A point \((x, y)\) in Cartesian coordinates may also be determined by

1) its distance \(r\) from \((0, 0)\),

and 2) its direction angle \(\theta\) from positive \(x\)-axis.

By trigonometry, this gives

\[
x = r \cos \theta \\
y = r \sin \theta \\
\]

and

\[
\theta = \arctan \left(\frac{y}{x}\right). \tag{2.3.1}
\]

We call \((r, \theta)\) the Polar Coordinates of \((x, y)\).

Ex: \((x, y) = (1, 1) \iff (r, \theta) = (\sqrt{2}, \frac{\pi}{4})\).

The Polar version is not unique:

\[
(r, \theta) = \left(\sqrt{2}, \frac{\pi}{4}\right) = \left(-\sqrt{2}, \frac{5\pi}{4}\right) = \left(\sqrt{2}, \frac{9\pi}{4}\right), \text{ etc.}
\]
-changing \( r \) to \(-r\) is OK if we add \( \pi \) to \( \theta \).

-changing \( \theta \) by adding a multiple of \(2\pi\) doesn't change the point.

Ex: Convert Cartesian \((1, \sqrt{3})\) to Polar:

\[ \rightarrow (2, \frac{\pi}{3}) \text{ or } (-2, \frac{4\pi}{3}) \text{, etc.} \]

Convert Polar \((2, \frac{3\pi}{4})\) to Cartesian:

\[ x = 2 \cos \left( \frac{3\pi}{4} \right) = -\sqrt{2} \]

\[ y = 2 \sin \left( \frac{3\pi}{4} \right) = \sqrt{2} \]

Curves in Polar:

\[ x^2 + y^2 = 4 \text{ is equivalent to } r^2 = 4 \]

or more simply \[ r = 2 \].
Ex: Convert \((x-2)^2 + (y-1)^2 = 9\)

to Polar:

\[
(x-2)^2 + (y-1)^2 = 9 \\ (r\cos\theta)^2 - 4r\cos\theta + (r\sin\theta)^2 - 2r\sin\theta = 4 \\
\sqrt{r^2 - 2r(2\cos\theta + \sin\theta) - 4} = 0.
\]

Ex: Convert \(r = \sin\theta\) to Cartesian:

First multiply by \(r\): \(r^2 = r\sin\theta\).

\[
x^2 + y^2 = y.
\]

What is it? Complete the square:

\[
x^2 + (y - \frac{1}{2})^2 = \frac{1}{4} = (\frac{1}{2})^2
\]

Circle of radius \(\frac{1}{2}\) at \((0, \frac{1}{2})\).

Ex: Convert \(\tan \theta = -2\) to Cartesian:

Then \(-2 = \frac{\sin\theta}{\cos\theta} = \frac{r\sin\theta}{r\cos\theta} = \frac{y}{x}\).

So this is a straight line \(y = -2x\).

Ex: \(r = \sec\theta \Rightarrow r\cos\theta = 1 \Rightarrow \boxed{x = 1}\).
Graphing: Make a table of $r$ vs. $\theta$ and plot on polar graph paper.

$r = 1 - \cos \theta$:

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$\pi/2$</td>
<td>1</td>
</tr>
<tr>
<td>$\pi$</td>
<td>2</td>
</tr>
<tr>
<td>$3\pi/2$</td>
<td>1</td>
</tr>
<tr>
<td>$2\pi$</td>
<td>0</td>
</tr>
</tbody>
</table>

How do we know it has a "cusp" at $\theta = 0$?

$y = r \sin \theta = (1 - \cos \theta) \sin \theta$.

\[
\frac{dx}{d\theta} = \sin^2 \theta + (1 - \cos \theta) \cos \theta
\]

$= \cos \theta + \sin^2 \theta - \cos^2 \theta$.

At $\theta = 0, 2\pi$, $\frac{dx}{d\theta} = 1 + 0 - 1 = 0$.

But $\frac{dx}{d\theta} = \frac{d}{d\theta} (r \cos \theta) = -\sin \theta + 2 \cos \theta \sin \theta$.

This is $> 0$ for $\theta > 0$, but $< 0$ for $\theta < 2\pi$, $\theta$ near $2\pi$. 
Ex: Find tangent line to \( r = 4 \cos \theta \) at \( \theta = \frac{\pi}{4} \).

\[
(x, y) = (r \cos \theta, r \sin \theta)
\]

\[
= (4 \cos^2 \theta, 4 \cos \theta \sin \theta)
\]

At \( \theta = \frac{\pi}{4} \), \( (x, y) = (2, 2) \).

\[
\frac{dy}{d\theta} \bigg|_{\theta=\frac{\pi}{4}} = \frac{\cos^2 \theta - \sin^2 \theta}{-2 \cos \theta \sin \theta}
\]

\[
= 0.
\]

So \( y = 2 \) is the tangent line.

Ex:

Sketch \( r = \sin 2\theta \)

4-leaved rose
\[ r = \sin 3\theta \]

Then it repeats the same 3-leaved rose.

\[ r = 1 + \theta \rightarrow \text{SPIRAL} \]