Math 5013: D-modules

The ring \( D \) of differential operators is one of the most important rings in all areas of mathematics. It is defined as follows. For a given positive integer \( n \), we let \( x_1, \ldots, x_n \) be \( n \) indeterminate variables. Then \( \mathbb{R}[x_1, \ldots, x_n] \) is the ring of polynomials in \( x_1, \ldots, x_n \). We also introduce \( n \) new symbols denoting the partial derivatives with respect to \( x_1, \ldots, x_n \), and we call these \( \partial_1, \ldots, \partial_n \). Then we define \( D = \mathbb{R}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] \) with the usual laws of polynomial addition but with the following axioms of multiplication:

- \( x_i x_j = x_j x_i \) for all \( i, j \).
- \( \partial_i \partial_j = \partial_j \partial_i \) for all \( i, j \).
- \( x_i \partial_j = \partial_j x_i \) for all \( i \neq j \).
- \( \partial_i x_i = 1 + x_i \partial_i \) for all \( i \).

The last law comes from thinking of these as derivatives applied to a module of functions \( f(x_1, \ldots, x_n) \). The product rule from calculus says that

\[
\partial_i (x_i f) = \partial_i (x_i) f + x_i \partial_i f
\]

\[
= f + x_i \partial_i f
\]

\[
= (1 + x_i \partial_i) f
\]

Thus, as operators we see that \( \partial_i x_i = 1 + x_i \partial_i \). These rules together with the distributive laws define the noncommutative ring \( D \) in \( n \) variables. There has been a vast amount of work on the theory of \( D \)-modules over the past 30 years. We also write \( D = \mathbb{R}[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n] \) when we want greater detail.

We can define a \( D \)-module simply by applying \( D \) to a function where the differentiation makes sense. Let \( P(x) \) be a polynomial in \( x_1, \ldots, x_n \). Then we define the action of \( D \) on powers \( P(x)^s \) as follows

- \( x_i P(x)^s = x_i P(x)^s \) (ordinary multiplication).
- \( \partial_i P(x)^s = (\partial_i P(x)) s P(x)^{s-1} \).

Then we can consider the module of functions \( DP^s \) of all elements of \( D \) applied to \( P^s \). We now extend the ring \( D \) slightly to include the new variable \( s \). Hereafter, we take \( D \) to be

\[
D = \mathbb{R}[x_1, \ldots, x_n, s, \partial_1, \ldots, \partial_n].
\]

That is, we allow polynomial coefficients in \( s \).

Using algebraic geometry first and later using the algebraic theory of \( D \)-modules which he largely created, Bernstein proved in 1969 and 1971 \[BG69, Ber71\] the following.
For any nonconstant polynomial \( P(x) \), there exists a nonzero \( d \in D \) and a nonconstant polynomial \( b(s) \) such that \( dP^s = b(s)P^{s-1} \).

Note that the operator \( d \) can depend on \( s \) in a polynomial way. Kashiwara later proved in [Kas77] that all the roots of the \( b \)-function are rational numbers. These numbers are still largely a mystery, but hold great significance for the geometry of the polynomial \( P(x) \).

1. For \( P = x_1^2 + \cdots + x_n^2 \) and \( d = \partial_1^2 + \cdots + \partial_n^2 \), prove that \( dP^s = b(s)P^{s-1} \) and determine the \( b \)-function and its roots.

2. Find (by trial and error) the \( b \)-functions and differential operators for the following polynomials.
   (a) \( x_1x_3 - 4x_2^2 \).
   (b) \( x^3 + y^3 \).
   (c) \( x^m + y^m \) for all integers \( m > 3 \).

References


[BG69] I. N. Bernšteǐn and S. I. Gel’fand, Meromorphy of the function \( P^\lambda \), Funkcional. Anal. i Priložen. 3 (1969), no. 1, 84–85. MR MR0247457 (40 #723)