Ch. 10 Modules

Let $R$ be a ring.

A left $R$-module is an abelian group $M$ together with an action

$$R \times M \to M$$

$$(r, m) \mapsto rm$$

satisfying:

a) $(r+s)m = rm + sm$

b) $(rs)m = r(sm)$

c) $r(m+n) = rm + rn$

for all $r, s \in R$, $m, n \in M$.

If $R$ has a 1, we also require $1.m = m$.

(Right $R$-modules are similarly defined)

If $R$ is a field, we say $M$ is a vector space.

The action $rm$ is also called scalar multiplication.
A ring $R$ is a left (or right) $R$-module too with scalar multiplication:

$$rm = \text{ordinary ring multiplication}$$

for $r \in R, m \in R$.

Given left $R$-modules $M, N$, the direct product $M \times N$ is the direct product as additive groups with scalar multiplication:

$$r(m, n) = (rm, rn)$$

The product $\prod_{i \in I} M_i$ of a family $\{M_i\}_{i \in I}$ is similarly defined.

$R^n = \text{direct product of } n \text{ copies of } R$

$$= \{ (x_1, \ldots, x_n) \mid x_i \in R, 1 \leq i \leq n \}$$

= free $R$-module of rank $n$. 
Any abelian group is also a $\mathbb{Z}$-module with scalar multiplication:

$$\begin{align*}
na &= \begin{cases} 
\frac{a+\ldots+a}{n \text{ times}} & \text{if } n > 0 \\
0 & \text{if } n = 0 \\
(-a)+\ldots+(-a) & \text{if } n < 0
\end{cases} \\
&= \lfloor n \rfloor \text{ times}
\end{align*}$$

An $R$-submodule $N$ of an $R$-module $M$ is an additive subgroup which is closed under scalar multiplication: $rx \in N$ for $r \in R$, $x \in N$.

An ideal of $R$ is the same as an $R$-submodule of $R$.

Submodule Criterion: A nonempty subset $N \subseteq M$ is a submodule iff $x+ry \in N$ for all $x, y \in N$, $r \in R$. (Assuming $R$ contains a $1$.)

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