8.2: PIDs

**Prop.** Every prime ideal \( \neq 0 \) in a PID \( R \) is maximal.

**Proof:** Let \( P = (p) \) be a prime ideal, \( p \neq 0 \).

Suppose \( (p) \subseteq (a) \).

Then \( a \divides p \). Hence \( p = ab \) for some \( b \in R \).

Then \( a \in (p) \) or \( b \in (p) \) since \( (p) \) is prime.

If \( a \in (p) \), then \( (p) = (a) \).

If \( b \in (p) \), then \( b = pc \) for \( c \in R \).

Hence, \( p = ab = pac \Rightarrow ac = 1 \)

\[ \Rightarrow a \in R^* \Rightarrow (a) = R \] \( \Box \)

**Corollary:** If \( R[x] \) is a PID, then \( R \) is a field.

**Proof:** In the polynomial ring \( (x) \) is a prime ideal \((x) = \{ f(x) \mid f(x) \in R[x] \} \)

\[ \Rightarrow x \divides f(x) \] or \( x \divides g(x) \)
8.2-2 (Also, \( \mathbb{R}[x]/(x) \cong R \) is an integral domain.)

Hence \((x)\) is maximal and

then \( \mathbb{R}[x]/(x) \cong R \) is a field.

Defn: \( N: R \to \{0, 1, 2, \ldots\} \) is a Dedekind-Hasse Norm if for every \( a, b \in R \setminus \{0\} \) either \( a \in (b) \) or \( \exists s \in \mathbb{R} \) s.t. \( 0 < N(sa - tb) < N(b) \).

Prop: \( R \) is a PID iff \( R \) has a Dedekind-Hasse Norm.

Proof: \((\Leftarrow)\) Let \( I \) be a nonzero ideal, and \( b \in I \) have minimal norm \( b \neq 0 \).

Given \( a \in I \), the ideal \((a, b) \subseteq I\),

and so there is no element \( \neq 0 \) with strictly smaller norm. Hence, \( a \in (b) \). \( \square \)

\((\Rightarrow)\) Left to textbook.
We can now prove \( \mathbb{O} = \mathbb{Z} \left[ \frac{1 + \sqrt{-19}}{2} \right] \) is a P.I.D.

Suppose \( \alpha, \beta \in \mathbb{O} \) and \( \alpha / \beta \notin \mathbb{O} \).

We must show \( \exists s, t \) with

\[
0 < N(s\alpha - t\beta) < N(\beta) \quad \text{(where } N(a) = a\overline{a}^*)
\]

or \( N\left(\frac{\alpha}{\beta} s - t\right) < 1 \).

Write \( \frac{\alpha}{\beta} = \frac{a + b\omega}{c}, \quad a, b, c \in \mathbb{Z}, \quad c > 1 \),

\[
(a, b, c) = 1 \quad \text{with } \omega = \frac{1 + \sqrt{-19}}{2}.
\]

Note \( \omega^2 = \omega - 5 \).

Since \( (a, b, c) = 1 \) \( \exists x, y, z \in \mathbb{Z} \) such that

\[
(a + b)x + by + cz = 1.
\]

Let \( q, r \) be such that \( a\gamma - 5b = qc + r \)

and \( \|r\| \leq \frac{c}{2} \).

Let \( s = y + x\omega \) and \( t = q - z\omega \).

Then

\[
s \frac{\alpha}{\beta} - t = (y + x\omega)\left(\frac{a + b\omega}{c}\right) - (q - z\omega)
\]

\[
= \frac{ay + (ax + by)}{c}\omega + \frac{b - q}{c}\omega^2 - q + z\omega
\]

\[
\omega - 5
\]
\[ a y - 5 b x - q c = \frac{(a+b)x + by + c z}{c} \]

Note: \(|\varepsilon| \leq \frac{1}{2}\) and \(\delta = \frac{1}{c}\).

So, \(N(\varepsilon + \delta w) = (\varepsilon + \delta w)(\varepsilon + \delta \bar{w})\)

\[ = \varepsilon^2 + \varepsilon \delta + 5 \delta^2 \]

\[ \leq \frac{1}{4} + \frac{1}{2c} + \frac{5}{c^2} \]

Decreasing function of \(c\).

For \(c = 3\) we get \(\frac{1}{4} + \frac{1}{6} + \frac{5}{9} = \frac{35}{36} < 1\).

Thus, this proves the claim for all \(c \geq 3\).

If \(c = 2\), then since \(\frac{a}{2} + \frac{b}{2} w \neq 0\),
either \(b\) is odd or \(a\) is odd.

In all cases, \(a^2 + ab + 5b^2\) is odd.

Choose \(q\) so that \(a^2 + ab + 5b^2 - 2q = 1\).

Then \(N\left(\frac{(a+bw)(a+b\bar{w}) - q}{2}\right) = N\left(\frac{a^2 + ab + 5b^2 - 2q}{2}\right)\)

\[ = N\left(\frac{1}{2}\right) = \frac{1}{4} < 1\]. QED.