Derivative Review

**Derivative** = What ordinary English phrase?

"rate of change" (instantaneous)

\[ f(x) = x^n \quad \text{(power function)} \]

\[ f'(x) = \frac{d}{dx} (x^n) = n \times x^{n-1} \]

Emphasize the variable.

So if our position at time \( t \) is \( x \),

the velocity is \( \frac{dx}{dt} (x^2) = 2t \).

So from \( t = 2 \) seconds to \( t = 2.01 \) seconds,

we should cover approximately

\[ 2.2 \text{ meters/second} \times .01 \text{ seconds} = .044 \text{ meters} \]

**Addition Rule:** \[ \frac{d}{dx} (f(x) + g(x)) = \frac{df}{dx} + \frac{dg}{dx} \]

**Product Rule:** \[ \frac{d}{dx} (f(x)g(x)) = f'(x)g(x) + f(x)g'(x) \]

*Do not over-simplify* \[ \frac{d}{dx} (x^9 \times x^3) = \frac{d}{dx} (x^9) \times \frac{d}{dx} (x^3) \]

\[ = 7x^6 \neq (4x^3)(3x^2) \]
Think of \( f(x)g(x) \) as the area of a rectangle.

\[
\begin{align*}
\text{df} & = \text{change in } f \\
\text{dg} & = \text{change in } g
\end{align*}
\]

Change in area \( \approx (df)g + f(dg) \).

Quotient Rule:

\[
\frac{d}{dx} \left( \frac{f(x)}{g(x)} \right) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}
\]

(In Chapter 7, we'll see a simpler variation.)

Do not over-simplify:

\[
\frac{d}{dx} \left( \frac{1}{x^2 + x^4} \right) \neq \frac{1}{2x + 4x^3}
\]

This is not the right answer:

\[
\frac{0 \cdot (x^2 + x^4) - 1 \cdot (2x + 4x^3)}{(x^2 + x^4)^2} = \frac{-(2x + 4x^3)}{(x^2 + x^4)^2}
\]
**Chain Rule:** Composition of functions

\[ \frac{d}{dx} \left( f(g(x)) \right) = f'(g(x)) \cdot g'(x). \]

Often we think of \( u = g(x) \) as an intermediate variable and then

\[ y = f(u). \]

Then the chain rule looks like

\[ \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \]

Here \( \frac{dy}{du} = \text{rate of change of } y \text{ relative to } u \)

\( \frac{du}{dx} = \text{rate of change of } u \text{ relative to } x \)

\[ \frac{d}{dx} \left( \left( \frac{x^2 + x^4}{x} \right)^3 \right) = 3 \left( \frac{x^2 + x^4}{x} \right)^2 \cdot \left( \frac{2x + 4x^3}{x} \right) \]

Trigonometric:

\[ \frac{d}{dx} \sin x = \cos x \] \( \text{as } x \)

\[ \frac{d}{dx} \cos x = -\sin x \] \( \text{as } x \)
7.1: Inverse Functions

Read and Review:

If \( y = f(x) \), \( x \) = independent variable

and \( y \) = dependent variable.

\( y = f(x) \) is invertible on \([a, b]\)

if it is one-to-one:

If \( f(x_1) = f(x_2) \), then \( x_1 = x_2 \).

One-to-one

Not one-to-one:

Failed horizontal line test.

If \( f(x) \) is invertible, its inverse function

is defined by \( g(x) = y \) if \( f(y) = x \).

We denote \( g \) by \( f^{-1} \) (does not mean power).
Ex: \( y = x^2 \) is one-to-one on \([0, \infty)\), NOT on \((\infty, \infty)\).

So we can define the inverse \( y = \sqrt{x} \) on \([0, \infty)\).

Since \( 3^2 = 9 \), we have \( \sqrt{9} = 3 \).

On the appropriate intervals, we have

\[ f(f^{-1}(x)) = x, \quad f^{-1}(f(x)) = x, \]

for all \( x \).

So \( (\sqrt{x})^2 = x \) for all \( x \geq 0 \).

\( \sqrt{(x^2)} = x \) for all \( x \geq 0 \).

Graph: \( y = f(x) \) flip around \( y = x \).
7.2 Exponential Functions

A power function is a fixed power of a variable base
\[ f(x) = x^n \]

An exponential function is a variable power of a fixed base: \[ f(x) = a^x \].

The two basic defining properties of \( f(x) = a^x \) are:

1) For a positive integer \( n \), \[ a^n = a \cdot a \cdot a \cdot \ldots \cdot a \quad n \text{ times} \]

2) For any \( x, y \) at all, \[ a^{x+y} = a^x a^y \].

Property (2) is easy to justify if \( x, y \) are positive integers. For example, \[ a^5 = a \cdot a \cdot a \cdot a \cdot a = (a \cdot a \cdot a)(a \cdot a) = a^3 a^2 \].

Consequences: 1) Since \( a \cdot a^x = a^{x+0} = a^x \),
\[ a^0 = \frac{a^x}{a^x} = 1 \].

2) Since \( a^x a^{-x} = a^{x-x} = a^0 = 1 \),
\[ a^{-x} = \frac{1}{a^x} \].
9) Since \((a^\frac{1}{n})^n = a^{\frac{1}{n} \cdot n} = a^1 = a\), \(a^{\frac{1}{n}}\) must be an \(n\)-th root of \(a\), also written \(\sqrt[n]{a} = a^{\frac{1}{n}}\).

The \(n\)-th root is the inverse function of \(f(x) = x^n\), i.e., \(f^{-1}(x) = x^{\frac{1}{n}}\).

This works only if \(n\) is odd and \(x \in (-\infty, \infty)\), or if \(n\) is even and \(x \geq 0\).

So we generally assume \(a \geq 0\) in order to define \(f(x) = a^x\).

Next we define \(a^{\frac{p}{q}} = (a^{\frac{1}{q}})^p = (a^p)^{\frac{1}{q}}\)

for any fraction \(\frac{p}{q}\).

Finally, for any irrational \(x\), we define \(a^x = \lim_{\frac{p}{q} \to x} a^{\frac{p}{q}}\) where \(\frac{p}{q}\) runs through fractions tending to \(x\). This takes some care to justify.
For $a > 1$, $a^x$ is increasing on $(-\infty, \infty)$.

For $a < 1$, $a^x$ is decreasing on $(-\infty, \infty)$.

Also, for $a > 1$, $\lim_{x \to \infty} a^x = \infty$.

Hence, $\lim_{x \to \infty} a^{-x} = \lim_{x \to \infty} \frac{1}{a^x} = 0$.

For $a < 1$, $\lim_{x \to 0} a^x = 0$, $\lim_{x \to -\infty} a^x = \infty$.

Application: $\lim_{x \to \infty} 2^x - 1 = \boxed{-1}$.
Rules of exponents: For all $a, b > 0$, $x, y \in (-\infty, \infty)$.

1) $a^{x+y} = a^x a^y$
2) $a^{x-y} = \frac{a^x}{a^y}$

3) $(a^x)^y = a^{xy} = (a^y)^x$
4) $(ab)^x = a^x b^x$

So $a^x = a^{-x} = (a^{-1})^{-x} = \left(\frac{1}{a}\right)^{-x}$

Example: $2^5 = \left(\frac{1}{2}\right)^{-5}$

Application: 
\[
\lim_{x \to \infty} \frac{3^x}{2^{2x}} = \lim_{x \to \infty} \left(\frac{3}{4}\right)^x = 0
\]
\[
\left(\frac{3}{2}\right)^x = \frac{3^x}{4^x} = \left(\frac{3}{4}\right)^x
\]

Application where exponential functions occur:
- Population change
- Epidemiology
- Chemical reactions
- Radioactive decay
Derivatives: For \( f(x) = a^x \),

\[
    f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} \\
    = \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} \\
    = a^x \cdot \lim_{h \to 0} \frac{a^h - 1}{h}
\]

The last limit \( \ell(a) = \lim_{h \to 0} \frac{a^h - 1}{h} \) is independent of \( x \).

It takes some care to justify this limit.

We shall see \( \ell(a) \) is the natural logarithm of \( a \), written \( \ln(a) \).

So \[
\left. \frac{d}{dx} (a^x) \right|_{a} = a^x \ln(a)
\]

For \( 0 < a < b, h > 0 \), \[
\frac{b^h - 1}{h} < \frac{a^h - 1}{h}
\]

So that proves \( \ln(a) \leq \ln(b) \).
(i.e. \( \ln(a) \) is an increasing function of \( a \)).

Also, \( \ln(1) = \lim_{h \to 0} \frac{1^h - 1}{h} = 0 \).

It can be shown that \( \lim_{a \to \infty} \ln(a) = \infty \).

Then there must be a number \( e \) such that \( \ln(e) \).

Check:

\[
\frac{2.001 - 1}{0.001} \approx 0.693
\]

\[
\frac{3.001 - 1}{0.001} \approx 1.099
\]

So \( 2 < e < 3 \). In fact, \( e = 2.718281828 \ldots \).

So,

\[
\frac{d}{dx} (a^x) = a^x \ln(a)
\]

\[
\frac{d}{dx} (e^x) = e^x
\]

WARNING:

\[
\frac{d}{dx} (a^x) = x a^{x-1}
\]

Big difference
Graphs of power functions:

- $y = x^3$
- $y = x^3$
- $y = x^2$
- $y = x^{1/2}$

Exponentials:

- $y = 2^x$
- $y = (0.5)^x$

Problems:

1. $\frac{d}{dx} (e^{\tan x}) = e^{\tan x} \frac{d}{dx} (\tan x)$
   
   $= e^{\tan x} \sec^2 x$

2. $\frac{d}{dx} (e^{-4x} \sin 2x)$
Ex: Find the max. \( f(x) = x e^{-x} \).

\[ f(0) = 0 \quad \text{and} \quad \lim_{x \to \infty} x e^{-x} = 0. \]

(expontential decay is always faster than polynomial growth)

Solve \( f'(x) = 0 \) for the critical point.

\[ f'(x) = 1 \cdot e^{-x} + x(e^{-x}(-1)) = e^{-x}(1 - x) \]

\[ = 0 \quad \text{implies} \quad x = 1. \]

So the max. is \( f(1) = 1 \cdot e^{-1} = \frac{1}{e} \).

Integrals: \( \int e^x \, dx = e^x + C \) (Total accumulation)

(Remember: if \( F'(x) = f(x) \), then \( \int f(x) \, dx = F(x) + C \))

Ex: \( \int_{1}^{2} e^{-3x} \, dx = \int_{-3}^{-6} e^u \, du = \frac{-1}{3} e^u \bigg|_{-3}^{-6} \)

Substitute \( u = -3x \), \( du = -3 \, dx \), \( = \frac{-1}{3} (e^{-6} - e^{-3}) \)