Fully discrete finite element approximation of a fluid-structure interaction problem

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Abstract: In this paper, we consider fully discrete finite element methods for the Fluid-Structure Interaction system. For the time discretization, backward difference algorithm and composite left rectangular methods are adopted to approximate the continuous derivative and integration with respect to \( t \), respectively. Existence and uniqueness of finite element solutions are proved, fully discrete error estimates are obtained.

Key words: fully discrete, fluid-structure interaction, finite element, error estimate

1 Introduction

The analysis of fluid-structure interaction (FSI) problems has attracted growing attention during recent years. The FSI modeling describes the dynamics of fluids in contact with the elastic structures with natural transmission conditions, coupled the solid unknown with the velocity of the fluid solution, at a common interface. There are many numerical studies of the FSI modeling in recent years. C. Hirt introduced the Arbitrary Lagrangian-Eulerian (ALE) method as a suitable procedure for the analysis of FSI modeling\(^1\). In the ALE method, the grid nodes may be moved with the fluid in normal Lagrangian way, or be held fixed in Eulerian manner. A series of detailed research\(^{2,3}\) of ALE finite element methods for FSI modeling are given. T. Tezduyar and S. Sathe\(^{1}\) developed space-time FSI techniques that have been applied to a wide range of 3D computation of FSI problems. These techniques enhanced the scope, accuracy, robustness and efficiency of traditional space-time methods.

In Ref. \([5]\), Q. Du, M. Gunzburger, L.
Hou and J. Lee explained the physical validity of the FSI modeling. In addition, they considered weak formulations for FSI modeling and established the existence of weak solutions. In Ref. [6], they further discussed a divergence-free formulation of FSI modeling which does not involve the fluid pressure field. Based on this formulation, they presented a semi-discrete finite element formulation of FSI modeling and derived error estimates.

The object of this paper is to discuss the fully discrete finite element approximation, prove the existence and uniqueness of finite element solutions, and derive the error estimates for the fully discrete finite element approximations. The rest of the paper is arranged as follows. In section 2, we recall some relevant results of Ref. [5,6]. In section 3, we discuss the fully discrete finite element approximation and establish the existence and uniqueness of the finite element solutions and derive the error estimates.

2 Results for semi-discrete finite element approximation

2.1 Weak formulation

Assume that the fluid and solid occupy two neighboring open Lipschitz domains, \( \Omega_1 \subset \mathbb{R}^d \) and \( \Omega_2 \subset \mathbb{R}^d \) where \( d = 2, 3 \). \( \Omega \) is the interior of \( \bar{\Omega}_1 \cup \bar{\Omega}_2 \), i.e. \( \Omega \) is the entire fluid—solid region. Moreover, we let \( \gamma = \partial \Omega_1 \cap \partial \Omega_2 \) denote the interface between fluid and solid, and let \( \Gamma_1 = \partial \Omega_1 \setminus \gamma \), \( \Gamma_2 = \partial \Omega_2 \setminus \gamma \), denote the parts of the fluid and solid boundaries respectively, excluding the interface \( \gamma \). Let \( n_i (i = 1, 2) \) denotes the unit outward normal vector to \( \Omega_i (i = 1, 2) \).

![Fig. 1 Geometric description of the fluid structure interaction model](image)

We consider the following FSI system, in the fluid region \( \Omega_1 \), we apply the Stokes system

\[
\begin{cases}
\rho_1 \mathbf{v} + \nabla \rho - \mu_1 \nabla \mathbf{v} = \rho_1 \mathbf{f}_1 & \text{in } \Omega_1, \\
\text{div} \mathbf{v} = 0 & \text{in } \Omega_1, \\
\mathbf{v} = 0 & \text{on } \Gamma_1, \mathbf{v} \mid_{r=0} = \mathbf{v}_0
\end{cases}
\]  

(2.1)

where \( \mathbf{v} = (v_1, \ldots, v_d)^T \) denotes the fluid velocity, \( \rho \) represents the fluid density, \( \mathbf{f} \) denotes the given force per unit mass, \( \rho_1 \) and \( \mu_1 \) represent the constant fluid density and viscosity, and \( \mathbf{v}_0 \) denotes the given initial velocity.

In the solid \( \Omega_2 \), we consider the equations of linear elasticity

\[
\begin{cases}
\rho_2 \mathbf{u}_s - 2\mu_2 \text{div}(\mathbf{u}_s) - \lambda_2 \text{div}(\text{div}(\mathbf{u}_s)) = \rho_2 \mathbf{f}_2 & \text{in } \Omega_2, \\
\mathbf{u}_s = 0 & \text{on } \Gamma_2, \\
\mathbf{u}_s \mid_{r=0} = \mathbf{u}_0 \text{ and } \mathbf{u}_s \mid_{r=0} = \mathbf{u}_i & \text{in } \Omega_2, 
\end{cases}
\]

(2.2)

where \( \mathbf{u}_s = (u_1, \ldots, u_d)^T \) and \( \mathbf{u}_s \) denote the displacement and velocity of the solid, respectively. \( \epsilon(\mathbf{u}) = \left[ \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \) represents the strain tensor. \( \mathbf{f}_2 \) represents the given loading force per unit mass, \( \mu_2 \) and \( \lambda_2 \) denote the Lame constants, \( \rho_2 \) represents the constant solid density, \( \mathbf{u}_0 \) and \( \mathbf{u}_i \) denote the given initial data. \( I \) denotes the unit matrix.

Across the fixed interface \( \gamma \) between the fluid and solid, the velocity and stress vector are both continuous, i.e.

\[
\mathbf{u}_s = \mathbf{v} \text{ on } \gamma,
\]

(2.3)

\[
2\mu_2 \epsilon(\mathbf{u}) \cdot \mathbf{n}_2 + \lambda_2 \text{div} \mathbf{u} \mathbf{n}_2 = \rho \mathbf{n}_3 - \mu_1 \nabla \mathbf{v} \cdot \mathbf{n}_1 \text{ on } \gamma,
\]

(2.4)

where \( \mathbf{n} \) denotes the outward-pointing unit normal vector along the boundary \( \partial \Omega_i, i = 1, 2 \).
Throughout this paper, $H^m(K), m \in \mathbb{R}$, denotes the standard Sobolev spaces with order $m$ defined on the region $K$ equipped with the standard norm $\| \cdot \|_{m,K}$. Vector-valued Sobolev spaces are denoted by $H^m(K)$. We use the following $L^2$ inner product notations on scalar and vector-valued $L^2$ spaces:

$$[p, q]_K = \int_K pq \, dK \quad \forall \, p, q \in L^2(K),$$

where the spatial set $K$ is $\Omega$ or $\gamma$ or $\Omega_i$, for $i = 1, 2$.

We define function spaces

$$H_i[\mathcal{H}_i(\Omega_i)] \quad \text{with the norm} \quad \| \cdot \|_{1,q_i} = \| \cdot \|_1, i = 1, 2$$

We define the bilinear forms:

$$a_1[u, v] = \int_{\Omega_1} \mu \sum_{i=1}^d \frac{\partial u_i}{\partial x_i} \frac{\partial v_i}{\partial x_i} \, d\Omega \quad \forall \, u, v \in H_1,$$

$$a_2[u, v] = \int_{\Omega_2} (\lambda_2 \text{div} u)(\text{div} v) + 2\mu_2 \sum_{i,j=1}^d e_{ij}(u) e_{ij}(v) \, d\Omega \quad \forall \, u, v \in H_2,$$

$$b[v, q] = -\int_{\Omega_1} q \text{div} v \, d\Omega \quad \forall \, v \in H_1, \forall \, q \in L^2(\Omega_1).$$

By using Korn’s inequalities it can be verified the bilinear forms $a_1[\cdot, \cdot], a_2[\cdot, \cdot]$ are coercive, i.e.

$$\begin{align*}
\forall \, u \in H_i, & \quad \text{if} \, \text{meas}(\Gamma_i) \neq 0 \\
[u, u]_1 + a_1[u, u] & \geq k_i \| u \|_{1,q_i}
\end{align*}$$

(2.7)

$$\begin{align*}
\forall \, u \in H_i, & \quad \text{if} \, \text{meas}(\Gamma_i) = 0 \\
[u, u]_2 + a_2[u, u] & \geq k_i \| u \|_{1,q_i}
\end{align*}$$

(2.8)

In this paper, $h$ and $r$, defined in the latter section, denote discretization parameter in space and time direction, respectively. The letter $C$ denotes a generic constant that may not be the same at different occurrences. For simplification, by $x \approx y (y \gg x)$ we mean that there exists a constant $C$ such that $x \leq C \gamma$ ($\gamma \geq C x$), where $C$ is independent with $h$ and $r$.

### 2.2 Semi-discrete approximation results

In this section, we will recall the weak formulations, and error estimates for the semi-discrete finite element approximation.

$$\begin{align*}
\xi & \equiv \begin{cases}
\psi \in \Omega_1, & u_1 \in \Omega_2,
\end{cases} \\
\xi & \equiv \begin{cases}
\psi \in \Omega_1, & u_1 \in \Omega_2,
\end{cases}
\end{align*}$$

(2.9)

and the weighted $L^2$ inner product $[[\cdot, \cdot, \cdot]]$:

$$[[\xi, \eta]_i] = [\rho_i \xi, \eta]_{a_i} + [\rho_i \xi, \eta]_{a_i}, \quad \forall \, \xi, \eta \in L^2(\Omega)$$

(2.10)

Thus the divergence—free weak formulation for (2.1)–(2.4) can be put conveniently into the following form: seek a $\xi$ such that

$$\xi \in L^2(0, T; L^2(\Omega)), \quad \partial_t \xi \in L^2(0, T; \mathcal{V}^*),$$

(2.11)

$$\xi \big|_{\partial \Omega_1} \in L^2(0, T; H_1), \quad \text{div} \xi \big|_{\partial \Omega_1} = 0, \quad \int_0^T \xi(s) \big|_{\partial \Omega_1} \, ds \in L^2(0, T; H_2),$$

(2.12)

$$[[\xi, \eta]_i] + a_i[\xi, \eta] + a_2[\int_0^T \xi(s) \, ds, \eta] = [[f, \eta]] - a_2[u_1, \eta] \quad \forall \, \eta \in \mathcal{V},$$

(2.13)

$$\xi(0) = \xi_0,$$

(2.14)

$$\int_0^T \langle \xi(s) \big|_{\partial \Omega_1} \rangle \, ds \big|_{\partial \Omega_1} = \int_0^T \langle \xi(s) \big|_{\partial \Omega_1} \rangle \, ds \big|_{\partial \Omega_1} \quad a.e. t,$$

(2.15)

The existence and uniqueness of the solution for the auxiliary problem (2.11)–(2.15) was proved in Ref. [6].

In what follows we assume that $\Omega_i \subset \mathbb{R}^2$ and
\( \Omega_k \subset \mathbb{R}^2 \) are both polyhedral parameter domains. Let \( h \) denote a discretization parameter associated with the triangulation \( \mathcal{T}_h \) of \( \Omega \), i.e., \( h = \max_{K \in \mathcal{T}_h} \{ \text{diam}(K) \} \). We also assume that \( K_i \) do not cross the interface \( \gamma \). For each \( 0 < h < 1 \), we choose that \( H^k \subset H^1(\Omega) \cap C(\Omega) \) and \( Q_h^k \subset L^2(\Omega) \) as finite element spaces that contain linear functions. We set \( H^k \rangle \mid_{\Omega_i} \),

\[
\inf_{v_h \in H^k \rangle \mid_{\Omega_i}} \| u - u_h \|_{0, \Omega_i} \leq h^{k+1} \| u \|_{r+1, \Omega_i} \\
\inf_{v_h \in H^k \rangle \mid_{\Omega_i}} \| u - u_h \|_{1, \Omega_i} \leq h^{r} \| u \|_{r+1, \Omega_i} \\
\inf_{q_h \in Q_h^k} \| q - q_h \|_{0, \Omega_i} \leq h^{r} \| q \|_{r, \Omega_i} \\
\forall u \in H^{r+1}(\Omega_i) \cap H_r, \quad r \in [0, k]. \tag{2.16}
\]

\[
\forall u \in H^{r+1}(\Omega_i) \cap H_r, \quad r \in [0, k]. \tag{2.17}
\]

\[
\forall q \in H^{r}(\Omega_i) \cap H_r, \quad r \in [0, k]. \tag{2.18}
\]

Under the definition \( \xi_h \defeq \begin{cases} v_h & \text{in } \Omega_i, \\ \partial_{\nu} u_h & \text{in } \Omega_2, \end{cases} \) \( \xi_{h,0} \), the semi-discrete finite element formulation can be put as follows

\[
\begin{align*}
&\{[\partial_{\nu} \xi_h \cdot \eta_h]\} + a_1 \{[\xi_h \cdot \eta_h]\} + \\
&a_2 \left[ \int_{\Omega} \eta_h (s) \partial_{\nu} \eta_h \right] = \\
&\{[f \cdot \eta_h]\} - a_2 \{[u_{0,h} \cdot \eta_h]\} \quad \forall \eta_h \in H^h, \tag{2.20}
\end{align*}
\]

\[
\xi_{h,0} \defeq \xi_h(0) = \begin{cases} v_{h,0} & \text{in } \Omega_i, \\ u_{1,h} & \text{in } \Omega_2, \end{cases} \tag{2.21}
\]

**Lemma 2.1** Assume that \( f_1, f_2, u_0, u_1 \) and \( v_0 \) satisfy (2.11) \(- (2.12). Then, there exists a unique solution \( \xi_h \in C^1(0, T; \Psi^k) \) which satisfies (2.13) \(- (2.15).

The error estimate between the continuous solution defined by (2.11) \(- (2.15) and the semi-discrete finite element solution defined by (2.20) \(- (2.21) are derived in Ref. 6, i.e.

**Lemma 2.2** Assume that \( f_1, f_2, u_0, u_1 \) and \( v_0 \) satisfy (2.11) \(- (2.12). Let \( (v, u) \) be the solution of (2.13) \(- (2.15), and \( (v_h, u_h) \) be the solution of (2.20) \(- (2.21). Assume that for some \( r \in [1, k], \nu \in L^2(0, T; H^{-1}(\Omega_1)), \partial_{\nu} \eta \in L^2(0, T; H^{r-1}(\Omega_1)), p \in L^2(0, T; H^r(\Omega_1)), \partial_{\nu} u \in L^2(0, T; H^{r-1}(\Omega_2)), v_h \in H^{r+1}(\Omega_1), u_h \in H^{r+1}(\Omega_2), \) and \( p_h \in H^r(\Omega_1). \) Then,

\[
\| v(t) - v_h(t) \|_{0, \Omega_i} + \\
\| v - v_h \|_{L^2(0, T; H^r(\Omega_1))} + \]

\[
\| \partial_{\nu} u_h(t) - \partial_{\nu} u_h(t) \|_{0, \Omega_i} + \\
\| u(t) - u_h(t) \|_{1, \Omega_i} \leq h^{r} \| v \|_{r+1, \Omega_i} + \\
\| \partial_{\nu} \eta \|_{L^2(0, T; H^{r-1}(\Omega_1))} + \\
\| \partial_{\nu} \eta \|_{L^2(0, T; H^{r-1}(\Omega_2))} + \\
\| \partial_{\nu} u \|_{L^2(0, T; H^{r-1}(\Omega_1))} + \\
\| \partial_{\nu} u \|_{L^2(0, T; H^{r-1}(\Omega_2))} \tag{2.22}
\]

**Remark 2.1** \( \epsilon \) depends on the region regularity assumption. In particular, if both \( \Omega_i \) and \( \Omega_2 \) are convex (i.e., \( \gamma \) is necessarily a straight line), then \( \epsilon \) can be chosen arbitrarily small.

3 Fully discrete finite element formulation and error estimate

We now consider a fully discrete scheme for (2.11) \(- (2.15). Suppose \( [0, T] \) is partitioned into equal subintervals with time step \( \tau = T/M \), where \( M \) is a positive integer. We denote \( t_n = n\tau \quad (0 \leq n \leq M) \) in the following discussion. The fully discrete form of (2.11) \(- (2.15) is as follows

Find \( (\xi_h)^M_{n=0} \subset \Psi^k \) such that

\[
\begin{align*}
&\{[\partial_{\nu} \xi_h \cdot \eta_h]\} + a_1 \{[\xi_h \cdot \eta_h]\} + \\
&a_2 \left[ \int_{\Omega} \eta_h (s) \partial_{\nu} \eta_h \right] = \\
&\{[f \cdot \eta_h]\} - a_2 \{[u_{0,h} \cdot \eta_h]\} \quad \forall \eta_h \in H^h, \tag{3.1}
\end{align*}
\]

\[
\xi_h = \xi_{h,0} = \begin{cases} v_{h,0} & \text{in } \Omega_i, \\ u_{1,h} & \text{in } \Omega_2, \end{cases} \tag{3.2}
\]
where the operator \( \overline{\partial} \) denotes the backward difference operator, i.e.,
\[
\overline{\partial} \xi = (\xi - \xi^{-1})/\tau,
\]
and \( L_\times \) denotes the composite left rectangular operator, i.e.,
\[
L_\times(\xi_k) = \xi_k^\times + \xi_k^\times + \cdots + \xi_k^\times.
\]
According to the fully-discrete form (3.1) – (3.2), \( u_\times^\mathrm{h} \) and \( v_\times^\mathrm{h} \) can be put as the following form:
\[
\begin{align*}
\begin{bmatrix}
\ddot{v}_\times^\mathrm{h} &= \xi_\times^\mathrm{h} |_{\partial_1} \\
u_\times^\mathrm{h} &= L_\times(\xi_\times^\mathrm{h} |_{\partial_2}) + \nu_{\times,\times}^\mathrm{h}
\end{bmatrix}
\end{align*}
\]
(3.5)

**Theorem 3.1** Assume that \( f_1, f_2, u_\times, u_\times, \) and \( v_\times \) satisfy (2.11) – (2.12). Then, there exists a unique series of \( \{\xi_\times^\mathrm{n}\}^\infty_{n=0} \subset H^k \) satisfying (3.1) – (3.2).

**Proof** According to the operator definitions of \( \overline{\partial} \) and \( L_\times \), the fully-discrete formulation (3.1) – (3.2) is equivalent to the following
\[
\begin{align*}
\frac{1}{\tau} \left[[\xi_\times, \nu] \right] &+ a_1 [\xi_\times, \nu] = \\
\frac{1}{\tau} \left[[\xi_\times^{-1}, \nu] \right] &- a_2 [\xi_\times^{-1} + \cdots + \xi_\times + \xi_\times, \nu] + \\
[[f, \nu]] &- a_2 [u_{\times,\times}, \nu],
\end{align*}
\]
(3.6)
\[
\forall \nu_\times \in H^k,
\]
(3.7)
Recall that \( \{\phi_j\} = 1 \) are a basis of \( H^k \), therefore
\[
\xi_\times = \sum_{j=1}^{M} \phi_j(x) g_j(t_\times).
\]
Let \( g = (g_1, \cdots, g_M)^T \), \( h = (h_1, \cdots, h_M)^T \), \( A_1 = (a_1 (\phi_j, \phi_i))^M_{i,j} \), and \( B = ([\phi_j, \phi_i])^M_{i,j} \), then we can use matrix format to rewrite (3.6) – (3.7) as follows
\[
\text{Find } g = (g_1, \cdots, g_M)^T \text{ such that:}
\]
\[
\left(\frac{1}{\tau} B + A_1\right) g = F,
\]
(3.8)
where \( F \) is the corresponding term of the right side of (3.6). Since matrices \( A_1, B \) are all positive definite, the coefficient matrix \( \frac{1}{\tau} B + A_1 \) is positive definite, therefore, invertible. Thus there exists a unique \( g = (g_1, \cdots, g_M)^T \) satisfying (3.8).

Noting the relations (3.5), we immediately obtain the existence of a series of \( (v_{\times}^\mathrm{h}, \nu_{\times}^\mathrm{h}) \) satisfying the fully discrete formulation (3.1) – (3.2), as follows

**Theorem 3.2** Assume that \( f_1, f_2, u_\times, u_\times, \) and \( v_\times \) satisfy (2.11) – (2.12). Then, there exists a unique series of \( \{v_{\times}^\mathrm{h}, \nu_{\times}^\mathrm{h}\}^M_{m=0} \subset H^k \times H^k \) satisfying (3.1) – (3.2).

Based on the fully discrete formulation (3.1) – (3.2), we derive the corresponding error estimates.

**Theorem 3.3** Assume that \( f_1, f_2, u_\times, u_\times, \) and \( v_\times \) satisfy (2.5). Let \( (v_\times, u_\times) \) be the solution of the continuous weak problem (2.6) – (2.8) and \( (v_{\times}^\mathrm{h}, \nu_{\times}^\mathrm{h}) \) be the solution of fully discrete problem (3.1) – (3.2). Assume for some \( r \in [1, k] \), \( v \in L^2(0, T; H^{r+1}(\Omega)), \partial_\nu v \in L^2(0, T; H^{r-1}(\Omega) \}) \), \( \partial_\nu u \in L^2(0, T; H^{r+1}(\Omega)), \partial_\nu u \in L^2(0, T; H^{r-1}(\Omega)) \), \( v_\times \in H^{r+1}(\Omega), u_\times \in H^{r+1}(\Omega) \) and \( u_\times \in H^{r+1}(\Omega) \). Then, the following error estimates hold for \( 0 \leq m \leq M \)
\[
\begin{align*}
\| v^\times - v_{\times}^\mathrm{h} \|_{0, \partial_1} &+ \sum_{m=1}^{M} \| v^\times - v_{\times}^\mathrm{h} \|_{1, \partial_1} \\
\| u^\times - u_{\times}^\mathrm{h} \|_{0, \partial_1} &+ \| u^\times - u_{\times}^\mathrm{h} \|_{1, \partial_1} \\
(\tau^r + \tau) (\| v_\times \| \partial_\nu u_{\times,\times} + \| u_\times \| \partial_\nu u_{\times,\times} + \\
\sum_{m=1}^{M} \| \partial_\nu u_{\times,\times} \|_{0, \partial_1} + \\
\| \partial_\nu v \|_{0, \partial_1} + \| \partial_\nu u \|_{0, \partial_1} + \\
(\tau^r + \tau) \| v_\times \| \partial_\nu u_{\times,\times} \|_{0, \partial_1} + \\
\sum_{m=1}^{M} \| \partial_\nu u_{\times,\times} \|_{0, \partial_1} + \\
\| \partial_\nu v \|_{0, \partial_1} + \| \partial_\nu u \|_{0, \partial_1} \)
\end{align*}
\]
(3.9)
where \( \tau \) is the time step, and \( v^\times = v(t_\times), u^\times = u(t_\times) \)

**Proof** Subtract (3.1) – (3.2) from (2.20) – (2.21), and let \( t = t_\times \), then we have
\[
\begin{align*}
0 &= [[\partial_\times \xi(t_\times) - \partial_\times \xi(t_\times), \nu]] + \\
& a_1 [\xi(t_\times) - \xi, \nu] + \\
a_2 \left[\int_{\Omega} \xi(s) \partial_\times v_{\times,\times} - L_\times(\xi(t_\times), \nu)] + \\
&[[\partial_\times \xi(t_\times) - \partial_\times \xi(t_\times), \nu]] + \\
a_1 [\xi(t_\times) - \xi, \nu] + \\
a_2 \left[\int_{\Omega} \xi(s) \partial_\times v_{\times,\times} - L_\times(\xi(t_\times), \nu)] + \\
a_1 [\xi(t_\times) - \xi, \nu] + \\
a_2 \left[\int_{\Omega} \xi(s) \partial_\times v_{\times,\times} - L_\times(\xi(t_\times), \nu)] + \\
a_2 \left[\int_{\Omega} \xi(s) \partial_\times v_{\times,\times} - L_\times(\xi(t_\times), \nu)] + \\
a_2 \left[\int_{\Omega} \xi(s) \partial_\times v_{\times,\times} - L_\times(\xi(t_\times), \nu)] + \\
a_2 \left[\int_{\Omega} \xi(s) \partial_\times v_{\times,\times} - L_\times(\xi(t_\times), \nu)] + \\
\forall \nu_\times \in H^k,
\end{align*}
\]
(3.10)
where
\[
L_\times(\xi(t_\times)) \overset{\text{def}}{=} \tau(\xi(t_\times) + \xi(t_\times) + \cdots + \xi(t_\times)).
\]
Set \( e^\times = \xi(t_\times) - \xi_\times \) and choose \( \eta_\times = e^\times \in H^k \) in (3.10), then we have
\[
\left[ \left[ \frac{e^r - e^{r-1}}{r}, e^r \right] \right]_a + a_1 [e^r, e^r] + \nonumber
a_2 \left[ \tau (e^r + e^{r-1} + \cdots + e^0), e^r \right] = \nonumber
- \left[ \left[ \partial_t \xi_a(t_s) - \partial_t \xi_a(t_s), e^r \right] \right] - \nonumber
a_2 \int_0^t \xi_a(s) ds - L_a(\xi_a), e^r \right] \nonumber
\]

Moreover, applying the interpolation error theory, the following operator error estimates hold:
\[
\left\| \int_0^t \xi_a(s) ds - L_a(\xi_a) \right\|_{\bar{a}, \bar{a}} \leq \nonumber
\tau \left\| \partial_t \xi \right\|_{L^2(0, T; H^1(a_0))}, \nonumber
\left\| \partial_t \xi_a(t_s) - \partial_t \xi_a(t_s) \right\|_{\bar{a}, \bar{a}} \leq \nonumber
\tau \left\| \partial_t \xi \right\|_{L^2(0, T; L^2(a_0))}. \nonumber
\]

Thus, the following inequality holds
\[
\frac{1}{\tau} \left\| e^r \right\|_{\bar{a}, \bar{a}} + \left\| e^r \right\|_{i, a} \leq \nonumber
\frac{1}{2\tau} \left\| e^{r+1} \right\|_{\bar{a}, \bar{a}} + \frac{1}{2\tau} \left\| e^r \right\|_{\bar{a}, \bar{a}} + \nonumber
\tau \sum_{j=0}^{r-1} \left\| e^r \right\|_{\bar{i}, a_j} + \frac{1}{2} \left\| e^r \right\|_{i, a_j} + \nonumber
\frac{1}{2} \left\| e^r \right\|_{\bar{i}, a_j} + \left\| e^r \right\|_{\bar{i}, a_j} + C \tau. \quad (3.11) \nonumber
\]

Simplifying (3.11), we have
\[
\left( \frac{1}{2\tau} - 1 \right) \left\| e^r \right\|_{\bar{a}, \bar{a}} + \left\| e^r \right\|_{i, a_j} \leq \nonumber
\frac{1}{2\tau} \left\| e^{r+1} \right\|_{\bar{a}, \bar{a}} + \left\| e^r \right\|_{\bar{i}, a_j} + \nonumber
\tau \sum_{j=0}^{r-1} \left\| e^r \right\|_{\bar{i}, a_j} + C \tau. \quad (3.12) \nonumber
\]

Noticing that \( \left\| e^r \right\|_{\bar{i}, a_j} \geq h^{-2} \left\| e^r \right\|_{i, a_j} \geq h^{-2} \left\| e^r \right\|_{i, a} \), for all \( e^r \in H_s \). Therefore,
\[
\left( \frac{1}{2\tau} - 2 \right) \left\| e^r \right\|_{\bar{a}, \bar{a}} + \left\| e^r \right\|_{i, a_j} \leq \nonumber
(h^{-2} - 1) \left\| e^r \right\|_{\bar{i}, a_j} \approx \nonumber
\frac{1}{2\tau} \left\| e^{r+1} \right\|_{\bar{a}, \bar{a}} + \nonumber
\tau \sum_{j=0}^{r-1} \left\| e^r \right\|_{\bar{i}, a_j} + C \tau. \quad (3.13) \nonumber
\]

Summing over \( n \) from 1 to \( M \) and multiplied by \( \tau \) on both side of (3.13), we get
\[
\left( \frac{1}{2\tau} - 2 \right) \left\| e^r \right\|_{\bar{a}, \bar{a}} + \nonumber
\tau \sum_{n=1}^{M} \left\| e^r \right\|_{\bar{i}, a_j} + \nonumber
(h^{-2} - 1) \sum_{n=1}^{M} \tau \left\| e^r \right\|_{i, a_j} \approx \nonumber
2 \sum_{n=1}^{M} \left\| e^{r+1} \right\|_{\bar{i}, a_j} + \nonumber
\tau \sum_{n=1}^{M} \tau \left\| e^r \right\|_{\bar{i}, a_j} + \tau. \quad (3.14) \nonumber
\]

Therefore,
\[
\left\| e^r \right\|_{\bar{a}, \bar{a}} + \tau \sum_{n=1}^{M} \left\| e^r \right\|_{i, a_j} + \nonumber
\sum_{n=1}^{M} \tau \left\| e^r \right\|_{\bar{i}, a_j} \leq \nonumber
\lambda \sum_{n=1}^{M} \left( \left\| e^r \right\|_{\bar{i}, a_j} + \right. \nonumber
\left. \tau \sum_{j=0}^{r-1} \left\| e^r \right\|_{\bar{i}, a_j} + \tau^2 \right), \quad (3.15) \nonumber
\]

where \( \lambda = \max \left[ \frac{2}{2\tau - 2\tau}, \frac{1}{h^{-2} - 1}, 1 \right] \).

Applying the discrete Gronwall inequality: \( a_s + b_s \lesssim c_s + \lambda \sum_{j=0}^{s-1} a_s \Rightarrow a_s + b_s \lesssim c_s \exp(n\lambda) \), yields
\[
\left\| e^r \right\|_{\bar{a}, \bar{a}} + \tau \sum_{n=1}^{M} \left\| e^r \right\|_{i, a_j} + \nonumber
\tau \sum_{n=1}^{M} \left\| e^r \right\|_{\bar{i}, a_j} \approx \tau^2. \quad (3.16) \nonumber
\]

Recalling the definition of \( e^r \), and applying the triangle inequality, we obtain
\[
\left\| \vec{u}_a(t_s) - \vec{u}_a \right\|_{i, a_j} \leq \nonumber
\left\| \int_0^t \xi_a(s) ds - L_a(\xi_a) \right\|_{\bar{a}, \bar{a}} + \nonumber
\left\| L_a(\xi_a) - L_a(\xi_a) \right\|_{i, a_j} \approx \nonumber
\tau + \tau \sum_{n=1}^{M} \left\| e^r \right\|_{i, a_j} \approx \tau. \quad (3.17) \nonumber
\]

\[
\sum_{n=1}^{M} \left\| \vec{v}_a(t_s) - \vec{v}_a \right\|_{i, a_j} \leq \nonumber
\sum_{n=1}^{M} \left\| e^r \right\|_{i, a_j} \approx \tau \quad (3.18) \nonumber
\]

\[
\sum_{n=1}^{M} \left\| \vec{u}_a(t_s) - \vec{u}_a \right\|_{i, a_j} + \left\| \vec{u}_a(t_s) - \vec{u}_a \right\|_{i, a_j} \approx \nonumber
\tau \sum_{n=1}^{M} \left\| e^r \right\|_{i, a_j} \approx \tau. \quad (3.19) \nonumber
\]

Combining (3.17) with (3.19), (2.22), with the triangle inequality, we obtain (3.9).

Reference:


