Sorted Putnam Problems and Hints

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Chapter 1

Extremal Problems

**Topics:** Critical points, gradient, derivative tests, variations


**General Hints:**

1. Don’t always try to use calculus. Start with an example and ask what can be done to decrease or increase the function.

2. Completing the square: Try to write the function as

\[ f(x) = C + h(|x - a|) \]

where \( h(u) \) is a positive increasing function. The minimum occurs at \( x = a \).

3. Replace the given function by a simpler one which you know has the same maximum or minimum.

4. In geometric problems, extrema occur when things are perpendicular.

5. In geometric problems, try to use convexity to prove that something is a minimum or maximum.

6. Positive second derivative means convex upward; negative second derivative means convex downward.
Example Problems:

1. Prove that the product of \( n \) successive integers is always divisible by \( n! \).

Hints: Reduce to \( n \) successive positive integers. Then choose the smallest case where it doesn’t happen and derive a contradiction.

2. Show that there exists a rational number, \( c/d \), with \( d < 100 \), such that

\[
\left\lfloor k \frac{c}{d} \right\rfloor = \left\lfloor k \frac{73}{100} \right\rfloor, \quad \text{for } k = 1, 2, 3, \ldots, 99.
\]

Here \( \lfloor x \rfloor \) means the greatest integer less than or equal to \( x \).

Hints: Choose \( c, d \) with \( \left( \frac{c}{d} - \frac{73}{100} \right) \) minimal.

3. Given a finite number of points in the plane, not all collinear, prove there is a straight line which passes through exactly two of them.

Hints: Minimize the distance \( d(P, L) \) where \( P \) ranges over the points and \( L \) ranges over the lines passing through at least two of the points but not through \( P \).

Problems to do:

1. Let \( S \) be a set of \( n \) distinct real numbers. Let \( A_S \) be the set of averages of two distinct elements of \( S \). For a given \( n \geq 2 \), what is the smallest possible number of distinct elements in \( A_S \)?

2. Which configurations of five (not necessarily distinct) points \( p_1, \ldots, p_5 \) on the circle \( x^2 + y^2 = 1 \) maximize the sum of the ten distances

\[
\sum_{i<j} d(p_i, p_j)?
\]

[Here \( d(p, q) \) denotes the straight line distance between \( p \) and \( q \).]

3. \( P \) is an interior point of the angle whose sides are the rays \( \overrightarrow{OA} \) and \( \overrightarrow{OB} \).

Locate \( X \) on \( \overrightarrow{OA} \) and \( Y \) on \( \overrightarrow{OB} \) so that the line segment \( XY \) contains \( P \) and so that the product of distances \( (PX)(PY) \) is a minimum.
4. Find the positive integers $n$ and $a_1, a_2, \ldots, a_n$ such that

$$a_1 + a_2 + \cdots + a_n = 1979$$

and the product $a_1a_2\cdots a_n$ is as large as possible.

**Hints:** Work on two numbers $a_i, a_j$ at a time to see what maximizes the product.

5. Find, with explanation, the maximum value of $f(x) = x^3 - 3x$ on the set of all real numbers $x$ satisfying $x^4 + 36 \leq 13x^2$.

6. Determine the minimum value of

$$(r - 1)^2 + \left( \frac{s}{r} - 1 \right)^2 + \left( \frac{t}{s} - 1 \right)^2 + \left( \frac{4}{t} - 1 \right)^2$$

for all real numbers $r, s, t$ with $1 \leq r \leq s \leq t \leq 4$.

7. Find the maximum value of

$$\int_0^y \sqrt{x^4 + (y-y^2)^2} \, dx$$

for $0 \leq y \leq 1$.

**Hints:** Replace the integral by something simpler that you can prove has the same maximum.

8. The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

9. Find the minimum value of

$$(u - v)^2 + \left( \sqrt{2 - u^2} - \frac{9}{v} \right)^2$$

for $0 < u < \sqrt{2}$ and $v > 0$.

**Hints:** Interpret the function as a distance and draw a revealing picture.
10. Let $T$ be an acute triangle. Inscribe a pair $R, S$ of rectangles in $T$ as shown:

Let $A(X)$ denote the area of polygon $X$. Find the maximum value, or show that no maximum exists, of $\frac{A(R)+A(S)}{A(T)}$, where $T$ ranges over all triangles and $R, S$ over all rectangles as above.
Chapter 2
Limits–Continuity

Topics: Squeezing theorem


General Hints:

1. In a limit as $n \to \infty$, try as many $n = 1, 2, 3, \ldots$ as practical to get an idea of the limiting value.

2. Strictly monotone (always increasing or always decreasing) sequences which are bounded must have limits.

Problems to do:

1. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt{n} - \sqrt{m}, (n, m = 0, 1, 2, \ldots)$?

Hints: Note that $\lim_{n \to \infty} \sqrt{n} = \infty$. Prove that $\lim_{n \to \infty} \sqrt{n + 1} - \sqrt{n} = 0$. Use these two facts to solve the problem.

2. Evaluate

$$\sqrt{\frac{2207 - \frac{1}{2207}}{2207 - \frac{1}{2207} - \ldots}}$$

Express your answer in the form $\frac{a + b\sqrt{c}}{d}$, where $a, b, c, d$ are integers.
Hints: Define the expression as the limit of a sequence \( x_1 = 2207, \ x_{n+1} = 2207 - \frac{1}{x_n} \). Ask natural questions about this sequence (increasing or decreasing?).

3. Let \((x_1, y_1) = (0.8, 0.6)\) and let \(x_{n+1} = x_n \cos y_n - y_n \sin y_n\) and \(y_{n+1} = x_n \sin y_n + y_n \cos y_n\) for \(n = 1, 2, 3, \ldots\). For each of \(\lim_{n \to \infty} x_n\) and \(\lim_{n \to \infty} y_n\), prove that the limit exists and find it or prove that the limit does not exist.

Hints: Use complex numbers \(z_n = x_n + iy_n\). What kind of numbers are the \(z_n\)?

4. Does there exist an infinite sequence of closed discs \(D_1, D_2, D_3, \ldots\) in the plane, with centers \(c_1, c_2, c_3, \ldots\), respectively, such that

(i) the \(c_i\) have no limit point in the finite plane,

(ii) the sum of the areas of the \(D_i\) is finite, and

(iii) every line in the plane intersects at least one of the \(D_i\)?

Hints: Think of area vs. length. What is a series whose convergence properties differ from that of the series of squares of the original terms?

5. For any pair \((x, y)\) of real numbers, a sequence \((a_n(x, y))_{n \geq 0}\) is defined as follows

\[
a_0(x, y) = x
\]

\[
a_{n+1}(x, y) = \frac{(a_n(x, y))^2 + y^2}{2}, \quad \text{for all } n \geq 0
\]

Find the area of \(\{(x, y) \mid (a_n(x, y))_{n \geq 0} \text{ converges}\}\).

Hints: Get a rough estimate of how large \(y\) may be.

6. Let \(d\) be a real number. For each integer \(m \geq 0\), define a sequence \(\{a_m(j)\}, j = 0, 1, 2, \ldots\) by the condition

\[
a_m(0) = d/2^m, \quad \text{and } a_m(j + 1) = (a_m(j))^2 + 2a_m(j), \quad j \geq 0.
\]

Evaluate \(\lim_{n \to \infty} a_n(n)\).
7. Let $D_n$ denote the value of the $(n - 1) \times (n - 1)$ determinant

$$
\begin{vmatrix}
3 & 1 & 1 & 1 & \cdots & 1 \\
1 & 4 & 1 & 1 & \cdots & 1 \\
1 & 1 & 5 & 1 & \cdots & 1 \\
1 & 1 & 1 & 6 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & n + 1
\end{vmatrix}
$$

Is the set $\left\{ \frac{D_n}{n!} \right\}_{n \geq 2}$ bounded?

**Hints:** Row-reduce to find the determinant.

8. Let $0 < a < b$. Evaluate $\lim_{t \to 0^+} \left\{ \int_0^1 [bx + a(1 - x)]^t \, dx \right\}^{1/t}$.

[The final answer should not involve any operations other than addition, subtraction, multiplication, division, and exponentiation.]

9. Find

$$
\lim_{t \to \infty} \left[ e^{-t} \int_0^t \int_0^t \frac{e^x - e^y}{x - y} \, dx \, dy \right]
$$

or show that the limit does not exist.

10. In three-dimensional Euclidean space, define a slab to be the open set of points lying between two parallel planes. The distance between the planes is called the thickness of the slab. Given an infinite sequence $S_1, S_2, \cdots$ of slabs of thickness $d_1, d_2, \cdots$, respectively, such that $\sum_{i=1}^{\infty} d_i$ converges, prove that there is some point in the space which is not contained in any of the slabs.
11. In the standard definition, a real-valued function of two real variables 
\( g : \mathbb{R}^2 \to \mathbb{R}^1 \) is \textit{continuous} if, for every point \( (x_0, y_0) \in \mathbb{R}^2 \) and every \( \epsilon > 0 \), there is a corresponding \( \delta > 0 \) such that \( \left| (x-x_0)^2 + (y-y_0)^2 \right|^{1/2} < \delta \) implies \( |g(x, y) - g(x_0, y_0)| < \epsilon \).

By contrast, \( f : \mathbb{R}^2 \to \mathbb{R}^1 \) is said to be \textit{continuous in each variable separately} if, for each fixed value \( y_0 \) of \( y \), the function \( f(x, y_0) \) is continuous in the usual sense as a function of \( x \), and similarly \( f(x_0, y) \) is continuous as a function of \( y \) for each fixed \( x_0 \).

Let \( f : \mathbb{R}^2 \to \mathbb{R}^1 \) be continuous in each variable separately. Show that there exists a sequence of continuous functions \( g_n : \mathbb{R}^2 \to \mathbb{R}^1 \) such that

\[
f(x, y) = \lim_{n \to \infty} g_n(x, y) \text{ for all } (x, y) \in \mathbb{R}^2.
\]

12. Let \( (r_n)_{n \geq 0} \) be a sequence of positive real numbers such that \( \lim_{n \to \infty} r_n = 0 \). Let \( S \) be the set of numbers representable as a sum

\[
r_i = r_{i_1} + r_{i_2} + \cdots + r_{i_{1994}}, \quad \text{with } i_1 < i_2 < \cdots < i_{1994}.
\]

Show that every nonempty interval \( (a, b) \) contains a nonempty subinterval \( (c, d) \) that does not intersect \( S \).

13. For each \( x > e^e \) define a sequence \( S_x = u_0, u_1, u_2, \ldots \) recursively as follows: \( u_0 = e \), while for \( n \geq 0 \), \( u_{n+1} \) is the logarithm of \( x \) to the base \( u_n \). Prove that \( S_x \) converges to a number \( g(x) \) and that the function \( g \) defined in this way is continuous for \( x > e^e \).

14. The sequence \( \{Q_n(x)\} \) of polynomials is defined by

\[
Q_1(x) = 1 + x, \quad Q_2(x) = 1 + 2x,
\]

and, for \( m \geq 1 \), by

\[
Q_{2m+1}(x) = Q_{2m}(x) + (m + 1)xQ_{2m-1}(x),
Q_{2m+2}(x) = Q_{2m+1}(x) + (m + 1)xQ_{2m}(x).
\]

Let \( x_n \) be the largest real solution of \( Q_n(x) = 0 \). Prove that \( \{x_n\} \) is an increasing sequence and that \( \lim_{n \to \infty} x_n = 0 \).
15. Let \( \sigma \) be a bijection of the positive integers, that is, a one-to-one function from \( \{1, 2, 3, \ldots \} \) onto itself. Let \( x_1, x_2, x_3, \ldots \) be a sequence of real numbers with the following three properties:

(a) \( |x_n| \) is a strictly decreasing function of \( n \);
(b) \( |\sigma(n) - n| \cdot |x_n| \to 0 \) as \( n \to \infty \);
(c) \( \lim_{n \to \infty} \sum_{k=1}^{n} x_k = 1 \).

Prove or disprove that these conditions imply that

\[
\lim_{n \to \infty} \sum_{k=1}^{n} x_{\sigma(k)} = 1.
\]

16. Let \( E(n) \) denote the largest integer \( k \) such that \( 5^k \) is an integral divisor of the product \( 1^1 2^2 3^3 \cdots n^n \). Calculate

\[
\lim_{n \to \infty} \frac{E(n)}{n^2}.
\]
Chapter 3

Polynomials

Topics: Factoring, Fundamental theorem of algebra, Rational functions

Problems: 89-A3

1. Every complex polynomial can be factored into linear polynomials:

\[ p(x) = (x - a_1) \ldots (x - a_n) \]

where the \( a \)'s are the complex roots.

2. If the coefficients of \( p(x) \) are real, the nonreal roots occur in conjugate pairs.

3. With coefficients in any field \( F \) (\( \mathbb{C}, \mathbb{R}, \mathbb{Q}, \mathbb{F}_p \), etc.), any polynomial may be factored as a product of irreducible polynomials:

\[ p(x) = f_1(x)f_2(x) \ldots f_r(x) \]

and this factorization is unique up to rearrangement.

4. A polynomial \( p(x) \) has a root of multiplicity \( k \) at \( x = a \) if and only if \( p^{(j)}(a) = 0 \) for \( 0 \leq j \leq k - 1 \) and \( p^{(k)}(a) \neq 0 \). Here \( p^{(j)}(x) \) denotes the \( j \)-th derivative of \( p(x) \).

5. Given the factorization in (1) above, the derivative of \( p(x) \) is

\[ p'(x) = \sum_{i=1}^{n} \prod_{\substack{1 \leq j \leq n \\text{if} \ j \neq i}} (x - a_j) \]
6. Linear Fractional Transformations are the simplest rational functions:

\[ T(z) = \frac{az + b}{cz + d} \]

These transformations are defined for all complex numbers \( z \) including \( \infty \) which is thought of as the “north pole” of the Riemann sphere sitting tangent to the complex plane at 0. The finite points of the plane correspond in a one-to-one way with the other points of the sphere by “stereographic projection,” i.e. drawing a line starting from \( \infty \) and passing through the sphere and the plane.

(a) \( T \) maps “circles” to “circles,” where a circle can mean an ordinary circle or a straight line (which can be viewed as a circle passing through infinity).

(b) \( T \) is a bijection of the Riemann sphere onto itself.

(c) Composition of two linear fractional transpositions corresponds to multiplication of the corresponding \( 2 \times 2 \) matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \).

**Problems to do:**

1. For which real numbers \( c \) is there a straight line that intersects the curve

\[ y = x^4 + 9x^3 + cx^2 + 9x + 4 \]

in four distinct points?

2. Find all real polynomials \( p(x) \) of degree \( n \geq 2 \) for which there exist real numbers \( r_1 < r_2 < \cdots < r_n \) such that

(i) \( p(r_i) = 0 \), for \( i = 1, 2, \ldots, n \),

and

(ii) \( p' \left( \frac{r_i + r_{i+1}}{2} \right) = 0 \), for \( i = 1, 2, \ldots, n - 1 \), where \( p'(x) \) denotes the derivative of \( p(x) \).
3. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 s + a_2 x^2 + \cdots + a_n x^n$$

has exactly $n$ distinct real roots?

4. Prove that if

$$11z^{10} + 10iz^9 + 10iz - 11 = 0,$$

then $|z| = 1$. (Here $z$ is a complex number and $i^2 = -1$.)

5. A composite (positive integer) is a product $ab$ with $a$ and $b$ not necessarily distinct integers in $\{2, 3, 4, \ldots\}$. Show that every composite is expressible as $xy + xz + yz + 1$, with $x$, $y$, and $z$ positive integers.

6. Let $P$ be a polynomial, with real coefficients, in three variables and $F$ a function of two variables such that

$$P(ux, uy, uz) = u^2 F(y - x, z - x)$$

for all real $x, y, z, u$,

and such that $P(1, 0, 0) = 4$, $P(0, 1, 0) = 5$, and $P(0, 0, 1) = 6$. Also let $A, B, C$ be complex numbers with $P(A, B, C) = 0$ and $|B - A| = 10$. Find $|C - A|$.

7. Curves $A$, $B$, $C$, and $D$ are defined in the plane as follows:

$$A = \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\},$$

$$B = \left\{(x, y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\},$$

$$C = \left\{(x, y) : x^3 - 3xy^2 + 3y = 1\right\},$$

$$D = \left\{(x, y) : 3x^2 y - 3x - y^3 = 0\right\}.$$

Prove that $A \cap B = C \cap D$. 

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8. Let \( f(x, y, z) = x^2 + y^2 + z^2 + xyz \). Let \( p(x, y, z), q(x, y, z), r(x, y, z) \)
be polynomials with real coefficients satisfying

\[
 f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).
\]

Prove or disprove the assertion that the sequence \( p, q, r \) consists of
some permutation of \( \pm x, \pm y, \pm z \), where the number of minus signs is
0 or 2.

9. Prove that there are only a finite number of possibilities for the ordered
triple \( T = (x - y, y - z, z - x) \) where \( x, y, z \) are complex numbers
satisfying the simultaneous equations

\[
 x(x - 1) + 2yz = y(y - 1) + 2zx = z(z - 1) + 2xy,
\]

and list all such triples \( T \).

10. Let \( a_1, a_2, \ldots, a_n \) be real numbers, and let \( b_1, b_2, \ldots, b_n \) be distinct posi-
tive integers. Suppose there is a polynomial \( f(x) \) satisfying the identity

\[
 (1 - x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.
\]

Find a simple expression (not involving any sums) for \( f(1) \) in terms of
\( b_1, b_2, \ldots, b_n \) and \( n \) (but independent of \( a_1, a_2, \ldots, a_n \)).

11. Let \( k \) be the smallest positive integer with the following property:

There are distinct integers \( m_1, m_2, m_3, m_4, m_5 \) such that the polyno-
mial

\[
 p(x) = (x - m_1)(x - m_2)(x - m_3)(x - m_4)(x - m_5)
\]

has exactly \( k \) nonzero coefficients.

Find, with proof, a set of integers \( m_1, m_2, m_3, m_4, m_5 \) for which this
minimum \( k \) is achieved.
12. If \( p(x) = a_0 + a_1 x + \cdots + a_m x^m \) is a polynomial with real coefficients \( a_i \), then set
\[
\Gamma(p(x)) = a_0^2 + a_1^2 + \cdots + a_m^2.
\]
Let \( f(x) = 3x^2 + 7x + 2 \). Find, with proof, a polynomial \( g(x) \) with real coefficients such that
(i) \( g(0) = 1 \), and
(ii) \( \Gamma(f(x)^n) = \Gamma(g(x)^n) \), for every integer \( n \geq 1 \).

13. Let \( P(x) \) be a polynomial with real coefficients and form the polynomial
\[
Q(x) = (x^2 + 1)P(x)P'(x) + x([P(x)]^2 + [P'(x)]^2).
\]
Given that the equation \( P(x) = 0 \) has \( n \) distinct real roots exceeding 1, prove or disprove that the equation \( Q(x) = 0 \) has at least \( 2n - 1 \) distinct real roots.

14. For \( k = 1, 2, \ldots, n \) let \( z_k = x_k + iy_k \), where the \( x_k \) and \( y_k \) are real and \( i = \sqrt{-1} \). Let \( r \) be the absolute value of the real part of
\[
\pm \sqrt{x_1^2 + z_2^2 + \cdots + z_n^2}.
\]
Prove that \( r \leq |x_1| + |x_2| + \cdots + |x_n| \).

15. Find the largest \( A \) for which there is a polynomial
\[
P(x) = Ax^4 + Bx^3 + Cx^2 + Dx + E
\]
with real coefficients, which satisfies
\[
0 \leq P(x) \leq 1 \quad \text{for} \quad -1 \leq x \leq 1.
\]

16. Determine all solutions in real numbers \( x, y, z, w \) of the system
\[
\begin{align*}
x + y + z &= w, \\
\frac{1}{2} + \frac{1}{y} + \frac{1}{z} &= \frac{1}{w}.
\end{align*}
\]
17. Consider all lines which meet the graph of
\[ y = 2x^4 + 7x^3 + 3x - 5 \]
in four distinct points, say \((x_i, y_i), \ i = 1, 2, 3, 4\). Show that \( \frac{x_1 + x_2 + x_3 + x_4}{4} \) is independent of the line and find its value.

18. Let \( r \) be a root of \( P(x) = x^3 + ax^2 + bx - 1 = 0 \) and \( r + 1 \) be a root of \( y^3 + cy^2 + dy + 1 = 0 \), where \( a, b, c, \) and \( d \) are integers. Also let \( P(x) \) be irreducible over the rational numbers. Express another root \( s \) of \( P(x) - 0 \) as a function of \( r \) which does not explicitly involve \( a, b, c, \) or \( d \).

19. Let \( P(x, y) = x^2y + xy^2 \) and \( Q(x, y) = x^2 + xy + y^2 \). For \( n = 1, 2, 3, \ldots \), let \( F_n(x, y) = (x + y)^n - x^n - y^n \) and \( G_n(x, y) = (x + y)^n + x^n + y^n \).
One observes that \( G^2 = 2Q, \ F_3 = 3P, \ G_4 = 2Q^2, \ F_5 = 5PQ, \ G_6 = 2Q^3 + 3P^2 \). Prove that, in fact, for each \( n \) either \( F_n \) or \( G_n \) is expressible as a polynomial in \( P \) and \( Q \) with integer coefficients.

20. Let \( s_k(a_1, \ldots, a_n) \) denote the \( k \)-th elementary symmetric function of \( a_1, \ldots, a_n \). With \( k \) held fixed, find the supremum (or least upper bound) \( M_k \) of
\[ s_k(a_1, \ldots, a_n)/[s_1(a_1, \ldots, a_n)]^k \]
for arbitrary \( n \geq k \) and arbitrary \( n \)-tuples \( a_1, \ldots, a_n \) of positive real numbers.

[The symmetric function \( s_k(a_1, \ldots, a_n) \) is the sum of all \( k \)-fold products of the variables \( a_1, \ldots, a_n \). Thus, for example:
\[ s_1(a_1, \ldots, a_n) = a_1 + a_2 + \cdots + a_n; \]
\[ s_3(a_1, a_2, a_3, a_4) = a_1a_2a_3 + a_1a_2a_4 + a_1a_3a_4 + a_2a_3a_4. \]
It should be remarked that the supremum \( M_k \) is never attained; it is approached arbitrarily closely when, for fixed \( k \), the number \( n \) of variables increases without bound, and the values \( a_i > 0 \) are suitably chosen.]
21. Let \( n = 2m \), where \( m \) is an odd integer greater than 1. Let \( \theta = e^{2\pi i/n} \).
Express \( (1 - \theta)^{-1} \) explicitly as a polynomial in \( \theta \),
\[
ak_k \theta^k + a_{k-1} \theta^{k-1} + \cdots + a_1 \theta + a_0,
\]
with integer coefficients \( a_k \).
[Note that \( \theta \) is a primitive \( n \)-th root of unity, and thus it satisfies all of the identities which hold for such roots.]

22. For which ordered pairs of real numbers \( b, c \) do both roots of the quadratic equation
\[
z^2 + bz + c = 0
\]
lie inside the unit disk \( \{ |z| < 1 \} \) in the complex plane?
Draw a reasonably accurate picture (i.e., ‘graph’) of the region in the real \( bc \)-plane for which the above condition holds. Identify precisely the boundary curves of this region.

23. (a) Let \( z \) be a solution of the quadratic equation
\[
a z^2 + bz + c = 0
\]
and let \( n \) be a positive integer. show that \( z \) can be expressed as a rational function of \( z^n, a, b, c \).
(b) Using (a) or by any other means, express \( x \) as a rational function of \( x^3 \) and \( x + \frac{1}{x} \). (Display your answer explicitly in a clearly visible form.)
[By a rational function of several variables, we mean a quotient of polynomials in those variables, the polynomials having rational numbers as coefficients, and the denominator being not identically zero. Thus to obtain \( x \) as a rational function of \( u = x^2 \) and \( v = x + (1/x) \), we could write \( x = (u + 1)/v \).]
Chapter 4

Set Theory

Topics: Venn diagrams, unions and intersections, countability, one-to-one correspondences
Problems: 85-A1, 90-A6, 91-A6

Problems to do:

1. Let \( S \) be a set of \( n \) distinct real numbers. Let \( A_S \) be the set of averages of two distinct elements of \( S \). For a given \( n \geq 2 \), what is the smallest possible number of distinct elements in \( A_S \)?

2. Let \( f_1, f_2, \ldots, f_{10} \) be bijections of the set of integers such that for each integer \( n \), there is some composition \( f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_{10}} \) of these functions (allowing repetitions) which maps \( 0 \) to \( n \). Consider the set of 1024 functions

\[
F = \{ f_1^{\epsilon_1} \circ f_2^{\epsilon_2} \circ \cdots \circ f_{10}^{\epsilon_{10}} \mid \epsilon_i = 0 \text{ or } 1 \text{ for } 1 \leq i \leq 10 \}
\]

(\( f_i^0 \) is the identity function and Show that if \( A \) is any nonempty finite set of integers, then at most 512 of the functions in \( F \) map \( A \) to itself.

3. Let \( \mathcal{P}_n \) be the set of subsets of \( \{1, 2, \ldots, n\} \). Let \( c(n, m) \) be the number of functions \( f : \mathcal{P}_n \to \{1, 2, \ldots, m\} \) such that \( f(A \cap B) = \min\{f(A), f(B)\} \). Prove that

\[
c(n, m) = \sum_{j=1}^{m} j^n.
\]
4. Can a countably infinite set have an uncountable collection of non-empty subsets such that the intersection of any two of them is finite?

5. Does there exist a subset $B$ of the unit circle $x^2 + y^2 = 1$ such that

   (a) $B$ is topologically closed, and
   (b) $B$ contains exactly one point from each pair of diametrically opposite points on the circle?

   (A set $B$ is topologically closed if it contains the limit of every convergent sequence of points in $B$.)
Chapter 5

Combinatorics

Topics: Choosing subsets of sets, pigeonhole principle, graphs and coloring

Problems to do:

1. For a partition $\pi$ of \{1, 2, 3, 4, 5, 6, 7, 8, 9\}, let $\pi(x)$ be the number of elements in the part containing $x$. Prove that for any two partitions $\pi$ and $\pi'$, there are two distinct numbers $x$ and $y$ in \{1, 2, 3, 4, 5, 6, 7, 8, 9\} such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$. [A partition of a set $S$ is a collection of disjoint subsets (parts) whose union is $S$.]

2. Suppose we have a necklace of $n$ beads. Each bead is labeled with an integer and the sum of all these labels is $n - 1$. Prove that we can cut the necklace to form a string whose consecutive labels $x_1, x_2, \ldots, x_n$ satisfy

$$\sum_{i=1}^{k} x_i \leq k - 1 \quad \text{for } k = 1, 2, \ldots, n.$$

3. Show that if the points of an isosceles right triangle of side length 1 are colored with one of four colors, then there must be two points of the same color which are at least a distance $2 - \sqrt{2}$ apart.

4. Let $A(n)$ denote the number of sums of positive integers

$$a_1 + a_2 + \cdots + a_r$$

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which add up to \( n \) with
\[
a_1 > a_2 + a_3, \ a_2 > a_3 + a_4, \ldots, \ a_{r-2} > a_{r-1} + a_r, \ a_{r-1} > a_r.
\]

Let \( B(n) \) denote the number of \( b_1 + b_2 + \cdots + b_s \) which add up to \( n \), with

(i) \( b_1 \geq b_2 \geq \cdots \geq b_s \),
(ii) each \( b_i \) is in the sequence \( 1, 2, 4, \ldots, g_j, \ldots \) defined by \( g_1 = 1 \),
\[ g_2 = 2, \text{ and } g_j = g_{j-1} + g_{j-2} + 1, \text{ and } \]
(iii) if \( b_1 = g_k \) then every element in \( \{1, 2, 4, \ldots, g_k\} \) appears at least once as a \( b_i \).

Prove that \( A(n) = B(n) \) for each \( n \geq 1 \).

(For example, \( A(7) = 5 \) because the relevant sums are \( 7, 6 + 1, 5 + 2, 4 + 3, 4 + 2 + 1 \), and \( B(7) = 5 \) because the relevant sums are \( 4 + 2 + 1, 2 + 2 + 1 2 + 2 + 1 + 1 + 2 + 1 + 1 + 1 + 1, 1 + 1 + 1 + 1 + 1 + 1 \).)

5. If \( X \) is a finite set, let \( |X| \) denote the number of elements in \( X \). Call an ordered pair \((S, T)\) of subsets of \( \{1, 2, \ldots, n\} \) admissible if \( s > |T| \) for each \( s \in S \), and \( t |S| \) for each \( t \in T \). How many admissible ordered pairs of subsets of \( \{1, 2, \ldots, 10\} \) are there? Prove your answer.

6. (a) If every point of the plane is painted one of three colors, do there necessarily exist two points of the same color exactly one inch apart?
(b) What if “three” is replaced by “nine”?

Justify your answers.

7. Let
\[
\begin{align*}
a_{1,1} & \quad a_{1,2} & \quad a_{1,3} & \quad \cdots \\
a_{2,1} & \quad a_{2,2} & \quad a_{2,3} & \quad \cdots \\
a_{3,1} & \quad a_{3,2} & \quad a_{3,3} & \quad \cdots \\
\vdots & \quad \vdots & \quad \vdots & \quad \ddots
\end{align*}
\]

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that \( a_{m,n} > mn \) for some pair of positive integers \((m,n)\).
8. Determine, with proof, the number of ordered triples \((A_1, A_2, A_3)\) of sets which have the property that

(i) \(A_1 \cup A_2 \cup A_3 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\), and

(ii) \(A_1 \cap A_2 \cap A_3 = \emptyset\),

where \(\emptyset\) denotes the empty set. Express the answer in the form \(2^a3^b5^c7^d\), where \(a, b, c\) and \(d\) are nonnegative integers.

9. Two distinct squares of the \(8 \times 8\) chessboard \(C\) are said to be adjacent if they have a vertex or side in common. Also, \(g\) is called a \(C\)-gap if for every numbering of the squares of \(C\) with all the integers \(1, 2, \ldots, 64\) there exist two adjacent squares whose numbers differ by at least \(g\). Determine the largest \(C\)-gap \(g\).

10. Let \(A_1, A_2, \ldots, A_{1066}\) be subsets of a finite set \(X\) such that \(|A_i| > \frac{1}{2}|X|\) for \(1 \leq i \leq 1066\). Prove that there exist ten elements \(x_1, \ldots, x_{10}\) of \(X\) such that every \(A_i\) contains at least one of \(x_1, \ldots, x_{10}\).

(Here \(|S|\) means the number of elements in the set \(S\).)

11. Let \(A\) be a set of \(2n\) points in the plane, no three of which are collinear. Suppose that \(n\) of them are colored red and the remaining \(n\) blue. Prove or disprove: there are \(n\) closed straight line segments, no two with a point in common, such that the endpoints of each segment are points of \(A\) having different colors.

12. Let \(f_0(x) = e^x\) and \(f_{n+1}(x) = x f_n'(x)\) for \(n = 0, 1, 2, \ldots\). Show that \(\sum_{n=0}^{\infty} \frac{f_n(1)}{n!} = e^x\).
Chapter 6

Recurrence

Topics: Solution of linear recurrence relations, induction proofs
Problems: 84-B1, 90-A1, 90-B2

1. The classic linear recurrence relation is \( F_{n+2} = F_{n+1} + F_n \), which with the initial conditions \( F_1 = F_2 = 1 \) defines the Fibonacci numbers.

2. For any linear recurrence with constant coefficients:
   \[
   T_{n+k} = c_1 T_{n+k-1} + \cdots + c_k T_n, \quad n \geq 1
   \]
solve by guessing a solution of the form \( T_n = \lambda^n \) leading to the characteristic equation
   \[
   \lambda^k = c_1 \lambda^{k-1} + \cdots + c_{k-1} \lambda + c_k
   \]
If the roots of this equation are all distinct (for Fibonacci they are \( \frac{1 \pm \sqrt{5}}{2} \)), the general solution is a linear combination of the powers \( \lambda^n \). If \( \lambda \) is a root of multiplicity \( l \), it contributes \( p(n)\lambda^n \) to the general solution where \( p \) is any polynomial of degree \( l - 1 \).

3. Generating functions: It is often useful to consider the formal power series
   \[
   f(x) = \sum_{n=0}^{\infty} T_n x^n
   \]
called the generating function. If the \( T_n \)'s satisfy a linear recurrence, \( f(x) \) is a rational function of \( x \) and the denominator gives the recurrence relation. (Exercise: find the generating function for the Fibonacci numbers.)
4. In solving a recurrence equation for $f(n)$, it helps to consider the recurrence satisfied by $g(n) = f(n) - f(n - 1)$ (the finite difference of $f(n)$).

5. Carry out the recurrence far enough that you might guess a formula for it. It is easy to prove the correct formula by induction.

6. If $n^{(k)} = n(n - 1)\ldots(n - k + 1)$, then

$$(n + 1)^{(k)} - n^{(k)} = kn^{(k-1)}$$

(analogous to $(x^k)' = kx^{k-1}$.)

Problems to do:

1. Let $(x_n)_{n\geq0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1, \quad \text{for } n = 1, 2, 3, \ldots.$$ 

Prove there exists a real number $a$ such that $x_{n+1} = ax_n - x_{n-1}$ for all $n \geq 1$.

2. Prove that for $|x| < 1$, $|z| > 1$,

$$1 + \sum_{j=1}^{\infty} (1 + x^j) \frac{(1 - z)(1 - zx)(1 - z^2x^2)\ldots(1 - z^jx^{j-1})}{(z - x)(z - x^2)(z - x^3)\ldots(z - x^j)} = 0.$$ 

3. Let

$$T_0 = 2, \ T_1 = 3, \ T_2 = 6,$$

and for $n \geq 3$,

$$T_n = (n + 4)T_{n-1} - 4nT_{n-2} + (4n - 8)T_{n-3}.$$ 

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$ 

Find, with proof, a formula for $T_n$ of the form $T_n = A_n + B_n$, where $(A_n)$ and $(N_n)$ are well-known sequences.

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4. Let \( n \) be a positive integer, and define

\[
f(n) = 1! + 2! + \cdots + n!.
\]

Find polynomials \( P(x) \) and \( Q(x) \) such that

\[
f(n + 2) = P(n)f(n + 1) + Q(n)f(n),
\]

for all \( n \geq 1 \).

5. For each \( x > e^e \) define a sequence \( S_x = u_0, u_1, u_2, \ldots \) recursively as follows: \( u_0 = e \), while for \( n \geq 0 \), \( u_{n+1} \) is the logarithm of \( x \) to the base \( u_n \). Prove that \( S_x \) converges to a number \( g(x) \) and that the function \( g \) defined in this way is continuous for \( x > e^e \).

6. For which real numbers \( a \) does the sequence defined by the initial condition \( u_0 = a \) and the recursion \( u_{n+1} = 2u_n - n^2 \) have \( u_n > 0 \) for all \( n \geq 0 \)?

(Express the answer in the simplest form.)

7. The sequence \( \{Q_n(x)\} \) of polynomials is defined by

\[
Q_1(x) = 1 + x, \quad Q_2(x) = 1 + 2x,
\]

and, for \( m \geq 1 \), by

\[
Q_{2m+1}(x) = Q_{2m}(x) + (m + 1)x Q_{2m-1}(x),
\]

\[
Q_{2m+2}(x) = Q_{2m+1}(x) + (m + 1)x Q_{2m}(x).
\]

Let \( x_n \) be the largest real solution of \( Q_n(x) = 0 \). Prove that \( \{x_n\} \) is an increasing sequence and that \( \lim_{n \to \infty} x_n = 0 \).
Chapter 7

Linear Algebra

Topics: Independence, bases, matrices, invertibility, determinants
(Vandermonde in particular), block decomposition

Problems:

Problems to do:

1. Let $A$ and $B$ be $2 \times 2$ matrices with integer entries such that $A$, $A + B$, $A + 2B$, $A + 3B$, and $A + 4B$ are all invertible matrices whose inverses have integer entries. Show that $A + 5B$ is invertible and that its inverse has integer entries.

2. Let $\mathcal{M}$ be a set of real $n \times n$ matrices such that

(a) $I \in \mathcal{M}$, where $I$ is the $n \times n$ identity matrix;

(b) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB \in \mathcal{M}$ or $-AB \in \mathcal{M}$, but not both;

(c) if $A \in \mathcal{M}$ and $B \in \mathcal{M}$, then either $AB = BA$ or $AB = -BA$;

(d) if $A \in \mathcal{M}$ and $A \neq I$, there is at least one $B \in \mathcal{M}$ such that $AB = -BA$.

Prove that $\mathcal{M}$ has at most $n^2$ matrices.

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3. Let $D_n$ denote the value of the $(n - 1) \times (n - 1)$ determinant

$$
\begin{vmatrix}
3 & 1 & 1 & 1 & \cdots & 1 \\
1 & 4 & 1 & 1 & \cdots & 1 \\
1 & 1 & 5 & 1 & \cdots & 1 \\
1 & 1 & 1 & 6 & \cdots & 1 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & 1 & \cdots & n + 1 \\
\end{vmatrix}
$$

Is the set $\left\{ \frac{D_n}{n!} \right\}_{n \geq 2}$ bounded?

4. Let $A$ and $B$ be different $n \times n$ matrices with real entries. If $A^3 = B^3$ and $A^2B = B^2A$, can $A^2 + B^2$ be invertible?

5. Let $S$ be a set of $2 \times 2$ integer matrices whose entries $a_{ij}$ (1) are all squares of integers, and, (2) satisfy $a_{ij} \leq 200$. Show that if $S$ has more than $50387 (= 15^4 - 15^2 - 15 + 2)$ elements, then it has two elements that commute.

6. If $A$ and $B$ are square matrices of the same size such that $ABAB = 0$, does it follow that $BAB A = 0$?

7. For positive integers $n$, let $M_n$ be the $2n + 1$ by $2n + 1$ skew-symmetric matrix for which each entry in the first $n$ subdiagonals below the main diagonal is 1 and each of the remaining entries below the main diagonal is $-1$. Find, with proof, the rank of $M_n$. (According to one definition the rank of a matrix is the largest $k$ such that there is a $k \times k$ submatrix with non-zero determinant.)

One may note that $M_1 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ and

$M_2 = \begin{pmatrix} 0 & -1 & -1 & 1 & 1 \\ 1 & 0 & -1 & -1 & 1 \\ 1 & 1 & 0 & -1 & -1 \\ -1 & -1 & 1 & 1 & 0 \end{pmatrix}$.
8. If a linear transformation $A$ on an $n$-dimensional vector space has $n+1$
e eigenvectors such that any $n$ of them are linearly independent, does it follow that $A$ is a scalar multiple of the identity? Prove your answer.

9. Let $O_n$ be the $n$-dimensional zero vector $(0,0,\ldots,0)$. Let $M$ be a $2n \times n$ matrix of complex numbers such that whenever $(z_1, z_2, \ldots, z_{2n})M = O_n$, with complex $z_i$, not all zero, then at least one of the $z_i$ is not real. Prove that for arbitrary real numbers $r_1, r_2, \ldots, r_{2n}$, there are complex numbers $w_1, w_2, \ldots, w_n$ such that

$$\Re \left[ M \begin{pmatrix} w_1 \\ \vdots \\ \vdots \\ w_n \end{pmatrix} \right] = \begin{pmatrix} r_2 \\ \vdots \\ \vdots \\ r_{2n} \end{pmatrix}.$$  

(note: If $C$ is a matrix of complex numbers, $\Re(C)$ is the matrix whose entries are the real parts of entries of $C$.)

10. Suppose $A$, $B$, $C$, $D$ are $n \times n$ matrices with entries in a field $F$, satisfying the conditions that $AB^t$ and $CD^t$ are symmetric and $AD^t - BC^t = I$. Here $I$ is the $n \times n$ identity matrix, and if $M$ is an $n \times n$ matrix, $M^t$ is the transpose of $M$. Prove that $A^tD - C^tB = I$.

11. A transversal of an $n \times n$ matrix $A$ consists of $n$ entries of $A$, no two in the same row or column. Let $f(n)$ be the number of $n \times n$ matrices $A$ satisfying the following two conditions:

(a) Each entry $a_{i,j}$ of $A$ is in the set $\{-1, 0, 1\}$.
(b) The sum of the $n$ entries of a transversal is the same for all transversals of $A$.

An example of such a matrix $A$ is

$$A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$  

Determine with proof a formula for $f(n)$ of the form

$$f(n) = a_1b_1^n + a_2b_2^n + a_3b_3^n + a_4,$$
where the $a_i$’s and $b_i$’s are rational numbers.

12. Let $n$ be a positive integer. Let $a, b, x$ be real numbers, with $a \neq b$, and let $M_n$ denote the $2n \times 2n$ matrix whose $(i, j)$ entry $m_{ij}$ is given by

$$m_{ij} = \begin{cases} 
    x & \text{if } i = j, \\
    a & \text{if } i \neq j \text{ and } i + j \text{ is even}, \\
    b & \text{if } i = n + j \text{ and } i + j \text{ is odd}.
\end{cases}$$

Thus, for example, $M_2 = \begin{pmatrix} x & b & a & b \\ a & b & x & b \\ b & a & b & x \end{pmatrix}$.

Express $\lim_{x \to a} \frac{\det M_n}{(x - a)^{2n-2}}$ as a polynomial in $a$, $b$, and $n$, where $\det M_n$ denotes the determinant of $M_n$.

13. Let $V$ be a set of $5$ by $7$ matrices, with real entries and with the property that $rA + sB \in V$ whenever $A, B \in V$ and $r$ and $s$ are scalars (i.e., real numbers). Prove or disprove the following assertion: If $V$ contains matrices of ranks $0, 1, 2, 4,$ and $5$, then it also contains a matrix of rank $3$.

[The rank of a nonzero matrix $M$ is the largest $k$ such that the entries of some $k$ rows and some $k$ columns form a $k$ by $k$ matrix with a nonzero determinant.]

14. Let $a, b, p_1, p_2, \ldots, p_n$ be real numbers with $a \neq b$. Define $f(x) = (p_1 - x)(p_2 - x)(p_3 - x)\ldots(p_n - x)$. Show that

$$\det \begin{pmatrix} 
    p_1 & a & a & \cdots & a & a \\
    b & p_2 & a & \cdots & a & a \\
    b & b & p_3 & \cdots & a & a \\
    b & b & b & \cdots & a & a \\
    \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
    b & b & b & \cdots & p_{n-1} & a \\
    b & b & b & \cdots & b & p_n \\
\end{pmatrix} = \frac{bf(a) - af(b)}{b - a}.$$
Chapter 8
Number Theory

Topics: Prime factorization, congruences, Fermat’s little theorem, quadratic reciprocity, generating functions, approximating irrational numbers by rationals

General Hints:

1. Learn how to use the prime factorization
\[ n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \]

2. Convert divisibility relations \( m \mid n \) into terms of modular arithmetic
\[ n \equiv 0 \pmod{m} \]
and use modular arithmetic to work on the problem.

3. Questions about digit representations of numbers should make use of the meaning of such a representation. If \( a_k a_{k-1} \cdots a_2 a_1 a_0 \) is a number \( n \) in base \( b \), this means
\[ n = a_k b^k + a_{k-1} b^{k-1} + \cdots + a_1 b + a_0 \]

4. The pigeonhole principle often comes up: if \( n + 1 \) things are sorted into \( n \) categories, then at least one category contains at least two things.

5. Fermat’s theorem: For any prime number \( p \) and any integer \( a \) not divisible by \( p \), we have \( a^{p-1} \equiv 1 \pmod{p} \).
6. Euler's theorem: If \( \phi(m) \) is the number of numbers \( 1 \leq n \leq m \) which are relatively prime to \( m \), then \( a^{\phi(m)} \equiv 1 \pmod{m} \) for any integer \( a \) which is relatively prime to \( m \).

**Digits Problems:**

1. Define a sequence \( \{a_i\} \) by \( a_1 = 3 \) and \( a_{i+1} = 3^{a_i} \) for \( i \geq 1 \). Which integers between 00 and 99 inclusive occur as the last two digits in the decimal expansion of infinitely many \( a_i \)?

**Hints:** The last two digits are the result of dividing the number by 100 and taking the remainder. This suggests keeping track of the powers modulo 100. What is the smallest power of 3 which leaves remainder 1 after division by 100?

2. The sequence of digits

\[
1 2 3 4 5 6 7 8 9 1 0 1 1 1 2 1 3 1 4 1 5 1 6 1 7 1 8 1 9 2 0 2 1 \cdots
\]

is obtained by writing the positive integers in order. If the 10\(n\)th digit in this sequence occurs in the part of the sequence in which the \( m \)-digit numbers are placed, define \( f(n) \) to be \( m \). For example, \( f(2) = 2 \) because the 100th digit enters the sequence in the placement of the two-digit integer 55. Find, with proof, \( f(1987) \).

3. For each positive integer \( n \), let \( a(n) \) be the number of zeros in the base 3 representation of \( n \). For which positive real numbers \( x \) does the series

\[
\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}
\]

converge?

4. How many primes among the positive integers, written as usual in base 10, are such that their digits are alternating 1's and 0's, beginning and ending with 1?

**Hints:** Write the numbers in question as

\[
10^{2n} + 10^{2n-2} + \cdots + 10^2 + 1
\]

and sum the geometric series. Then look for proper factors.
5. The number $d_1d_2\ldots d_9$ has nine (not necessarily distinct) decimal digits. The number $e_1e_2\ldots e_9$ is such that each of the nine-digit numbers formed by replacing one of the digits $d_i$ in $d_1d_2\ldots d_9$ by the corresponding digit $e_i$ ($1 \leq i \leq 9$) is divisible by 7. The number $f_1f_2\ldots f_9$ is related to $e_1e_2\ldots e_9$ in the same way: that is, each of the nine numbers formed by replacing one of the $e_i$ by the corresponding $f_i$ is divisible by 7. Show that, for each $i$, $d_i - f_i$ is divisible by 7. [For example, if $d_1d_2\ldots d_9 = 199501996$, then $e_6$ may be 2 or 9, since 199502996 and 199509996 are multiples of 7.]

**Divisibility:**

1. Call a set of positive integers “conspiratorial” if no three of them are pairwise relatively prime. (A set of integers is “pairwise relatively prime” if no pair of them has a common divisor greater than 1.) What is the largest number of elements in any “conspiratorial” subset of the integers 1 through 16?

2. As usual, let $\sigma(N)$ denote the sum of all the (positive integral) divisors of $N$. (Including among these divisors are 1 and $N$ itself.) For example, if $p$ is a prime, the $\sigma(p) = p + 1$. Motivated by the notion of a “perfect” number, a positive integer $N$ is called “quasiperfect” if $\sigma(N) = 2N + 1$. Prove that every quasiperfect number is the square of an odd integer.

**Hints:** For the prime factorization $n = p_1^{a_1} \cdots p_k^{a_k}$, all divisors are of the form $p_1^{b_1} \cdots p_k^{b_k}$ with exponents at most equal to the earlier exponents. The goal is to prove all the $p$’s are odd and all the $a$’s are even. Find an expression for the sum of all those divisors, and then assume it’s odd.

3. Let $E(n)$ denote the largest integer $k$ such that $5^k$ is an integral divisor of the product $1^12^23^3\cdots n^n$. Calculate

$$\lim_{n \to \infty} \frac{E(n)}{n^2}.$$ 

4. How many positive integers $n$ are there such that $n$ is an exact divisor of at least one of the numbers

$$10^{10}, 20^{30}?$$

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5. For a given positive integer $m$, find all triples $(n, x, y)$ of positive integers, with $n$ relatively prime to $m$, which satisfy $(x^2 + y^2)^m = (xy)^n$.

**Polynomial Equations**

1. Supposing that an integer $n$ is the sum of two triangular numbers,

\[
    n = \frac{a^2 + a}{2} + \frac{b^2 + b}{2},
\]

write $4n + 1$ as the sum of two squares,

\[
    4n + 1 = x^2 + y^2,
\]

and show how $x$ and $y$ can be expressed in terms of $a$ and $b$. Show that, conversely, if $4n + 1 = x^2 + y^2$, then $n$ is the sum of two triangular numbers.

[Of course, $a$, $b$, $x$, $y$ are understood to be integers.]

2. Let $p_n$ be the probability that $c + d$ is a perfect square when the integers $c$ and $d$ are selected independently at random from the set \{1, 2, 3, \ldots, n\}. Show that \(\lim_{n \to \infty} (p_n \sqrt{n})\) exists and express this limit in the form \(r(\sqrt{s} - t)\), where $s$ and $t$ are integers and $r$ is a rational number.

3. Prove that there exist an infinite number of ordered pairs $(a, b)$ of integers such that for every positive integer $t$ the number $at + b$ is a triangular number if and only if $t$ is a triangular number. (The triangular numbers are the $t_n = n(n + 1)/2$ with $n$ in \{0, 1, 2, \ldots\}.)

**Rational Approximation**

1. (a) Prove that there exist integers $a, b, c$, not all zero and each of absolute value less than one million, such that

\[
    |a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}.
\]
(b) Let \( a, b, c \) be integers, not all zero and each of absolute value less than one million. Prove that
\[
|a + b\sqrt{2} + c\sqrt{3}| > 10^{-21}.
\]

2. For a positive real number \( r \), let \( G(r) \) be the minimum value of \( |r - \sqrt{m^2 + 2n^2}| \) for all integers \( m \) and \( n \). Prove or disprove the assertion that \( \lim_{r \to \infty} G(r) \) exists and equals 0.

3. For every \( n \) in the set \( \mathbb{Z}^+ = \{1, 2, \ldots\} \) of positive integers, let \( r_n \) be the minimum value of \( |c - d\sqrt{3}| \) for all nonnegative integers \( c \) and \( d \) with \( c + d = n \). Find, with proof, the smallest positive real number \( g \) with \( r_n \leq g \) for all \( n \) in \( \mathbb{Z}^+ \).

4. Find the smallest positive integer \( n \) such that for every integer \( m \), with \( 0 < m < 1993 \), there exists an integer \( k \) for which
\[
\frac{m}{1993} \leq \frac{k}{n} \leq \frac{m + 1}{1994}.
\]

**Miscellaneous**

1. In the additive group of ordered pairs of integers \((m, n)\) [with addition defined componentwise: \((m, n) + (m', n') = (m + m', n + n')\)] consider the subgroup \( H \) generated by the three elements
\[
(3, 8), \quad (4, -1), \quad (5, 4).
\]
Then \( H \) has another set of generators of the form
\[
(1, b), \quad (0, a)
\]
for some integers \( a, b \) with \( a > 0 \). Find \( a \).

[Elements \( g_1, \ldots, g_k \) are said to generate a subgroup \( H \) if (i) each \( g_i \in H \), and (ii) every \( h \in H \) can be written as a sum \( h = n_1 g_1 + \cdots + n_k g_k \) where the \( n_i \) are integers (and where, for example, \( 3g_1 - 2g_2 \) means \( g_1 + g_1 + g_1 - g_2 - g_2 \)).]
2. Let \( A \) be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7, \ldots, 100. Prove that there must be two distinct integers in \( A \) whose sum is 104.

3. Let \( x_1, x_2, \ldots, x_{19} \) be positive integers each of which is less than or equal to 93. Let \( y_1, y_2, \ldots, y_{93} \) be positive integers each of which is less than or equal to 19. Prove that there exists a (nonempty) sum of some \( x_i \)'s equal to a sum of some \( y_j \)'s.

4. Consider a paper punch that can be centered at any point of the plane and that, when operated, removes from the plane precisely those points whose distance from the center is irrational. How many punches are needed to remove every point?

5. For each integer \( n \geq 0 \), let \( S(n) = n - m^2 \), where \( m \) is the greatest integer with \( m^2 \leq n \). Define a sequence \( (a_k)_{k=0}^{\infty} \) by \( a_0 = A \) and \( a_{k+1} = a_k + S(a_k) \) for \( k \geq 0 \). For what positive integers \( A \) is this sequence eventually constant?

6. Does there exist a real number \( L \) such that, if \( m \) and \( n \) are integers greater than \( L \), then an \( m \times n \) rectangle may be expressed as a union of \( 4 \times 6 \) and \( 5 \times 7 \) rectangles, any two of which intersect at most along their boundaries?

7. Find all positive integers that are within 250 of exactly 15 perfect squares. (Note: A **perfect square** is the square of an integer; that is, a member of the set \( \{0, 1, 4, 9, 16, \ldots\} \). \( a \) is **within** \( n \) of \( b \) if \( b - n \leq a \leq b + n \).)
Chapter 9

Binomials

Topics: Pascal’s triangle, expansion of \((x + y)^n\), binomial coefficients.

Problems:

1. The main tool in analyzing combinations of binomial coefficients is the binomial expansion:

\[
(1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k
\]

Clearly, the coefficients are positive integers. Differentiating \(k\) times and plugging in \(x = 0\), we obtain

\[
n(n - 1) \ldots (n - k + 1)(1 + x)^{n-k}\bigg|_{x=0}
= \left[ \sum_{j=k}^{n} \binom{n}{j} j(j - 1) \ldots (j - k + 1)x^{j-k} \right]_{x=0}
\]

\[
n(n - 1) \ldots (n - k + 1) = \binom{n}{k} k!
\]

This proves the classic formulas

\[
\binom{n}{k} = \frac{n(n - 1) \ldots (n - k + 1)}{k!} = \frac{n!}{(n - k)!k!}
\]

This formula for the binomial coefficient is of surprisingly little use in many problems. It is often simpler to consider the original binomial expansion itself.

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2. From the fact that \((1+x)^{n+1} = (1+x)^n \cdot (1+x)\), we obtain the recurrence

\[
\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}
\]

This makes it very easy to compute the binomial coefficients in a tabular way known as Pascal’s Triangle:

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
\end{array}
\]

3. Sums of binomial coefficients: If we plug in \(x = 1\), we obtain

\[
2^n = \sum_{k=0}^{n} \binom{n}{k}
\]

If we plug in \(x = -1\), we obtain

\[
0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k
\]

which together with the previous sum implies that

\[
2^{n-1} = \sum_{k \leq n/2} \binom{n}{2k} = \sum_{k \leq (n-1)/2} \binom{n}{2k+1}
\]

Considering the binomial expansion for \((1+x)^{2n}\) as well as the square \((1+x)^n \cdot (1+x)^n\), we obtain

\[
\binom{2n}{n} = \text{coefficient of } x^n
\]

\[
= \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}
\]

\[
= \sum_{k=0}^{n} \binom{n}{k}^2
\]

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4. Choosing subsets: The binomial expansion results from multiplying out 

\[(x + y)(x + y) \ldots (x + y)\]

where the product has \(n\) factors. To get one term of the form \(x^k y^{n-k}\), we must choose an \(x\) from \(k\) of the factors and a \(y\) from the remaining \(n-k\) factors. Thus, the binomial coefficient counts the number of ways of choosing a subset of \(k\) objects from a set of \(n\) objects. In this way, the binomial coefficient commonly appears in counting problems.

5. Prime exponents: Various special phenomena occur when the exponent \(n\) is a prime \(p\). Then all the interior coefficients are divisible by \(p\). This may be summarized by the congruence

\[(1 + x)^p \equiv 1 + x^p \pmod{p}\]

In abstract algebra, this is equivalent to the fact that the \(p\)-th power map \(x \mapsto x^p\) is a field automorphism for any finite field of order \(p^f\).

**Problems to do:**

1. For nonnegative integers \(n\) and \(k\), define \(Q(n, k)\) to be the coefficient of \(x^k\) in the expansion of \((1 + x + x^2 + x^3)^n\). Prove that

\[Q(n, k) = \sum_{j=0}^{n} \binom{n}{j} \binom{n}{k - 2j},\]

where \(\binom{a}{b}\) is the standard binomial coefficient. (Reminder: For integers \(a\) and \(b\) with \(a \geq 0\), \(\binom{a}{b} = \frac{a!}{b!(a-b)!}\) for \(0 \leq b \leq a\), and \(\binom{a}{b} = 0\) otherwise.)

2. Suppose \(p\) is an odd prime. Prove that

\[\sum_{j=0}^{p} \binom{p}{j} \binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}.\]

3. Let \(r, s, t\) be integers with \(0 \leq r, 0 \leq s, \) and \(r + s \leq t\). Prove that

\[
\binom{s}{0} \binom{s}{1} \binom{s}{2} + \cdots + \binom{s}{s} = \frac{t + 1}{(t + 1 - s)(t - s)}.\]
(Note: \( \binom{n}{k} \) denotes the binomial coefficient \( \frac{n(n-1)\cdots(n+1-k)}{(k-1)\cdots3\cdot2\cdot1} \).)

4. Let \( k \) be a positive integer and let \( m = 6k - 1 \). Let

\[
S(m) = \sum_{j=1}^{2k-1} (-1)^{3j+1} \binom{m}{3j-1}.
\]

For example with \( k = 3 \),

\[
S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.
\]

Prove that \( S(m) \) is never zero. (As usual, \( \binom{m}{r} = \frac{m!}{r!(m-r)!} \).)

5. Prove that \( \binom{pa}{pb} \equiv \binom{a}{b} \pmod{p} \) for all integers \( p, a, \) and \( b \) with \( p \) a prime, \( p > 0 \), and \( a \leq b \leq 0 \).

Notation: \( \binom{m}{n} \) denotes the binomial coefficient \( \frac{m!}{n!(m-n)!} \).

6. Evaluate \( \sum_{k=0}^{n} (-1)^{k} \binom{n}{k} (x - k)^n \).

7. For a set with \( n \) elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively \( \equiv 0 \pmod{3} \), \( \equiv 1 \pmod{3} \), \( \equiv 2 \pmod{3} \)? In other words, calculate

\[
s_{i,n} = \sum_{k \equiv i \pmod{3}} \binom{n}{k} \quad \text{for } i = 0, 1, 2.
\]

Your result should be strong enough to permit direct evaluation of the numbers \( s_{i,n} \) and to show clearly the relationship of \( s_{0,n} \) and \( s_{1,n} \) and \( s_{2,n} \) to each other for all positive integers \( n \). In particular, show the relationships among these three sums for \( N = 1000 \). (An illustration of the definition of \( s_{i,n} \) is \( s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22 \).)
Chapter 10

Abstract Algebra

Topics: Ring theory, isomorphisms

Problems:

Problems to do:

1. Let $S$ be a set of real numbers which is closed under multiplication (that is, if $a$ and $b$ are in $S$, then so is $ab$). Let $T$ and $U$ be disjoint subsets of $S$ whose union is $S$. Given that the product of any three (not necessarily distinct) elements of $T$ is in $T$ and that the product of any three elements of $U$ is in $U$, show that at least one of the two subsets $T, U$ is closed under multiplication.

2. Let $S$ be a non-empty set with an associative operation that is left and right cancellative ($sy - sz$ implies $y = z$, and $yz = zx$ implies $y = z$). Assume that for every $a$ in $S$ the set $\{a^n : n = 1, 2, 3, \ldots \}$ is finite. Must $S$ be a group?

3. Prove or disprove the following statement. If $F$ is a finite set with two or more elements, then there exists a binary operation $\ast$ on $F$ such that for all $x, y, z$ in $F$,

   (i) $x \ast z = y \ast z$ implies $x = y$ (right cancellation holds), and

   (ii) $x \ast (y \ast z) \neq (x \ast y) \ast z$ (no case of associativity holds).
4. A “bypass” operation on a set $S$ is a mapping from $S \times S$ to $S$ with the property

$$B(B(w, x), B(y, z)) = B(w, z)$$

for all $w, x, y, z$ in $S$.

(a) Prove that $B(a, b) = c$ implies $B(c, c) = c$ when $B$ is a bypass.

(b) Prove that $B(a, b) = c$ implies $B(a, x) = B(c, x)$ for all $x$ in $S$ when $B$ is a bypass.

(c) Construct a table for a bypass operation on a finite set $S$ with the following three properties

i. $B(x, x) = x$ for all $x$ in $S$.

ii. There exist $d$ and $e$ in $S$ with $B(d, e) = d \neq e$.

iii. There exist $f$ and $g$ in $S$ with $B(f, g) \neq f$.

5. In the additive group of ordered pairs of integers $(m, n)$ [with addition defined componentwise: $(m, n) + (m', n') = (m + m', n + n')$] consider the subgroup $H$ generated by the three elements

$$(3, 8), \quad (4, -1), \quad (5, 4).$$

Then $H$ has another set of generators of the form

$$(1, b), \quad (0, a)$$

for some integers $a, b$ with $a > 0$. Find $a$.

[Elements $g_1, \ldots, g_k$ are said to generate a subgroup $H$ if (i) each $g_i \in H$, and (ii) every $h \in H$ can be written as a sum $h = n_1 g_1 + \cdots + n_k g_k$ where the $n_i$ are integers (and where, for example, $3g_1 - 2g_2$ means $g_1 + g_1 + g_1 - g_2 - g_2$).]
Chapter 11
Field Theory

Topics: Finite fields

Problems:

Problems to do:

1. Let $F$ be the field of $p^2$ elements where $p$ is an odd prime. Suppose $S$ is a set of $\frac{p^2 - 1}{2}$ distinct nonzero elements of $F$ with the property that for each $a = n \neq 0$ in $F$, exactly one of $a$ and $-a$ is in $S$. Let $N$ be the number of elements in the intersection $S \cap \{2a : a \in S\}$. Prove that $N$ is even.

2. Let $F$ be a field in which $1 + 1 \neq 0$. Show that the set of solutions to the equation $x^2 + y^2 = 1$ with $x$ and $y$ in $F$ is given by $(x, y) = (1, 0)$ and $(x, y) = \left(\frac{x^2 - 1}{r^2 + 1}, \frac{2x}{r^2 + 1}\right)$, where $r$ runs through the elements of $F$ such that $r^2 \neq -1$.

3. Let $F$ be a finite field having an odd number $m$ of elements. Let $p(x)$ be a irreducible (i.e., nonfactorable) polynomial over $F$ of the form

$$x^2 + bx + c, \quad b, c \in F.$$

For how many elements $k$ in $F$ is $p(x) + k$ irreducible over $F$?
Chapter 12

Group Theory

Topics: Basic group theory, Lagrange’s theorem, characters

Problems:

Problems to do:

1. Let $G$ be a finite group of order $n$ generated by $a$ and $b$. Prove or disprove: there is a sequence

$$g_1, g_2, g_3, \ldots, g_{2n}$$

such that

(1) every element of $G$ occurs exactly twice, and

(2) $g_{i+1}$ equals $g_i b$, for $i = 1, 2, \ldots, 2n$. (Interpret $g_{2n+1}$ as $g_1$.)

2. Let $G$ be a finite set of real $n \times n$ matrices $(M_i)$, $1 \leq i \leq r$, which form a group under matrix multiplications. Suppose that $\sum_{i=1}^{r} tf(M_i) = 0$, where $tr(A)$ denotes the trace of the matrix $A$. Prove that $\sum_{i=1}^{r} M_i$ is the $n \times n$ zero matrix.

3. Let $H$ be a subgroup with $h$ elements in a group $G$. Suppose that $G$ has an element $a$ such that for all $x$ in $H$, $(xa)^3 = 1$, the identity. In $G$, let $P$ be the subset of all products $x_1 ax_2 a \ldots x_n a$, with $n$ a positive integer and the $x_i$ in $H$. 

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(a) Show that $P$ is a finite set.
(b) Show that, in fact, $P$ has no more than $3k^2$ elements.

4. Suppose that $G$ is a group generated by elements $A$ and $B$, that is, every element of $G$ can be written as a finite “word” $A^{n_1}B^{n_2}A^{n_3}\ldots B^{n_k}$, where $n_1, \ldots, n_k$ are any integers, and $A^0 = B^0 = 1$ as usual. Also, suppose that $A^4 = B^7 = ABA^{-1}B = 1$, $A^2 \neq 1$, and $B \neq 1$.

(a) How many elements of $G$ are of the form $C^2$ with $C$ in $G$?
(b) Write each such square as a word in $A$ and $B$. 

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Chapter 13

Geometry

Topics: analytic geometry, conic sections, inequalities, circles, triangles
(special points: incenter, circumcenter, centroid)

Problems:

1. Let \( V \) be the region in the cartesian plane consisting of all points \((x, y)\)
satisfying the simultaneous conditions

\[
|x| \leq y \leq |x| + 3 \text{ and } y \leq 4.
\]

Find the centroid \((\bar{x}, \bar{y})\) of \( V \).

2. Find the area of a convex octagon that is inscribed in a circle and has
four consecutive sides of length 3 units and the remaining four sides of
length 2 units. Give the answer in the form \( r + s\sqrt{t} \) with \( r, s, \) and \( t \)
positive integers.

3. In the plane, let \( C \) be a closed convex set that contains \((0, 0)\) but no
other point with integer coordinates. Suppose that \( A(C) \), the area of \( C \),
is equally distributed among the four quadrants. Prove that \( A(C) \leq 4 \).

4. Inscribed a rectangle of base \( b \) and height \( h \) and an isosceles triangle of
base \( b \) in a circle of radius one as shown.
For what values of $h$ do the rectangle and triangle have the same area?

5. Let $R$ be the region consisting of the points $(x, y)$ of the cartesian plane satisfying both $|x| - |y| \leq 1$ and $|y| \leq 1$. Sketch the region $R$ and find its area.

6. A $2 \times 3$ rectangle has vertices at $(0, 0), (2, 0), (0, 3), \text{and} (2, 3)$. It rotates $90^\circ$ clockwise about the point $(2, 0)$. It then rotates $90^\circ$ clockwise about the point $(5, 0)$ then $90^\circ$ about the point $(7, 0)$, and finally, $90^\circ$ clockwise about the point $(10, 0)$. (the side originally on the $x$-axis is not back on the $x$-axis.) Find the area of the region above the $x$-axis and below the curve traced out by the point whose initial position is $(1, 1)$.

7. Prove that it is impossible for seven distinct straight lines to be situated in the euclidean plane so as to have at least six points where exactly three of these lines intersect and at least four points where exactly two of these lines intersect.

8. A convex pentagon $P = ABCDE$ with vertices labeled consecutively, is inscribed in a circle of radius 1. Find the maximum area of $P$ subject to the condition that the chords $AC$ and $BD$ be perpendicular.

9. A sequence of convex polygons $\{P_n\}, n \geq 0$, is defined inductively as follows. $P_0$ is an equilateral triangle with sides of length 1. Once $P_n$ has been determined, its sides are trisected; the vertices of $P_{n+1}$ are the interior trisection points of the sides of $P_n$. Thus, $P_{n+1}$ is obtained by
cutting corners off $P_n$ and $P_n$ has $3 \cdot 2^n$ sides. ($P_1$ is a regular hexagon with sides of length 1/3.)

Express $\lim_{n \to \infty} \text{Area } (P_n)$ in the form $\sqrt{a}/b$, where $a$ and $b$ are positive integers.

10. Let $m$ be a positive integer and let $G$ be a regular $(2m_1)$-gon inscribed in the unit circle. Show that there is a positive constant $A$, independent of $m$, with the following property. For any point $p$ inside $G$ there are two distinct vertices $v_1$ and $v_2$ of $G$ such that

$$||p - v_1|| - ||p - v_2|| < \frac{1}{m} - \frac{A}{m^2}.$$ 

Here $|s - t|$ denotes the distance between the points $s$ and $t$.

11. Label the vertices of a trapezoid $T$ (quadrilateral with two parallel sides) inscribed in the unit circle as $A, B, C, D$ so that $AB$ is parallel to $CD$ and $A, B, C, D$ are in counterclockwise order. Let $s_1, s_2,$ and $d$ denote the lengths of the line segments $AB, CD,$ and $OE,$ where $E$ is the point of intersection of the diagonals of $T,$ and $O$ is the center of the circle. Determine the least upper bound of $\frac{\Delta T}{d}$ over all such $T$ for which $d \neq 0,$ and describe all cases, if any, in which it is attained.

12. Prove that any convex pentagon whose vertices (no three of which are collinear) have integer coordinates must have area $\geq 5/2$.

13. Suppose that each of the vertices of $\triangle ABC$ is a lattice point in the $(x, y)$-plane and that there is exactly one lattice point $P$ in the interior of the triangle. The line $AP$ is extended to meet $BC$ at $E$. Determine the largest possible value for the ratio of lengths of segments

$$\frac{|AP|}{|EP|}.$$ 

(A lattice point is a point whose coordinates $x$ and $y$ are integers.)

14. Let $M$ be the midpoint of side $BC$ of a general triangle $\triangle ABC$. Using the smallest possible $n$, describe a method for cutting $\triangle AMB$ into $n$ triangles which can be reassembled to form a triangle congruent to $\triangle ABC$. 

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Chapter 14

Inequalities

Topics: Triangle inequality, Cauchy-Schwarz, inequality of arithmetic and geometric means

Problems:

Problems to do:

1. Let $a$ and $b$ be positive numbers. Find the largest number $c$, in terms of $a$ and $b$ such that

\[ a^x b^{1-x} \leq \frac{\sinh u x}{\sinh u} + b \frac{\sinh u(1 - x)}{\sinh u} \]

for all $u$ with $0 < |u| \leq c$ and for all $x$, $0 < x < 1$. (Note: $\sinh u = (e^u - e^{-u})/2$).

2. Prove or disprove: If $x$ and $y$ are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

3. Let $K(x, y, z)$ denote the area of a triangle whose sides have lengths $x$, $y$, and $z$. For any two triangles with sides $a, b, c$ and $a', b', c'$, respectively, prove that

\[ \sqrt{K(a, b, c)} + \sqrt{K(a', b', c')} \leq \sqrt{K(a + a', b + b', c + c')} \]

and determine the cases of equality.
4. Let $a$, $b$, $c$ and $d$ be positive integers and
\[ r = 1 - \frac{a}{b} - \frac{c}{d}. \]
Given that $a + c \leq 1982$ and $r > 0$, prove that
\[ r > \frac{1}{1983^2}. \]

5. For which real numbers $c$ is $(e^x + e^{-x})/2 \leq e^{cx^2}$ for all real $x$?

6. Let $0 \leq P_i \leq 1$ for $i = 1, 2, \ldots, n$. Show that
\[ \sum_{i=1}^{n} \frac{1}{|x - P_i|} \leq 8n \left( 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n - 1} \right) \]
for some $x$ satisfying $0 \leq x \leq 1$.

7. Let $p$ and $n$ be positive integers. Suppose that the numbers $c_{h,k}$ ($h = 1, 2, \ldots, n; k = 1, 2, \ldots, ph$) satisfy $0 \leq c_{h,k} \leq 1$. Prove that
\[ \left( \sum \frac{c_{h,k}}{h} \right)^2 \leq 2p \sum c_{h,k} \]
where each summation is over all admissible ordered pairs $(h, k)$.

8. Let $0 < x_i < \pi$ for $i = 1, 2, \ldots, n$ and set
\[ x = \frac{x_1 + x_2 + \cdots + x_n}{n}. \]
Prove that
\[ \prod_{i=1}^{n} \frac{\sin x_i}{x_i} \leq \left( \frac{\sin x}{x} \right)^n. \]

9. Suppose that $a_1, a_2, \ldots, a_n$ are real, $(n > 1)$, and
\[ A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n - 1} \left( \sum_{i=1}^{n} a_i \right)^2. \]
Prove that $A < 2a_i a_j$ for $1 \leq i \leq j \leq n$. 

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10. Show that if \( s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \), then
   
   (a) \( n(n + 1)^{1/n} < n + s_n \) for \( n > 1 \), and
   (b) \( (n - 1)n^{-1/(n-1)} < n - s_n \) for \( n > 2 \).

11. Show that \( 1 + \frac{n}{n!} + \frac{n^2}{2!} + \cdots + \frac{n^n}{n!} > \frac{e^n}{2} \) for every integer \( n \geq 0 \).

   Remarks. You may assume as known Taylor’s remainder formula:
   
   \[
   e^x - \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{1}{n!} \int_0^x (x-t)^n e^t \, dt,
   \]
   as well as the fact that
   
   \[
   n! = \int_0^\infty t^n e^{-t} \, dt.
   \]

12. (a) On \([0, 1]\), let \( f \) have a continuous derivative satisfying \( 0 < f'(x) \leq 1 \). Also suppose that \( f(0) = 0 \). Prove that

   \[
   \left[ \int_0^1 f(x) \, dx \right]^2 \geq \int_0^1 [f(x)]^3 \, dx.
   \]

   [Hint: Replace the inequality by one involving the inverse function to \( f \).]

   (b) Show an example in which equality occurs.

13. Let \( a_1, a_2, \ldots, a_{2n+1} \) be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of \( n \) integers with equal sums. Prove \( a_1 = a_2 = \cdots = a_{2n+1} \).

14. (a) Let \( ABC \) be any triangle. Let \( X, Y, Z \) be points on the sides \( BC, CA, AB \) respectively. Suppose the distances \( BX \leq XC \), \( CY \leq YA \), \( AZ \leq ZB \) (see Figure 1). Show that the area of the triangle \( XYZ \) is \( \geq (1, 4)(\text{area of triangle } ABC) \).
(b) Let $ABC$ be any triangle, and let $X, Y, Z$ be points on the sides $BC, CA, AB$ respectively (but without any assumption about the ratios of the distances $BX/XC$ etc.; see Figures 1 and 2). Using (a) or by any other method, show: One of the three corner triangles $AZY, BXZ, CYX$ has an area $\leq$ area of triangle $XYZ$. 

Figure 1

Figure 2
Chapter 15

Functional Equations

Topics: Iterating functions
Problems:
Problems to do:

1. Establish necessary and sufficient conditions on the constant $k$ for the existence of a continuous real valued function $f(x)$ satisfying $f(f(x)) = kx^9$ for all real $x$.

2. Let $S$ be a set of three, not necessarily distinct, positive integers. Show that one can transform $S$ into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say $x$ and $y$, where $x \leq y$, and replace them with $2x$ and $y - x$.

3. Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions.
   (a) $f(f(n)) = n$, for all integers $n$;
   (b) $f(f(n + 2) + 2) = n$ for all integers $n$;
   (c) $f(0) = 1$

4. Suppose $f$ and $g$ are non-constant, differentiable, real-valued functions of $x$ on $(-\infty, \infty)$. Furthermore, suppose that for each pair of real numbers $x$ and $y$

\[
\begin{align*}
f(x + y) & = f(x)f(y) - g(x)g(y), \\
g(x + y) & = f(x)g(y) + g(x)f(y).
\end{align*}
\]
If $f'(0) = 0$, prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x$.

5. Prove that there exists a unique function $f$ from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x)$$
and $f(x) > 0$ for all $x > 0$.

6. Let $u$, $f$, and $g$ be functions, defined for all real numbers $x$, such that

$$\frac{u(x + 1) + u(x - 1)}{2} = f(x)$$
and $$\frac{u(x + 4) + u(x - 4)}{2} = g(x).$$

Determine $u(x)$ in terms of $f$ and $g$. 
Chapter 16

Differential Calculus

Topics: Mean value theorem, Chain rule in several variables

Problems:

Problems to do:

1. Find the set of all real numbers \( k \) with the following property: For any positive, differentiable function \( f \) that satisfies \( f'(x) > f(x) \) for all \( x \), there is some number \( N \) such that \( f(x) > e^{kx} \) for all \( x > N \).

2. Let \( f \) be an infinitely differentiable real-valued function defined on the real numbers. If

\[
f \left( \frac{1}{n} \right) = \frac{n^2}{n^2 + 1}, \quad n = 1, 2, 3, \ldots
\]

compute the values of the derivatives \( f^{(k)}(0) \), \( k = 1, 2, 3, \ldots \).

3. For each \( t \geq 0 \), let \( S_t \) be the set of all nonnegative, increasing, convex, continuous, real-valued functions \( f(x) \) defined on the closed interval \([0, 1]\) for which

\[
f(1) - 2f(2/3) + f(1/3) \geq t[f(2/3) - 2f(1/3) + f(0)].
\]

Develop necessary and sufficient conditions on \( t \) for \( S_t \) to be closed under multiplication.

(This closure means that, if the functions \( f(x) \) and \( g(x) \) are in \( S_t \), so is their product \( f(x)g(x) \). A function \( f(x) \) is convex if and only if \( f(su + (1 - s)v) \leq sf(u) + (1 - s)f(v) \) whenever \( 0 \leq s \leq 1 \)).
4. Suppose \( f(x) \) is a twice continuously differential real valued function defined for all real numbers \( x \) and satisfying \( |f(x)| \leq 1 \) for all \( x \) and \( (f(0))^2 + (f'(0))^2 = 4 \). Prove that there exists a real number \( x_0 \) such that \( f(x_0) + f''(x_0) = 0 \).

5. Let \( y(x) \) be a continuously differentiable real-valued function of a real variable \( x \). Show that if \( (y')^2 + y^3 \to 0 \) as \( x \to +\infty \), then \( y(x) \) and \( y'(x) \to 0 \) as \( x \to +\infty \).

6. How many zeros does the function \( f(x) = 2^x - 1 - x^2 \) have on the real line? [By a “zero” of a function \( f \), we mean a value \( x_0 \) in the domain of \( f \) (here the set of all real numbers) such that \( f(x_0) = 0 \).]
Chapter 17

Integral Calculus

Topics: substitution, partial fractions, integration by parts, special functions, Riemann sums

Problems:

Problems to do:

1. An ellipse, whose semi-axes have lengths $a$ and $b$, rolls without slipping on the curve $y = c \sin \left( \frac{x}{a} \right)$. How are $a$, $b$, $c$ related, given that the ellipse completes one revolution when it traverses one period of the curve?

2. For what pairs $(a, b)$ of positive real numbers does the improper integral

$$\int_{0}^{\infty} \left( \sqrt{x + a} - \sqrt{x} - \sqrt{x - \sqrt{x - b}} \right) \, dx$$

converge?

3. Let $A$ be the area of the region in the first quadrant bounded by the line $y = \frac{1}{2}x$, the $x$-axis, and the ellipse $\frac{1}{b}x^2 + y^2 = 1$. Find the positive number $m$ such that $A$ is equal to the area of the region in the first quadrant bounded by the line $y = mx$, the $y$-axis, and the ellipse $\frac{1}{b}x^2 + y^2 = 1$.

4. Show that

$$\int_{-10}^{0} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx + \int_{0}^{\frac{1}{10}} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx + \int_{\frac{1}{10}}^{\frac{1}{100}} \left( \frac{x^2 - x}{x^3 - 3x + 1} \right)^2 \, dx$$

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is a rational number.

5. The horizontal line \( y = c \) intersects the curve \( y = 2x - 3x^3 \) in the first quadrant as in the figure. Find \( c \) so that the areas of the two shaded regions are equal.

\[
\begin{align*}
\int_0^1 C(-y - 1) \left( \frac{1}{y + 1} + \frac{1}{y + 2} + \frac{1}{y + 3} + \cdots + \frac{1}{y + 1992} \right) dy
\end{align*}
\]

6. Define \( C(\alpha) \) to be the coefficient of \( x^{1992} \) in the power series expansion of about \( x = 0 \) of \( (1 + x)^\alpha \). Evaluate

\[
\int_0^1 C(-y - 1) \left( \frac{1}{y + 1} + \frac{1}{y + 2} + \frac{1}{y + 3} + \cdots + \frac{1}{y + 1992} \right) dy
\]

7. Find all real-valued continuously differentiable functions \( f \) on the real line for all \( x \)

\[
(f(x))^2 = \int_0^x ((f(t))^2 + (f'(t))^2) \, dt + 1990.
\]

8. Evaluate

\[
\int_0^a \int_0^b e^{\max\{b^2x^2, a^2y^2\}} \, dy \, dx
\]

where \( a \) and \( b \) are positive.
9. Evaluate \[ \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}} \, dx. \]

10. Evaluate \[ \int_0^\infty t^{-1/2} e^{-1985(t+t^{-1})} \, dt. \] You may assume that \[ \int_\infty^\infty e^{-x^2} \, dx = \sqrt{\pi}. \]

11. Let \( I_m = \int_0^{2\pi} \cos(x) \cos(2x) \cdots \cos(mx) \, dx. \) For which integers \( m, \ 1 \leq m \leq 10, \) is \( I_m \neq 0? \)

12. Find, with proof, all real-valued functions \( y = g(x) \) defined and continuous on \([0, \infty), \) positive on \((0, \infty), \) such that for all \( x > 0 \) the \( y \)-coordinate of the centroid of the region

\[ R_x = \{(s,t)|0 \leq s \leq x, \ 0 \leq t \leq g(s)\} \]

is the same as the average value of \( g \) on \([0, x].\)

13. Let \( R \) be the region consisting of all triples \((x, y, z)\) of nonnegative real numbers satisfying \( x + y + z \leq 1. \) Let \( w = 1 - x - y - z. \) Express the value of the triple integral

\[ \int \int \int_R x^y z^w x^4 \, dx \, dy \, dz \]

in the form \( a! b! c! d!/n! \), where \( a, b, c, d, \) and \( n \) are positive integers.

14. Let \( \|u\| \) denote the distance from the real number \( u \) to the nearest integer. (For example, \( \|2.8\| = .2 = \|3.2\| \).) For positive integers \( n, \) let

\[ a_n = \frac{1}{n} \int_1^n \frac{||x||}{x} \, dx. \]

Determine \( \lim_{n \to \infty} a_n. \) You may assume the identity

\[ \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots = \frac{\pi}{2}. \]
15. Let \( \exp(t) \) denote \( e^t \) and

\[
F(x) = \frac{x^4}{\exp(x^3)} \int_0^x \int_0^{x-u} \exp(u^3 + v^3) \, dv \, du.
\]

Find \( \lim_{x \to \infty} F(x) \) or prove that it does not exists.

16. Let \( A(x, y) \) denote the number of points \((m, n)\) in the plane with integer coordinates \(m\) and \(n\) satisfying \(m^2 + n^2 \leq x^2 + y^2\). Let \( g = \sum_{k=0}^{\infty} e^{-k^2} \).

Express

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(x, y) e^{-x^2-y^2} \, dx \, dy
\]
as a polynomial in \( g \).

17. Evaluate

\[
\int_0^{\infty} \frac{\arctan(\pi x) - \arctan(x)}{x} \, dx.
\]

18. Let \( C \) be a fixed circle in the Cartesian plane. For any convex polygon \( P \) each of whose sides is tangent to \( C \), let \( N(P, h, k) \) be the number of points common to \( P \) and the unit circle with center at \((h, k)\). Let \( H(P) \) be the region of all points \((x, y)\) for which \( N(P, x, y) \geq 1 \) and \( F(P) \) be the area of \( H(P) \). Find the smallest number \( u \) with

\[
\frac{1}{F(P)} \int \int N(P, x, y) \, dx \, dy < u
\]
for all polygons \( P \), where the double integral is taken over \( H(P) \).

19. Find

\[
\lim_{t \to \infty} \left[ e^{-t} \int_0^t \int_0^t \frac{e^x - e^y}{x-y} \, dx \, dy \right]
\]
or show that the limit does not exist.

20. Let \( C \) be the class of all real valued continuously differentiable functions \( f \) on the interval \( 0 \leq x \leq 1 \) with \( f(0) = 0 \) and \( f(1) = 1 \). Determine the largest real number \( u \) such that

\[
u \leq \int_0^1 |f'(x) - f(x)| \, dx
\]
for all \( f \) in \( C \).

21. Let \( P(t) \) be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

\[
0 = \int_0^x P(t) \sin t \, dt = \int_0^x P(t) \cos t \, dt
\]

has only finitely many real solutions \( x \).

22. Evaluate

\[
\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^2}.
\]

23. Let \( p(x) = 2 + 4x + 3x^2 + 5x^3 + 3x^4 + 4x^5 + 2x^6 \). For \( k \) with \( 0 < k < 5 \), define

\[
I_k = \int_0^\infty \frac{x^k}{p(x)} \, dx
\]

For which \( k \) is \( I_k \) smallest?

24. Let \( f(x, y) \) be a continuous function on the square

\[
S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.
\]

For each point \( (a, b) \) in the interior of \( S \), let \( S_{(a,b)} \) be the largest square that is contained in \( S \), is centered at \( (a, b) \), and has sides parallel to those of \( S \). If the double integral \( \iint f(x, y) \, dx \, dy \) is zero when taken over each square \( S_{(a,b)} \), must \( f(x, y) \) be identically zero on \( S \)?

25. Evaluate \( \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^n \left( \frac{2n}{k} - 2 \left[ \frac{n}{k} \right] \right) \) and express your answer in the form \( \log a - b \), with \( a \) and \( b \) positive integers.

Here \( [x] \) is defined to be the integer such that \( [x] \leq x < [x] + 1 \) and \( \log x \) is the logarithm of \( x \) to base \( e \).

26. In the \((x, y)\)-plane, if \( R \) is the set of points inside and on a convex polygon, let \( D(x, y) \) be the distance from \((x, y)\) to the nearest point of \( R \).
(a) Show that there exist constants $a$, $b$, and $c$, independent of $R$, such that

\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-D(x,y)} dx \; dy = a + bL + cA, \]

where $L$ is the perimeter of $R$ and $A$ is the area of $R$.

(b) Find the values of $a$, $b$, and $c$. 
Chapter 18

Probability

Topics: Probabilities arising as integrals or areas, expectation, independence
Problems:

Problems to do:

1. Suppose that each of $n$ people writes down the numbers 1, 2, 3 in random order in one column of a $3 \times n$ matrix, with all orders equally likely and with the orders for different columns independent of each other. Let the row sums $a, b, c$ of the resulting matrix be rearranged (if necessary) so that $a \leq b \leq c$. Show that for some $n \geq 1995$, it is at least four times as likely that both $b = a + 1$ and $c = a + 2$ as that $a = b = c$.

2. Two real numbers $x$ and $y$ are chosen at random in the interval $(0, 1)$ with respect to the uniform distribution. What is the probability that the closest integer to $x/y$ is even? Express the answer in the form $r + s\pi$, where $r$ and $s$ are rational numbers.

3. Consider the following game played with a deck of $2n$ cards numbered from 1 to $2n$. The deck is randomly shuffled and $n$ cards are dealt to each of two players $A$ and $B$. Beginning with $A$, the players take turns discarding one of their remaining cards and announcing its number. The game ends as soon as the sum of the numbers on the discarded cards is divisible by $2n + 1$. The last person to discard wins the game.
If we assume optimal strategy by both A and B, what is the probability that A wins?

4. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

5. Let \((x_1, x_2, \ldots, x_n)\) be a point chosen at random from the \(n\)-dimensional region defined by \(0 < x_1 < x_2 < \cdots < x_n < 1\). Let \(f\) be a continuous function on \([0, 1]\) with \(f(1) = 0\). Set \(x_0 = 0\) and \(x_{n+1} = 1\). Show that the expected value of the Riemann sum

\[
\sum_{i=0}^{n} (x_{i+1} - x_i) f(x_{i+1})
\]

is \(\int_0^1 f(t) P(t) \, dt\), where \(P\) is a polynomial of degree \(n\), independent of \(f\), with \(0 \leq P(t) \leq 1\) for \(0 \leq t \leq 1\).

6. A dart, thrown at random, hits a square target. Assuming that any two parts of the target of equal area are equally likely to be hit, find the probability that the point hit is nearer to the center than to any edge. Express your answer in the form \(\frac{a\sqrt{b} + c}{d}\), where \(a, b, c, d\) are integers.

7. If \(\alpha\) is an irrational number, \(0 < \alpha < 1\), is there a finite game with an honest coin such that the probability of one player winning the game is \(\alpha\)? (An honest coin is one for which the probability of heads and the probability of tails are both \(\frac{1}{2}\). A game is finite if with probability 1 it must end in a finite number of moves.)

8. Let \(C\) be the unit circle \(x^2 + y^2 = 1\). A point \(p\) is chosen randomly on the circumference \(C\) and another point \(q\) is chosen randomly from the interior of \(C\) (these points are chosen independently and uniformly over their domains). Let \(R\) be the rectangle with sides parallel to the \(x\)- and \(y\)-axes with diagonal \(pq\). What is the probability that no point of \(R\) lies outside of \(C\)?
9. Let \( p_n \) be the probability that \( c + d \) is a perfect square when the integers \( c \) and \( d \) are selected independently at random from the set \( \{1, 2, 3, \ldots, n\} \). Show that \( \lim_{n \to \infty} (p_n \sqrt{n}) \) exists and express this limit in the form \( r(\sqrt{s} - t) \), where \( s \) and \( t \) are integers and \( r \) is a rational number.

10. Suppose that we have \( n \) events \( A_1, \ldots, A_n \), each of which has probability at least \( 1 - a \) of occurring, where \( a < 1/4 \). Further suppose that \( A_i \) and \( A_j \) are mutually independent if \( |i - j| > 1 \), although \( A_i \) and \( A_{i+1} \) may be dependent. Assume as known that the recurrence \( u_{k+1} = u_k - au_{k-1} \),
\[ u_0 = 1, \ u_1 = 1 - a, \] defines positive real numbers \( u_k \) for \( k = 0, 1, \ldots \). Show that the probability of all of \( A_1, \ldots, A_n \) occurring is at least \( u_n \).

11. An unbiased coin is tossed \( n \) times. What is the expected value of \( |H - T| \), where \( H \) is the number of heads and \( T \) is the number of tails? In other words, evaluate in closed form:
\[
\frac{1}{2^{n-1}} \sum_{k<n/2} (n - 2k) \binom{n}{k}.
\]

(In this problem, “closed form” means a form not involving a series. The given series can be reduced to a single term involving only binomial coefficients, rational functions of \( n \) and \( 2^n \), and the greatest integer function \( \lfloor x \rfloor \).)
Chapter 19

Series

Techniques:

**Arithmetic progressions:** The general formula is

\[ a + (a + d) + (a + 2d) + \cdots + (a + (n-1)d) = \sum_{j=0}^{n-1} a + jd \]

\[ = na + \frac{n(n-1)}{2} d \]

\[ = \frac{n(2a + (n-1)d)}{2} \]

- \(1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}\) (triangular numbers)
- General rule: add first term to last term, multiply by \(n\) (number of terms), and then divide by 2.

**Geometric series:** General formula:

\[ a + ar + ar^2 + \cdots + ar^{n-1} = \sum_{j=0}^{n-1} ar^j \]

\[ = \frac{ar^n - a}{r - 1} \]

- \(1 + 2 + 4 + 8 + \cdots + 2^{n-1} = 2^n - 1\) (sum of powers of 2)
• If the number of terms is infinite, the sum converges if and only if the common ratio $r$ has absolute value less than 1. Then

$$
\sum_{j=0}^{\infty} ar^j = \frac{a}{1 - r}
$$

**Telescoping series:** General formula:

$$(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_{n-1} - a_n) = \sum_{j=1}^{n-1} a_j - a_{j+1}
$$

$$
= a_1 - a_n
$$

• Principle: all the middle terms cancel out.

• Example:

$$
\sum_{j=1}^{\infty} \frac{1}{j(j + 1)} = \sum_{j=1}^{\infty} \frac{1}{j} - \frac{1}{j + 1} = 1
$$

• An infinite telescoping series converges if and only if $|a_n| \to 0$ as $n \to \infty$.

**Convergence:** The main tool in convergence proofs is comparison

• If $|a_n| \leq b_n$ for all $n$ and $\sum b_n$ converges, then $\sum a_n$ converges absolutely.

• If $|a_n| \geq b_n$ for all $n$ and $\sum b_n$ diverges, then $\sum a_n$ diverges absolutely.

• Sometimes we compare the series with an integral. If $f(x)$ is nonnegative and decreasing, then

$$
\int_{1}^{\infty} f(x) \, dx \leq \sum_{j=1}^{\infty} f(j) \leq f(1) + \int_{1}^{\infty} f(x) \, dx
$$

**Rearrangement:** Often the key to using the above techniques is to reorder the terms in some way, if that is possible.

**Taylor Series:** For suitable functions $f(x)$, we have

$$
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n
$$

convergent if $|x - a|$ is small enough.

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Example Problems:

1. For $0 < x < 1$, express $\sum_{n=0}^{\infty} \frac{x^{2n}}{1-x^{2n+1}}$ as a rational function of $x$.

2. Evaluate the infinite product $\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$

3. Express
\[
\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2n + mn^2 + 2mn}
\]
as a rational number.

4. Find
\[
\lim_{N \to \infty} \left[ \frac{1}{n^b} \sum_{h=1}^{n} \sum_{k=1}^{n} (5h^4 - 18h^2k^2 + 5k^4) \right].
\]

5. Express $\sum_{k=1}^{\infty} \frac{6^k}{(3k+1 - 2k+1)(3k - 2k)}$ as a rational number.

6. What is the units (i.e., rightmost) digit of $\left[ \frac{10^{20000}}{10^{100} + 3} \right]$? Here $[x]$ is the greatest integer $\leq x$.

7. Evaluate $\sum_{n=0}^{\infty} \arccot(n^2 + n + 1)$, where $\arccot t$ for $t \geq 0$ denotes the number $\theta$ in the interval $0 < \theta \leq \pi/2$ with $\cot \theta = t$.

8. Let $B(n)$ be the number of ones in the base two expression for the positive integer $n$. For example, $B(6) = B(110_2) = 2$ and $B(15) = B(1111_2) = 4$. Determine whether or not
\[
\exp \left( \sum_{n=1}^{\infty} \frac{B(n)}{n(n+1)} \right)
\]
is a rational number. Here $\exp(x)$ denotes $e^x$.  

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9. Consider an infinite series whose \( n \)-th term is \( \pm (1/n) \), the \( \pm \) signs being determined according to a pattern that repeats periodically in blocks of eight. [There are \( 2^8 \) possible patterns of which two examples are:
\[
\begin{align*}
+ & + - - - - + +, \\
+ & - - - + - - - -.
\end{align*}
\]

The first example would generate the series
\[
1 + (1/2) - (1/3) - (1/4) - (1/5) - (1/6) + (1/7) + (1/8) + (1/9) + (1/10) - (1/11) - (1/12) - \cdots
\]

(a) Show that a sufficient condition for the series to be conditionally convergent is that there be four “+” signs and four “−” signs in the block of eight.

(b) Is this sufficient condition also necessary?

[Here “convergent” means “convergent to a finite limit”.

10. For which real numbers \( c \) is \( (e^x + e^{-x})/2 \leq e^{cx^2} \) for all real \( x \)?

11. For positive real \( x \), let
\[
B_n(x) = 1^x + 2^x + 3^x + \cdots + n^x.
\]

Prove or disprove the convergence of
\[
\sum_{n=2}^{\infty} \frac{B_n(\log n)}{(n \log n)^2}.
\]

12. Determine, with proof, the set of real numbers \( x \) for which
\[
\sum_{n=1}^{\infty} \left( \frac{1}{n} \csc \frac{1}{n} - 1 \right)^x
\]
converges.

13. Prove that if \( \sum_{n=1}^{\infty} a_n \) is a convergent series of positive real numbers, then
\[
\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}
\]
so is
\[
\sum_{n=1}^{\infty} (a_n)^{n/(n+1)}.
\]

14. Suppose that a sequence \( a_1, a_2, a_3, \ldots \) satisfies \( 0 < a_n \leq a_{2n} + a_{2n+1} \) for all \( n \geq 1 \). Prove that the series \( \sum_{n=1}^{\infty} a_n \) diverges.
Chapter 20

Vector Calculus

Topics: Stokes’ theorem, Green’s theorem, potentials

Problems:

Problems to do:

1. Let $\vec{G}(x, y) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$. Prove or disprove that there is a vector-valued function

$$\vec{F}(x, y, z) = (M(x, y, z), N(x, y, z), P(x, y, z))$$

with the following properties:

(i) $M, N, P$ have continuous partial derivatives for all $(x, y, z) \neq (0, 0, 0)$;
(ii) $\text{Curl } \vec{F} = \vec{0}$ for all $(x, y, z) \neq (0, 0, 0)$;
(iii) $\vec{F}(x, y, 0) = \vec{G}(x, y)$.

2. Suppose $f_1(x), f_2(x), \ldots, f_n(x)$ are functions of $n$ real variables $x = (x_1, \ldots, x_n)$ with continuous second order partial derivatives everywhere on $\mathbb{R}^n$. Suppose further that there are constants $c_{ij}$ such that

$$\frac{\partial f_i}{\partial x_j} - \frac{\partial f_j}{\partial x_i} = c_{ij} \text{ for all } i \text{ and } j, \ 1 \leq i \leq n, \ 1 \leq j \leq n.$$ 

Prove that there is a function $g(x)$ on $\mathbb{R}^n$ such that $f_i + \frac{\partial g}{\partial x_i}$ is linear for all $i, \ 1 \leq i \leq n$.

(A linear function is one of the form $a_0 + a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$.)
3. A particle moves in 3-space according to the equations:

\[
\frac{dx}{dt} = yz, \quad \frac{dy}{dt} = zx, \quad \frac{dz}{dt} = xy.
\]

[Here \(x(t), y(t), z(t)\) are real-valued functions of the real variable \(t\).]

Show that

(a) If two of \(x(0), y(0), z(0)\) equal zero, then the particle never moves.

(b) If \(x(0) = y(0) = 1, z(0) = 0\), then the solution is:

\[
x = \sec t, \quad y = \sec t, \quad z = \tan t
\]

whereas if \(x(0) = y(0) = 1, z(0) = -1\), then

\[
x = 1/(t + 1), \quad y = 1/(t + 1), \quad z(t) = -1/(t + 1).
\]

(c) If at least two of the values \(x(0), y(0), z(0)\) are different from zero, then either the particle moves to infinity at some point in the future, or it came from infinity at some finite time in the past. [A point \((x, y, z)\) in 3-space ”moves to infinity” if it’s distance from the origin approaches infinity.]
Chapter 21

Differential Equations

Topics: Linear equations, constant coefficient equations, Laplace equation

Problems:

Problems to do:

1. Let $x_1, x_2, \ldots, x_n$ be differentiable (real-valued) functions of a single variable $t$ which satisfy

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$
$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n$$
$$\vdots \quad \vdots \quad \vdots$$
$$\frac{dx_n}{dt} = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n$$

for some constants $a_{ij} \geq 0$. Suppose that for all $i$, $x_i(t) \to 0$ as $t \to \infty$. Are the functions $x_1, x_2, \ldots, x_n$ necessarily linearly dependent?

2. The function $K(x,y)$ is positive and continuous for $0 \leq x \leq 1$, $0 \leq y \leq 1$, and the functions $f(x)$ and $g(x)$ are positive and continuous for $0 \leq x \leq 1$. Suppose that for all $x$, $0 \leq x \leq 1$,

$$\int_0^1 f(y)K(x,y) \, dy = g(x) \quad \text{and} \quad \int_0^1 g(y)K(x,y) \, dy = f(x)$$

Show that $f(x) = g(x)$ for $0 \leq x \leq 1$. 

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3. Let \( p(x) \) be a nonzero polynomial of degree less than 1992 having no nonconstant factor in common with \( x^3 - x \). Let
\[
\frac{d^{1992}}{dx^{1992}} \left( \frac{p(x)}{x^3 - x} \right) = \frac{f(x)}{g(x)}
\]
for polynomials \( f(x) \) and \( g(x) \). Find the smallest possible degree of \( f(x) \).

4. Let \( f \) be a function on \([0, \infty)\), differentiable and satisfying
\[
f'(x) = -3f(x) + 6f(2x)
\]
for \( x > 0 \). Assume that \(|f(x)| \leq e^{-\sqrt{x}}\) for \( x \leq 0 \) (so that \( f(x) \) tends rapidly to 0 as \( x \) increases). For \( n \) non-negative integer, define
\[
\mu_n = \int_0^\infty x^n f(x) \, dx
\]
(sometimes called the \( n \)th moment of \( f \)).

(a) Express \( \mu_n \) in terms of \( \mu_0 \).

(b) Prove that the sequence \( \left\{ \mu_n \frac{3^n}{n!} \right\} \) always converges, and that the limit is 0 only if \( \mu_0 = 0 \).

5. A not uncommon calculus mistake is to believe that the product rule for derivatives says that \((fg)' = f'g'\). If \( f(x) = e^{x^2} \), determine, with proof, whether there exists an open interval \((a, b)\) and a non-zero function \( g \) define on \((a, b)\) such that this wrong product rule is true for \( x \) in \((a, b)\).

6. For all real \( x \), the real valued function \( y = f(x) \) satisfies
\[
y'' - 2y' + y = 2e^x.
\]
(a) If \( f(x) > 0 \) for all real \( x \), must \( f'(x) > 0 \) for all real \( x \)? Explain.

(b) If \( f(x) > 0 \) for all real \( x \), must \( f(x) > 0 \) for all real \( x \)? Explain.
7. Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$, and
\[
\frac{d}{dx}(f_{n+1}(x)) = (n + 1)f_n(x + 1)
\]
for $n \geq 1$. Find, with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

8. Assume that the differential equation
\[
y''' + P(x)y'' + q(x)y' + r(x)y = 0
\]
has solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ on the whole real line such that
\[
y_1^2(x) + y_2^2(x) + y_3^2(x) = 1
\]
for all real $x$. Let
\[
f(x) = (y_1'(x))^2 + (y_2'(x))^2 + (y_3'(x))^2.
\]
Find constants $A$ and $B$ such that $f(x)$ is a solution to the differential equation
\[
y' + A p(x)y = B r(x).
\]

9. Assume that the system of simultaneous differential equations
\[
y' = -z^3, \quad z' = y^3
\]
with the initial conditions $y(0) = 1$, $z(0) = 0$ has a unique solution $y = f(x)$, $z = g(x)$ defined for all real $x$. Prove that there is a positive constant $L$ such that for all real $x$,
\[
f(x + L) = f(x), \quad g(x + L) = g(x).
\]

10. (a) Find a solution that is not identically zero, of the homogeneous linear differential equation
\[
(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 0.
\]
Intelligent guessing of the form of a solution may be helpful.
(b) Let \( y = f(x) \) be the solution of the nonhomogeneous differential equation

\[
(3x^2 + x - 1)y'' - (9x^2 + 9x - 2)y' + (18x + 3)y = 6(6x + 1)
\]

that has \( f(0) = 1 \) and \((f(-1) - 2)(f(1) - 6) = 1\). Find integers \(a, b, c\) such that \((f(-2) - a)(f(2) - b) = c\).

11. On some interval \( I \) of the real line, let \( y_1(x) \) and \( y_2(x) \) be linearly independent solutions of the differential equation

\[
y'' - f(x)y,
\]

where \( f(x) \) is a continuous real-valued function. Suppose that \( y_1(x) > 0 \) and \( y_2(x) > 0 \) on \( I \). Show that there exists a positive constant \( c \) such that, on \( I \), the function

\[
z(x) = c \sqrt{y_1(x)y_2(x)}
\]

satisfies the equation

\[
z'' + \frac{1}{z^3} = f(x)z.
\]

State clearly the manner in which \( c \) depends on \( y_1(x) \) and \( y_2(x) \).