (d = 2, 3)$ is a closed domain with Lipschitz continuous boundary, and $f \in L^2(\Omega)$.

For any subdomain $K \subset \Omega$ with Lipschitz continuous boundary, we use the standard definition of Sobolev spaces $H^s(K)$ with $s \geq 0$ (e.g., see [1, 5] for details). The associated inner product, norm, and seminorms in $H^s(K)$ are denoted by $(\cdot, \cdot)_{s,K}$, $\|\cdot\|_{s,K}$, and $|\cdot|_{s,K}$, respectively. When $s = 0$, $H^0(K)$ coincides with the space of square integrable functions $L^2(K)$. In this case, the subscript s is suppressed from the notation of norm, semi-norm, and inner products. Furthermore, the subscript K is also suppressed when $K = \Omega$. Finally, all above notations can easily be extended to any $e \subset \partial K$. For the L^2 inner product on e , we usually denote it as $\langle \cdot, \cdot \rangle_e$ instead of $(\cdot, \cdot)_e$, as it can be replaced by the duality pair when needed.

For simplicity, we assume that Ω satisfy certain conditions such that Equation (2.1) has at least H^r regularity with $r > 3/2$, that is, the solution to Equation (2.1) satisfies $u \in H^r(\Omega)$ and

$$(2.2) \quad \|u\|_r \leq C_R \|f\|.$$

This assumption is standard in the practice of interior penalty discontinuous Galerkin methods, as it ensures that the exact solution u also satisfies the discontinuous Galerkin formulation, and thus the a priori error estimation can be easily derived in a Lax-Milgram framework. However, such a regularity assumption is not necessary in the practice of interior penalty methods. A well-known technique [12, 4] is to use a posteriori error estimation to derive an a priori error estimation for the interior penalty method, with only minimum regularity requirement $u \in H^1(\Omega)$. We believe that the same technique applies for the general polygonal and polyhedral meshes, as long as a working a posteriori error estimation is available. However, here we choose to completely skip this issue, as it is not the main purpose of this paper.

Assume that for all set K discussed in this paper, including Ω itself, the unit outward normal vector \mathbf{n} is defined almost everywhere on ∂K . Note this is true for all polygonal and polyhedral elements with Lipschitz continuous boundaries. Since the exact solution $u \in H^r(\Omega)$ with $r > 3/2$, it is clear that for any smooth function v defined on K ,

$$(\nabla u, \nabla v)_K - \langle \nabla u \cdot \mathbf{n}, v \rangle_{\partial K} = (f, v)_K,$$

where $(\cdot, \cdot)_K$ is the L^2 -inner product in $L^2(K)$ and $\langle \cdot, \cdot \rangle_{\partial K}$ is the L^2 -inner product in $L^2(\partial K)$

Let \mathcal{T}_h be a partition of the domain Ω into non-overlapping subdomains/elements, each with Lipschitz continuous boundary. Here h denotes the characteristic size of the partition, which will be defined in details later. The interior interfaces are denoted by $e = \bar{K}_1 \cap \bar{K}_2$, where $K_1, K_2 \in \mathcal{T}_h$. Boundary segments are similarly denoted by $e = \bar{K} \cap \partial\Omega$, where $K \in \mathcal{T}_h$. Denote by \mathcal{E}_h the set of all interior interfaces and boundary segments in \mathcal{T}_h , and by $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ the set of all interior interfaces. For every $K \in \mathcal{T}_h$, let $|K|$ be the area/volume of K , and for every $e \in \mathcal{E}_h$, let $|e|$ be its

length/area. Denote h_e the diameter of $e \in \mathcal{E}_h$ and h_K the diameter of $K \in \mathcal{T}_h$. Clearly, when $e \subset \partial K$, we have $h_e \leq h_K$. Finally, define $h = \max_{K \in \mathcal{T}_h} h_K$ to be the characteristic mesh size.

Notice that \mathcal{T}_h defined above is a very general mesh/partition on Ω , as we do not specify the shape and conformal property of $K \in \mathcal{T}_h$. The interior penalty discontinuous Galerkin (IPDG) method can be extended to such a general mesh, without any modification of the formulation. However, to ensure its approximation rate, certain conditions must be imposed on \mathcal{T}_h and the discrete function spaces. In this paper, we are interested in discussing the minimum requirements of such conditions. First, we shall give the formulation of the interior penalty discontinuous Galerkin method.

Let V_K be a finite dimensional space of smooth functions defined on $K \in \mathcal{T}_h$. Define

$$V_h = \{v \in L^2(\Omega) : v|_K \in V_K, \text{ for all } K \in \mathcal{T}_h\},$$

and

$$V(h) = V_h + \left(H_0^1(\Omega) \cap \prod_{K \in \mathcal{T}_h} H^r(K) \right), \quad \text{where } r > \frac{3}{2}.$$

For any internal interface $e = \bar{K}_1 \cap \bar{K}_2 \in \mathcal{E}_h$, let \mathbf{n}_1 and \mathbf{n}_2 be the unit outward normal vectors on e , associated with K_1 and K_2 , respectively. For $v \in V(h)$, define the average $\{\nabla v\}$ and jump $[v]$ on e by

$$\{\nabla v\} = \frac{1}{2} (\nabla v|_{K_1} + \nabla v|_{K_2}), \quad [v] = v|_{K_1} \mathbf{n}_1 + v|_{K_2} \mathbf{n}_2.$$

On any boundary segment $e = \bar{K} \cap \partial\Omega$, the above definitions of average and jump need to be modified:

$$\{\nabla v\} = \nabla v|_K, \quad [v] = v|_K \mathbf{n}_K,$$

where \mathbf{n}_K is the unit outward normal vector on e with respect to K .

Define a bilinear form on $V(h) \times V(h)$ by

$$\begin{aligned} A(u, v) = & \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{E}_h} \langle \{\nabla u\}, [v] \rangle_e \\ & - \delta \sum_{e \in \mathcal{E}_h} \langle \{\nabla v\}, [u] \rangle_e + \alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \langle [u], [v] \rangle_e, \end{aligned}$$

where $\delta = \pm 1, 0$ and $\alpha > 0$. when $\delta = 1$, the bilinear form $A(\cdot, \cdot)$ is symmetric. The constant α is usually required to be large enough, but still independent of the mesh size h , in order to guarantee the well-posedness of the discontinuous Galerkin formulation. Details will be given later.

It is clear that the exact solution u to Equation (2.1) satisfies

$$(2.3) \quad A(u, v) = (f, v) \quad \text{for all } v \in V_h,$$

as $[u]$ vanishes on all $e \in \mathcal{E}_h$. Hence the following interior penalty discontinuous Galerkin formulation is consistent with Equation (2.1): find $u_h \in V_h$ satisfying

$$(2.4) \quad A(u_h, v) = (f, v) \quad \text{for all } v \in V_h.$$

Finally, we would like to point out that the formulation (2.4) is computable, as long as each finite dimensional space V_K has a clearly defined and computable basis.

3. Abstract theory. Define a norm $\|\cdot\|$ on $V(h)$ as following:

$$(3.1) \quad \|v\|^2 = \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2 + \sum_{e \in \mathcal{E}_h} h_e \|\{\nabla v\}\|_e^2 + \alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v]\|_e^2.$$

By the Poincaré inequality, $\|\cdot\|$ is obviously a well-defined norm on $V(h)$.

Next, we give a set of assumptions, which form the minimum requirements guaranteeing the well-posedness and the approximation properties of the interior penalty discontinuous Galerkin method.

I1 (The trace inequality) There exists a positive constant C_T such that for all $K \in \mathcal{T}_h$ and $\theta \in H^1(K)$, we have

$$(3.2) \quad \|\theta\|_{\partial K}^2 \leq C_T (h_K^{-1} \|\theta\|_K^2 + h_K \|\nabla \theta\|_K^2).$$

I2 (The inverse inequality) There exists a positive constant C_I such that for all $K \in \mathcal{T}_h$, $\phi \in V_K$ and $\phi \in \frac{\partial}{\partial x_i} V_K$ where $i = 1, \dots, d$, we have

$$(3.3) \quad \|\nabla \phi\|_K \leq C_I h_K^{-1} \|\phi\|_K.$$

I3 (The approximability) There exist positive constants s and C_A such that for all $v \in H^{s+1}(\Omega)$, we have

$$(3.4) \quad \inf_{\chi_h \in V_h} \|v - \chi_h\| \leq C_A \left(\sum_{K \in \mathcal{T}_h} h_K^{2s} \|v\|_{s+1,K}^2 \right)^{1/2}.$$

Note that in Assumption **I2**, the condition $\phi \in \frac{\partial}{\partial x_i} V_K$ ensures that the inverse inequality holds not only for functions in V_K , but also for the gradient of functions in V_K . This condition is redundant if $\frac{\partial}{\partial x_i} V_K$ is a subspace of V_K , for example, when V_K consists of polynomials, etc. But in the most abstract setting, we do not assume that $\frac{\partial}{\partial x_i} V_K \subseteq V_K$.

The abstract theory of the interior penalty discontinuous Galerkin method can be entirely based on Assumptions **I1-I3**.

LEMMA 3.1. *Assume **I1-I2** hold. The bilinear form $A(\cdot, \cdot)$ is bounded in $V(h)$, with respect to the norm $\|\cdot\|$. Indeed,*

$$A(u, v) \leq \frac{1+\alpha}{\alpha} \|u\| \|v\| \quad \text{for all } u, v \in V(h).$$

Furthermore, denote $C_1 = C_T(1 + C_I)^2$. Then for any constant $0 < C < 1$ and $\alpha \geq \frac{(1+\delta)^2 C_1}{4(1-C)^2}$, the bilinear form $A(\cdot, \cdot)$ is coercive on V_h . That is,

$$A(v, v) \geq \frac{C}{1+C_1} \|v\|^2 \quad \text{for all } v \in V_h.$$

Proof. The boundedness of $A(\cdot, \cdot)$ follows immediately from the Schwarz inequality. Here we only prove the coercivity. First, notice that for all $v \in V_h$, by assumptions

I1-I2 and the fact that $h_e \leq h_K$ for all $e \in \partial K \cap \mathcal{E}_h$,

$$\begin{aligned}
\sum_{e \in \mathcal{E}_h} h_e \|\{\nabla v\}\|_e^2 &\leq \sum_{K \in \mathcal{T}_h} \left(\sum_{e \in \partial K \cap \mathcal{E}_h} h_e \|\nabla v\|_e^2 \right) \\
&\leq \sum_{K \in \mathcal{T}_h} h_K \|\nabla v\|_{\partial K}^2 \\
&\leq \sum_{K \in \mathcal{T}_h} h_K \left(C_T (1 + C_I^2) h_K^{-1} \|\nabla v\|_K^2 \right) \\
&= C_1 \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2.
\end{aligned}$$

Then, by the Schwarz inequality, the Young's inequality and assumptions **I1-I2**, we have

$$\begin{aligned}
\sum_{e \in \mathcal{E}_h} \langle \{\nabla v\}, [v] \rangle_e &\leq \varepsilon \sum_{e \in \mathcal{E}_h} h_e \|\{\nabla v\}\|_e^2 + \frac{1}{4\varepsilon} \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v]\|_e^2 \\
&\leq \varepsilon C_1 \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2 + \frac{1}{4\varepsilon\alpha} \left(\alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v]\|_e^2 \right),
\end{aligned}$$

where ε is chosen to be $\frac{1-C}{(1+\delta)C_1}$ for any given constant $0 < C < 1$. Clearly, for such an ε , we have $1 - (1 + \delta)\varepsilon C_1 = C$ and

$$1 - \frac{1 + \delta}{4\varepsilon\alpha} \geq C \iff \alpha \geq \frac{(1 + \delta)^2 C_1}{4(1 - C)^2}.$$

Combining the above and letting $\alpha \geq \frac{(1+\delta)^2 C_1}{4(1-C)^2}$, we have

$$\begin{aligned}
A(v, v) &= \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2 - (1 + \delta) \sum_{e \in \mathcal{E}_h} \langle \{\nabla v\}, [v] \rangle_e + \alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v]\|_e^2 \\
&\geq \left(1 - (1 + \delta)\varepsilon C_1 \right) \sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2 + \left(1 - \frac{1 + \delta}{4\varepsilon\alpha} \right) \alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v]\|_e^2 \\
&\geq C \left(\sum_{K \in \mathcal{T}_h} \|\nabla v\|_K^2 + \alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v]\|_e^2 \right) \\
&\geq \frac{C}{1 + C_1} \|v\|^2.
\end{aligned}$$

□

Lemma 3.1 guarantees the existence and uniqueness of the solution to Equation (2.4). In the rest of this paper, we shall always assume $\alpha \geq \frac{(1+\delta)^2 C_1}{4(1-C)^2}$. Let u and u_h be the solution of equations (2.1) and (2.4), respectively. By subtracting (2.3) from (2.1), one gets the standard orthogonality property of the error,

$$A(u - u_h, v_h) = 0 \quad \text{for all } v \in V_h.$$

Then clearly, for all $\chi_h \in V_h$,

$$\begin{aligned}\|\chi_h - u_h\|^2 &\leq \frac{1+C_1}{C} A(\chi_h - u_h, \chi_h - u_h) \\ &= \frac{1+C_1}{C} A(\chi_h - u, \chi_h - u_h) \\ &\leq \frac{(1+C_1)(1+\alpha)}{C\alpha} \|\chi_h - u\| \|\chi_h - u_h\|.\end{aligned}$$

Then, using the triangle inequality,

$$\begin{aligned}\|u - u_h\| &\leq \inf_{\chi_h \in V_h} \left(\|u - \chi_h\| + \|\chi_h - u_h\| \right) \\ &\leq \left(1 + \frac{(1+C_1)(1+\alpha)}{C\alpha} \right) \inf_{\chi_h \in V_h} \|u - \chi_h\|.\end{aligned}$$

Combine this with assumption **I3**, we get the following abstract error estimation:

THEOREM 3.2. *Assume **I1-I3** hold, C be a given constant in $(0, 1)$ and $\alpha \geq \frac{(1+\delta)^2 C_1}{4(1-C)^2}$. Let u and u_h be the solution to equations (2.1) and (2.4), respectively. Then*

$$\|u - u_h\| \lesssim C_A \left(1 + \frac{(1+C_1)(1+\alpha)}{C\alpha} \right) \left(\sum_{K \in \mathcal{T}_h} h_K^{2s} \|u\|_{s+1, K}^2 \right)^{1/2}.$$

Finally, we derive the L^2 error estimation by using the standard duality argument. Let $\delta = 1$, that is, the bilinear form $A(\cdot, \cdot)$ is symmetric. Consider the following problem

$$\begin{cases} -\Delta \phi = u - u_h & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Here again, we assume that the domain Ω satisfies certain condition such that ϕ has H^r regularity, with $r > 3/2$. Let $\phi_h \in V_h$ be an approximation to ϕ such that they satisfy Assumption **I3**. Clearly

$$\begin{aligned}\|u - u_h\|^2 &= (-\Delta \phi, u - u_h) = \sum_{K \in \mathcal{T}_h} (\nabla \phi, \nabla(u - u_h))_K - \sum_{e \in \mathcal{E}_h} \langle \{\nabla \phi\}, [u - u_h] \rangle_e \\ &= A(\phi, u - u_h) = A(\phi - \phi_h, u - u_h) \\ &\leq \frac{1+\alpha}{\alpha} \|\phi - \phi_h\| \|u - u_h\| \\ &\leq \frac{1+\alpha}{\alpha} C_A \left(\sum_{K \in \mathcal{T}_h} h_K^{2\min\{r-1, s\}} \|\phi\|_{\min\{r, s+1\}, K}^2 \right)^{1/2} \|u - u_h\|.\end{aligned}$$

This gives the following theorem

THEOREM 3.3. *Assume **I1-I3** hold, $\delta = 1$, C be a given constant in $(0, 1)$, $\alpha \geq \frac{(1+\delta)^2 C_1}{4(1-C)^2}$, and the elliptic equation (2.1) has H^r regularity with $r > 3/2$. Let u and u_h be the solution to equations (2.1) and (2.4), respectively. Then*

$$\|u - u_h\| \leq \frac{1+\alpha}{\alpha} C_A C_R h^{\min\{r-1, s\}} \|u - u_h\|.$$

4. Requirements on meshes and discrete spaces. On triangular or quadrilateral meshes, the usual tool for proving assumptions **I1-I3** is to use a scaling argument built on affine transformations. However, on general polygons and polyhedrons, it is not clear how to define such affine transformations. The assumptions **I1-I3** were first validated in [18] for general polygonal and polyhedral meshes that satisfy a set of conditions introduced in [18]. Such conditions can be stated as follows.

All the elements of \mathcal{T}_h are assumed to be closed and simply connected polygons or polyhedrons. We make the following shape regularity assumptions for the partition \mathcal{T}_h .

A1: Assume that there exist two positive constants ρ_v and ρ_e such that for every element $K \in \mathcal{T}_h$ and $e \in \mathcal{E}_h$, we have

$$(4.1) \quad \rho_v h_K^d \leq |K|, \quad \rho_e h_e^{d-1} \leq |e|.$$

A2: Assume that there exists a positive constant κ such that for every element $K \in \mathcal{T}_h$ and $e \in \partial K \cap \mathcal{E}_h$, we have

$$(4.2) \quad \kappa h_K \leq h_e.$$

A3: Assume that for every $K \in \mathcal{T}_h$, and $e \in \partial K \cap \mathcal{E}_h$, there exists a pyramid $P(e, K, A_e)$ contained in K such that its base is identical with e , its apex is $A_e \in K$, and its height is proportional to h_K with a proportionality constant σ_e bounded away from a fixed positive number σ^* from below. In other words, the height of the pyramid is given by $\sigma_e h_K$ such that $\sigma_e \geq \sigma^* > 0$. The pyramid is also assumed to stand up above the base e in the sense that the angle between the vector $\mathbf{x}_e - A_e$, for any $\mathbf{x}_e \in e$, and the outward normal direction of e is strictly acute by falling into an interval $[0, \theta_0]$ with $\theta_0 < \pi/2$.

A4: Assume that each $K \in \mathcal{T}_h$ has a circumscribed simplex $S(K)$ that is shape regular and has a diameter $h_{S(K)}$ proportional to the diameter of K ; i.e., $h_{S(K)} \leq \gamma_* h_K$ with a constant γ_* independent of K . Furthermore, assume that each circumscribed simplex $S(K)$ intersects with only a fixed and small number of such simplexes for all other elements $K' \in \mathcal{T}_h$.

Under the above assumptions, the following results have been proved in [18]:

LEMMA 4.1. *(The trace inequality). Assume **A1-A3** hold on a polygonal or polyhedral mesh. Then **I1** is true.*

LEMMA 4.2. *(The inverse Inequality). Assume **A1-A4** hold on a polygonal or polyhedral mesh and each V_K is the space of polynomials with degree less than or equal to n . Then **I2** is true with C_I depending on n , but not on h_K or $|K|$.*

LEMMA 4.3. *Assume **A1-A4** hold on a polygonal or polyhedral mesh and each V_K is the space of polynomials with degree less than or equal to n . Let Q_h be the L^2 projection onto V_h . Then for all $0 \leq s \leq n$ and $v \in H^{s+1}(\Omega)$,*

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \|v - Q_h v\|_K^2 &\leq C_{Q0} h^{2(s+1)} \|v\|_{s+1}^2. \\ \sum_{K \in \mathcal{T}_h} \|\nabla(v - Q_h v)\|_K^2 &\leq C_{Q1} h^{2s} \|v\|_{s+1}^2. \end{aligned}$$

It is not hard to see that **I3** follows immediately from Lemma 4.3. Indeed, notice that as long as **I1** and **I2** are true and $v \in H^r(\Omega)$ with $r > 3/2$, we have

$$\|v - Q_h v\| \leq (1 + C_1) \sum_{K \in \mathcal{T}_h} \|\nabla(v - Q_h v)\|_K^2 + \alpha \sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v - Q_h v]\|_e^2,$$

where $C_1 = C_T(1 + C_I)^2$. Next, notice that by **A2**, **I1** and Lemma 4.3,

$$\begin{aligned}
\sum_{e \in \mathcal{E}_h} \frac{1}{h_e} \|[v - Q_h v]\|_e^2 &\leq \sum_{K \in \mathcal{T}_h} \left(\sum_{e \in \partial K \cap \mathcal{E}_h} \frac{1}{h_e} \|(v - Q_h v)|_K\|_e^2 \right) \\
&\leq \sum_{K \in \mathcal{T}_h} \left(\sum_{e \in \partial K \cap \mathcal{E}_h} \frac{1}{\kappa h_K} \|(v - Q_h v)|_K\|_e^2 \right) \\
&= \sum_{K \in \mathcal{T}_h} \frac{1}{\kappa h_K} \|(v - Q_h v)|_K\|_{\partial K}^2 \\
&\leq \sum_{K \in \mathcal{T}_h} \frac{C_T}{\kappa h_K} (h_K^{-1} \|v - Q_h v\|_K^2 + h_K \|\nabla(v - Q_h v)\|_K^2) \\
&\leq \frac{C_T}{\kappa} (C_{Q0} + C_{Q1}) h^{2s} \|v\|_{s+1}^2.
\end{aligned}$$

Combine the above, we have

LEMMA 4.4. *(The approxiability) Assume **A1-A4** hold on a polygonal or polyhedral mesh and each V_K is the space of polynomials with degree less than or equal to n . Then for all $\frac{1}{2} < s \leq n$ and $v \in H^{s+1}(\Omega)$, there exists a constant C_A independent of h such that*

$$\inf_{\chi_h \in V_h} \|v - \chi_h\| \leq C_A h^s \|v\|_{s+1}.$$

Here $s > \frac{1}{2}$ is added so that $\|v - \chi_h\|$ is well-defined.

5. Numerical Examples. Finally, we present numerical results that support the theoretical analysis of this paper. We fix the coefficients $\delta = 1$ and $\alpha = 10$, since the purpose of the numerical experiments is to examine the accuracy of the interior penalty discontinuous Galerkin method on arbitrary polygonal meshes, not for different coefficients. Consider the Poisson's equation on $\Omega = (0, 1) \times (0, 1)$ with the exact solution $u = \sin(2\pi x) \cos(2\pi y)$. Clearly $u = 0$ on $\partial\Omega$. For simplicity of the notation, we denote

$$|u - u_h|_{1,h} = \left(\sum_{K \in \mathcal{T}_h} |\nabla(u - u_h)|_K^2 \right)^{1/2}.$$

The first test is performed on a non-conformal triangular-quadrilateral hybrid mesh. The initial mesh and the mesh after one uniform refinement are given in Figure 5.1. A sequence of uniform refinements are then applied to generate a set of nested meshes. Notice that the meshes are non-conformal and there are hanging nodes. However, the interior penalty discontinuous Galerkin method can deal with such meshes without special treatments. We solve the Poisson equation using the interior penalty discontinuous Galerkin formulation (2.4) on these meshes, where the local discrete spaces V_K are taken to be P_1 polynomials on each $K \in \mathcal{T}_h$, no matter whether K is a triangle or quadrilateral. The H^1 semi-norm and the L^2 norm of the errors are reported in Table 5.1 and Figure 5.2. These errors are computed using a 5th order Gaussian quadrature on triangles. For quadrilateral elements, the errors can be conveniently computed by dividing the quadrilateral into two triangles and then applying the Gaussian quadrature. Our results show that the H^1 semi-norm has

FIG. 5.1. Initial and refined mesh for test 1.

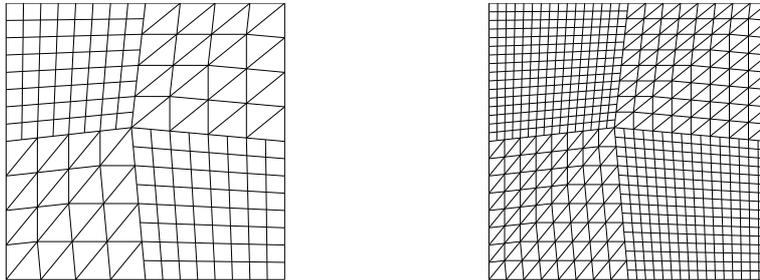


TABLE 5.1
Convergence rates for test 1.

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$O(h^r), r =$
$ u - u_h _{1,h}$	1.2006	0.5904	0.2917	0.1452	0.0725	1.0124
$\ u - u_h\ $	0.0551	0.0159	0.0042	0.0011	0.0003	1.9270

an approximate order of $O(h)$, while the L^2 norm has an approximate order of $O(h^2)$, as predicted by the theoretical analysis.

In the first test, hanging nodes are treated as vertices with internal angles equal to 180° . We would like to point out that meshes shown in Figure 5.1 satisfy Assumptions **A1-A4**. Note that **A2** specifies that edges in the mesh can not be too “short” comparing to the size of associated polygons, which imposes a restriction on the position of hanging nodes. This restriction can not be removed as long as one uses Assumption **A2**. In Section 4, it has been shown that Assumptions **A1-A4** are sufficient conditions for deriving the well-posedness and error estimates of the interior penalty discontinuous Galerkin method. However, we do not claim that they are necessary conditions. Interested readers may use the theoretic framework provided in this paper to explore the possibility of dropping Assumption **A2** or deriving new sets of assumptions on meshes.

In the second test, we consider a hybrid mesh containing mainly hexagons, but with a few quadrilaterals and pentagons. Indeed, it is derived by taking the dual mesh of a simple triangular mesh. In Figure 5.3, the initial triangular mesh and its dual mesh are shown. By refining the triangular mesh and computing its dual mesh, we get a sequence of hexagon hybrid meshes. Again, we solve the interior penalty discontinuous Galerkin formulation (2.4) on these hexagon hybrid meshes, with the local discrete spaces V_K of P_1 polynomials. The H^1 semi-norm and the L^2 norm of the errors are reported in Table 5.2 and Figure 5.4. Optimal convergence rates are achieved.

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FIG. 5.2. Convergence rates for test 1.

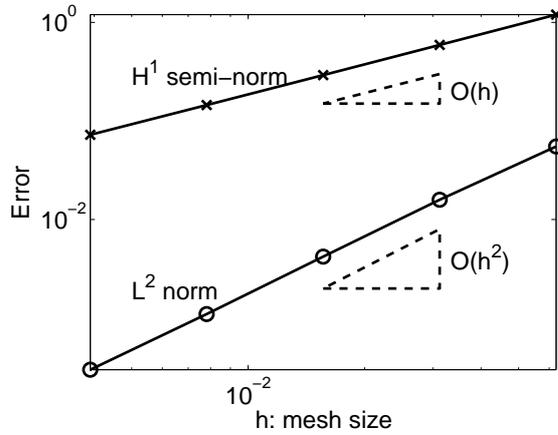
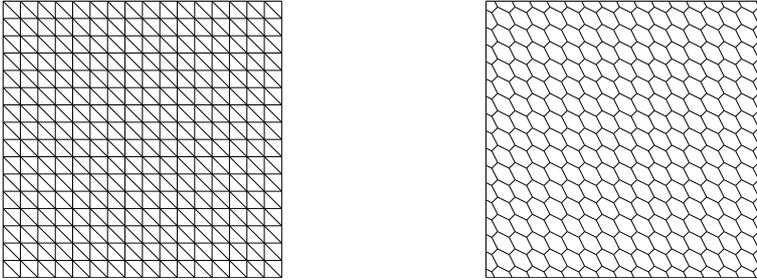


FIG. 5.3. The original triangular mesh and its dual mesh used in test 2.

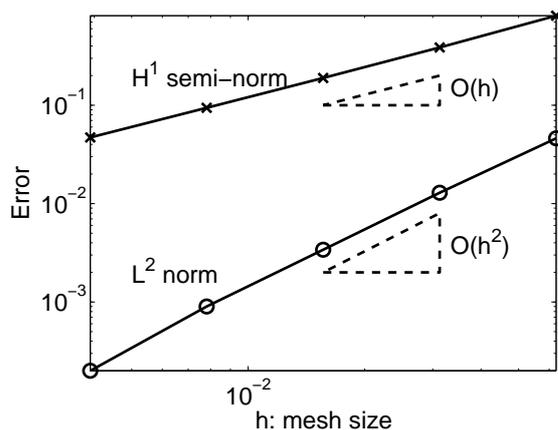


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TABLE 5.2
Convergence rates for test 2.

h	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$	$O(h^r)$, $r =$
$ u - u_h _{1,h}$	0.8139	0.3868	0.1894	0.0941	0.0470	1.0270
$\ u - u_h\ $	0.0461	0.0129	0.0034	0.0009	0.0002	1.9393

FIG. 5.4. Convergence rates for test 2.



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