Regular triangulations and the index of a cusped hyperbolic 3-manifold

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The 3D index of an ideal triangulation

Dimofte-Gaiotto-Gukov define the 3D index of an ideal triangulation:

$$\begin{aligned} I : \{\text{oriented ideal triangulations}\} &\to \mathbb{Z}((q^{1/2})) \\ \mathcal{T} &\mapsto I_{\mathcal{T}} \end{aligned}$$

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The index $I_{\mathcal{T}}$ is built from the tetrahedron index $I_\Delta(m,e)(q) \in \mathbb{Z}[[q^{1/2}]]$, for $m, e \in \mathbb{Z}$.

$$I_\Delta(m,e)(q) = \sum_{n=(-e)_+}^{\infty} (-1)^n \frac{q^{\frac{1}{2}n(n+1)-\left(n+\frac{1}{2}e\right)m}}{(q)_n (q)_{n+e}}$$

where $e_+ = \max\{0, e\}$ and $(q)_n = \prod_{i=1}^{n}(1-q^i)$.

Very roughly speaking, if the manifold has r cusps and $\mathcal{T}$ has N edges (and so N tetrahedra), then

$$I_{\mathcal{T}}(q) = \sum_{k \in \mathbb{Z}^{N-r} \subset \mathbb{Z}^N} (-q^{1/2})^{*} \prod_{j=1}^{N} I_\Delta(*, *)(q)$$

Here:

- k is a vector, with an integer "number of copies" of each of the $N - r$ edges
- j runs over the tetrahedra of $\mathcal{T}$
- $*$ terms are "dot products" between k and how the edges are incident to the tetrahedra

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Problem 1

The sum in the definition of $I_{\mathcal{T}}$ may not converge, so $I_{\mathcal{T}}$ is not well-defined for all $\mathcal{T}$.

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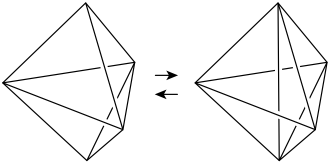
Theorem (Garoufalidis, Hodgson, Rubinstein, S)

$I_{\mathcal{T}}$ is well-defined $\Leftrightarrow$ $\mathcal{T}$ is 1-efficient $\Leftarrow$ M is atoroidal and $\mathcal{T}$ admits a semi-angle structure.

(1-efficiency is a property about normal surfaces in the triangulation. We will come back to what a semi-angle structure is shortly.)

Theorem (Garoufalidis)

If $\mathcal{T}, \mathcal{T}'$ are two triangulations of M connected by a 2-3 move and $I_{\mathcal{T}}, I_{\mathcal{T}'}$ are well-defined, then $I_{\mathcal{T}} = I_{\mathcal{T}'}$.



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Any two ideal triangulations of M are connected by a sequence of 2-3 moves.

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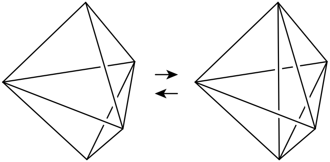

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Problem 2

We have no idea if the 1-efficient or semi-angle structure admitting triangulations are connected by 2-3 moves.

The plan

Construct a set of triangulations χ_M of M so that:

- χ_M depends only on the topology of M.
- χ_M is connected by 2-3 moves.
- Every member of χ_M admits a semi-angle structure.

Then define I_M to be $I_{\mathcal{T}}$ for any $\mathcal{T} \in \chi_M$. This would promote the 3D index to a topological invariant of M.

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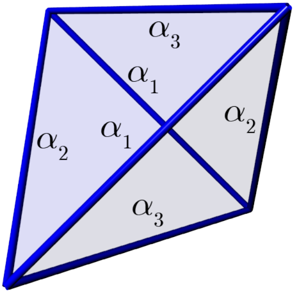
Theorem (Garoufalidis, Hodgson, Rubinstein, S)

This works.

Semi-angle structures

Associate angles (real numbers) to the edges of the tetrahedra of $\mathcal{T}$, so that:

1. In each tetrahedron, angles at opposite edges are the same.
2. In each tetrahedron, $\alpha_1 + \alpha_2 + \alpha_3 = \pi$.
3. Around each edge of $\mathcal{T}$, $\sum \alpha = 2\pi$.



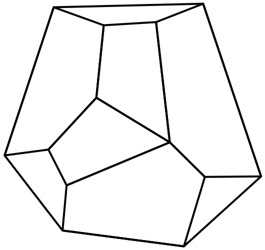
If all angles are in $[0, \pi]$ then this is a *semi-angle structure*.

The dihedral angles of an ideal hyperbolic tetrahedron satisfy (1) and (2), and if many ideal hyperbolic tetrahedra fit together nicely in $\mathbb{H}^3$ then their angles also satisfy (3).

So we use hyperbolic geometry to build our triangulations with semi-angle structures.

Theorem (Epstein-Penner)

Let M be a hyperbolic 3-manifold with one torus boundary component. Then there is a canonical subdivision of M into convex ideal hyperbolic polyhedra.



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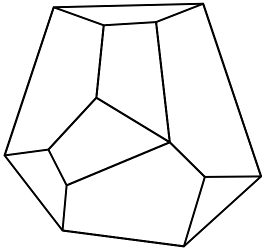

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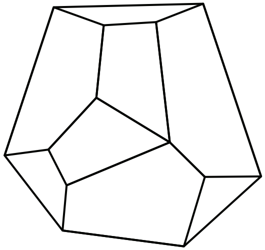

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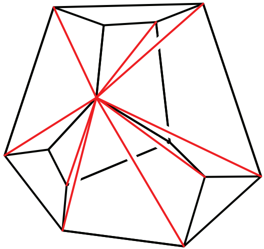

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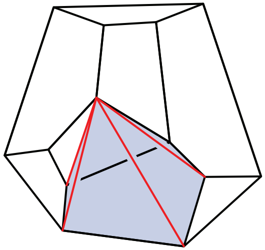

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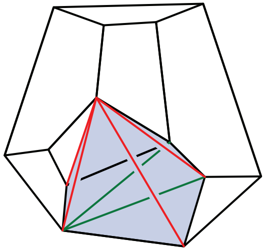

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We cannot just take *all* subdivisions of the polyhedra into ideal hyperbolic tetrahedra since:

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It isn't known if the set of all geometric triangulations of a convex ideal polyhedron is connected by 2-3 moves. (True in dimension 2, false in dimension 5.)

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However,

Theorem (Gelfand-Kapranov-Zelevinsky)

Regular triangulations of a convex polytope in $\mathbb{R}^n$ are connected by geometric bistellar flips.

Regular triangulations of a polytope in $\mathbb{R}^n$

Here a *polytope* is the convex hull of a set of points in $\mathbb{R}^n$.

Roughly speaking, a triangulation of the polytope is *regular* if it is isomorphic to the lower faces of a convex polytope in $\mathbb{R}^{n+1}$.

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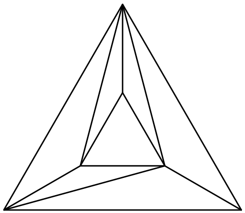

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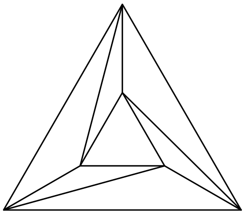

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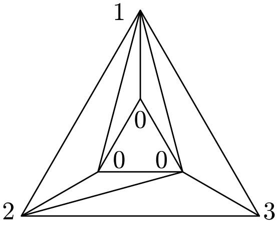

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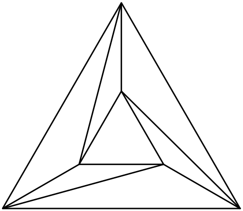

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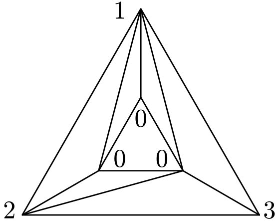

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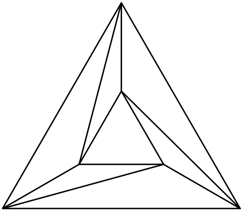
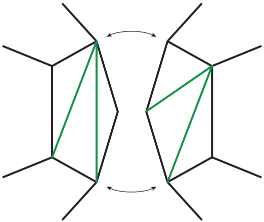


Using the Klein model of $\mathbb{H}^3$, we have a correspondence

convex ideal hyperbolic polyhedron	$\leftrightarrow$	convex Euclidean polyhedron with vertices on S^2
"regular ideal triangulation"	$\leftrightarrow$	regular triangulation
2-3 move or sequence of moves	$\leftrightarrow$	geometric bistellar flip

Finally, the triangulations of the faces of our polyhedra may not match.



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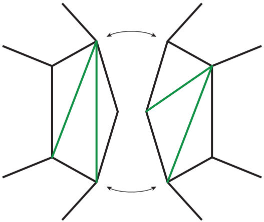
This can be fixed by inserting flat tetrahedra to bridge between the triangulations.



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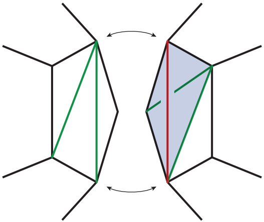

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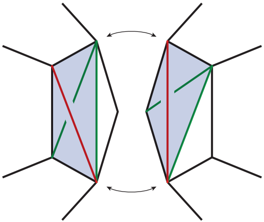

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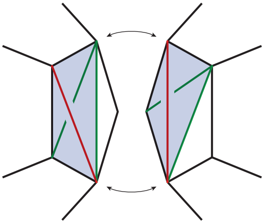

Our set χ_M consists of all triangulations of M constructed by the following:

1. Insert a regular ideal triangulation into each polyhedron of the Epstein-Penner decomposition.
2. Insert any sequence of flat tetrahedra that bridges between the induced triangulations on each pair of glued faces of the polyhedra.

All such triangulations have semi-angle structures, and the set is connected by 2-3 moves. (In fact, we also need 0-2 moves, we also prove that these do not change $I_{\mathcal{T}}$.)

Thanks!