8.8: Improper Integrals

Any integral \( \int_a^b f(x) \, dx \) with

1. one endpoint \( a \) or \( b \) equal to \( \infty \) or \(-\infty \),

or
2. a vertical asymptote \( x = c \) for \( f(x) \) with \( c \in [a, b] \),

is called improper.

They are always evaluated as limits of proper integrals.

Example: \( \int_{-\infty}^{\infty} \frac{1}{x} \, dx \)

\[
= \lim_{N \to \infty} \int_{-N}^{N} \frac{1}{x} \, dx
= \lim_{N \to \infty} \left[ \ln|N| - \ln(1) \right]
= \lim_{N \to \infty} \ln(N) = \infty.
\]

This integral diverges to \( \infty \).

Example: \( \int_{1}^{\infty} \frac{1}{x^2} \, dx = \left[ -\frac{1}{x} \right]_{1}^{\infty} = \lim_{N \to \infty} \left( -\frac{1}{N} - (-1) \right) = 1 \)
The area under \( y = \frac{1}{x^2} \) for \( x \geq 1 \) is finite. \[ \int_{1}^{\infty} \frac{1}{x^p} \, dx = \frac{x^{1-p}}{1-p} \bigg|_1^\infty \]

\[ = \left( \lim_{N \to \infty} \frac{N^{1-p}}{1-p} \right) - \frac{1}{1-p} \]

\[ = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \vspace{1em} \\ \infty & \text{if } p < 1 \end{cases} \]

Ex: \[ \int_{0}^{\infty} e^{-x} \, dx = \left. -e^{-x} \right|_0^\infty \]

\[ = \lim_{N \to \infty} (-e^{-N}) - (-1) = 1 \]
\[ \int_{0}^{1} \frac{dx}{\sqrt{x}} \]

Improper at \( x = 0 \), because there is a vertical asymptote.

\[ = 2 \sqrt{x} \bigg|_{0}^{1} = 2 - \lim_{h \to 0^+} (2 \sqrt{h}) \]

The limit approaches the improper endpoint

\[ = 2 - 0 = 2. \]

If the limit exists and is a finite number, we say the improper integral converges.

If the limit is \( \infty \) (resp. \( -\infty \)), we say it diverges to \( \infty \) (resp. \( -\infty \)).

Otherwise, we simply say the integral is divergent or "Does Not Exist."

\[ \text{Ex:} \int_{1}^{\infty} \frac{dx}{\sqrt{1+x^3}} \]

We cannot integrate this in elementary terms, but
we can say it converges to a finite number

since \( \sqrt{\ln x^3} \geq \sqrt{x^3} = x^{3/2} \) and so

\[ \frac{1}{\sqrt{1+x^3}} \leq \frac{1}{x^{3/2}}. \]

Then, \( \int_{1}^{\infty} \frac{dx}{\sqrt{1+x^3}} \leq \int_{1}^{\infty} \frac{dx}{x^{3/2}} = 2 \)

by our previous formula \( \frac{1}{p-1} \) with \( p = \frac{3}{2} \).

This is the **Comparison Theorem**;

If \( f(x) \leq g(x) \) for \( x \in [a, b] \),

then \( \int_{a}^{b} f(x) \, dx \leq \int_{a}^{b} g(x) \, dx \).

In our example,

\[ \int_{1}^{\infty} \frac{dx}{\sqrt{1+x^3}} \] is an increasing function

of \( N \) bounded above by 2.

Thus, it does have a limit.
**Comparison Test:**

If \( f(x) \geq g(x) \geq 0 \) for \( x \geq a \),

1. if \( \int_a^\infty f(x) \, dx \) converges, so does \( \int_a^\infty g(x) \, dx \).
2. if \( \int_a^\infty g(x) \, dx \) diverges, so does \( \int_a^\infty f(x) \, dx \).

**Example:** \( x^2 \geq x \) for all \( x \geq 1 \).

So \( -x^2 \leq -x \) for all \( x \geq 1 \).

So \( e^{-x^2} \leq e^{-x} \) for all \( x \geq 1 \).

Thus \( \int_1^\infty e^{-x^2} \, dx \) converges since we showed \( \int_1^\infty e^{-x} \, dx \) converges.

Also, \( \int_0^\infty e^{-x^2} \, dx = \int_0^1 e^{-x^2} \, dx + \int_1^\infty e^{-x^2} \, dx \)

converges.

Finally, \( \int_{-\infty}^\infty e^{-x^2} \, dx = \int_{-\infty}^0 e^{-x^2} \, dx + \int_0^\infty e^{-x^2} \, dx \)

\( = 2 \int_0^\infty e^{-x^2} \, dx \) converges.

**Amazing Fact:** \( \int_0^\infty e^{-x^2} \, dx = \sqrt{\pi} \)
The error function:

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \]

Note that \( \text{erf}(0) = 0 \) and \( \text{erf}(\infty) = 1 \).

In statistics we talk about the mean or average value of a measurement and the standard deviation (or spread).

For example, the mean height of OSU male students might be 6'2" = 74" with a standard deviation of 3".

Then under the normal bell curve assumption, the probability that a random male student has height in \((74 - 3, 74 + 3)\) is \( \text{erf}\left(\frac{3}{\sqrt{2}}\right) = 0.68 \) \( \text{erf}\left(\frac{3}{\sqrt{2}}\right) = 0.95 \).
Ex: \( \int_{-\infty}^{\infty} \frac{7x^9}{1+x^2} \, dx \)

Split into: \( \int_{-\infty}^{0} \frac{7x^9}{1+x^2} \, dx + \int_{0}^{\infty} \frac{7x^9}{1+x^2} \, dx \).

Both are \boxed{\text{divergent}}.

Be careful: \( \int_{-N}^{N} 7x^9 \, dx = 0 \) for all \( N \), so we do not use that to decide convergence.

Ex: \( \int_{-\infty}^{\infty} \frac{1}{1+x^2} \, dx = \int_{-\infty}^{0} \frac{1}{1+x^2} \, dx + \int_{0}^{\infty} \frac{1}{1+x^2} \, dx \)

\[ = \arctan(x) \bigg|_{-\infty}^{0} + \arctan(x) \bigg|_{0}^{\infty} \]

\[ = 0 - \lim_{x \to -\infty} \arctan(x) + \lim_{x \to \infty} \arctan(x) - 0 \]

\[ = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{\pi}. \]

Ex: \( \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \, dx \)

Improper at both endpoints.

\[ = \int_{-1}^{0} \frac{1}{\sqrt{1-x^2}} \, dx + \int_{0}^{1} \frac{1}{\sqrt{1-x^2}} \, dx \]

\[ = \arcsin(x) \bigg|_{-1}^{0} + \arcsin(x) \bigg|_{0}^{1} \]

\[ = -\arcsin(-1) + \arcsin(1) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \boxed{\pi}. \]
\[ \int_0^1 \ln x \, dx = x (\ln x - 1) \bigg|_0^1 \]
\[ = -1 - \lim_{x \to 0^+} x \ln x \bigg|_0^1 \]
\[ = -1 - \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \]
\[ = -1 - 0 \quad \text{by L'Hôpital} \]
\[ = -1. \]