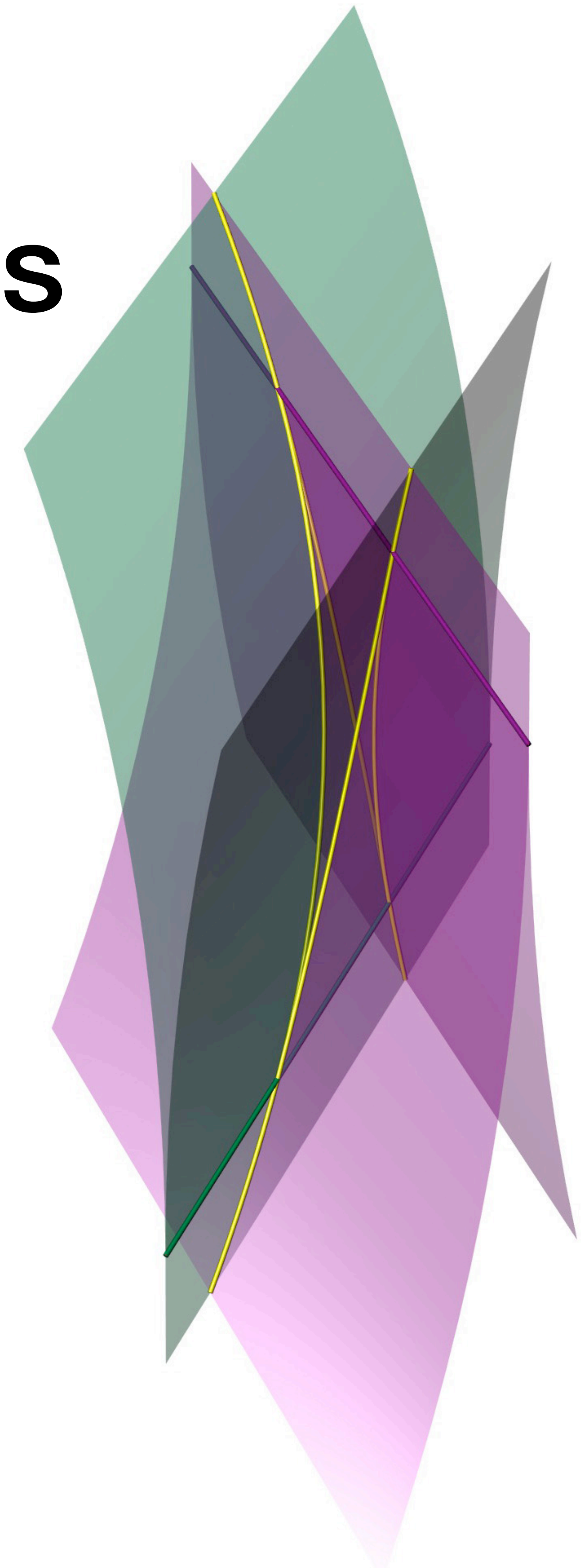
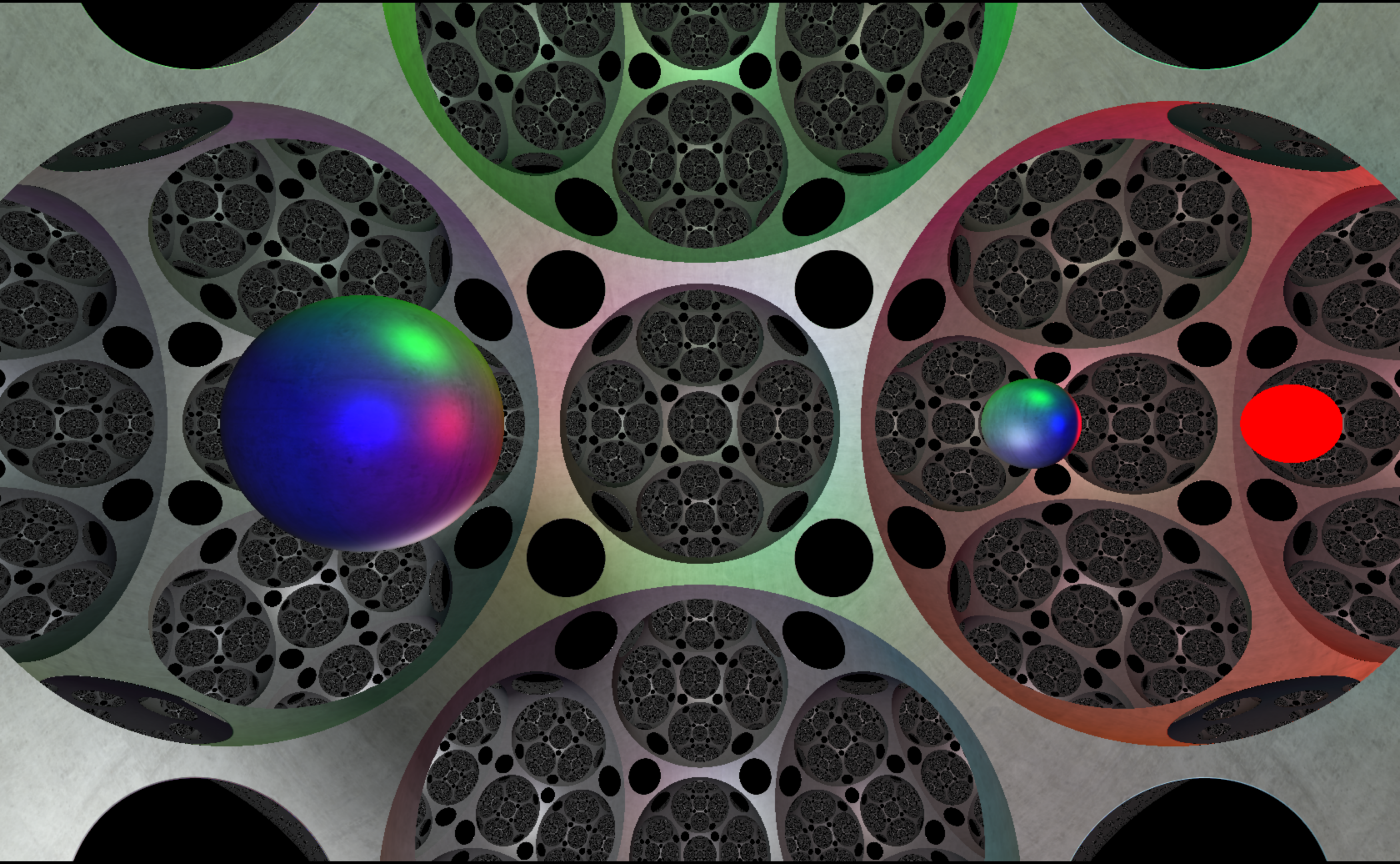
From veering triangulations to pseudo-Anosov flows



Henry Segerman
Oklahoma State University

joint work with
Saul Schleimer

Research supported by Australian Research Council grant DP1095760 and National Science Foundation grants DMS-1308767 and DMS-1708239.



michaelwoodard.net/hypVR-Ray

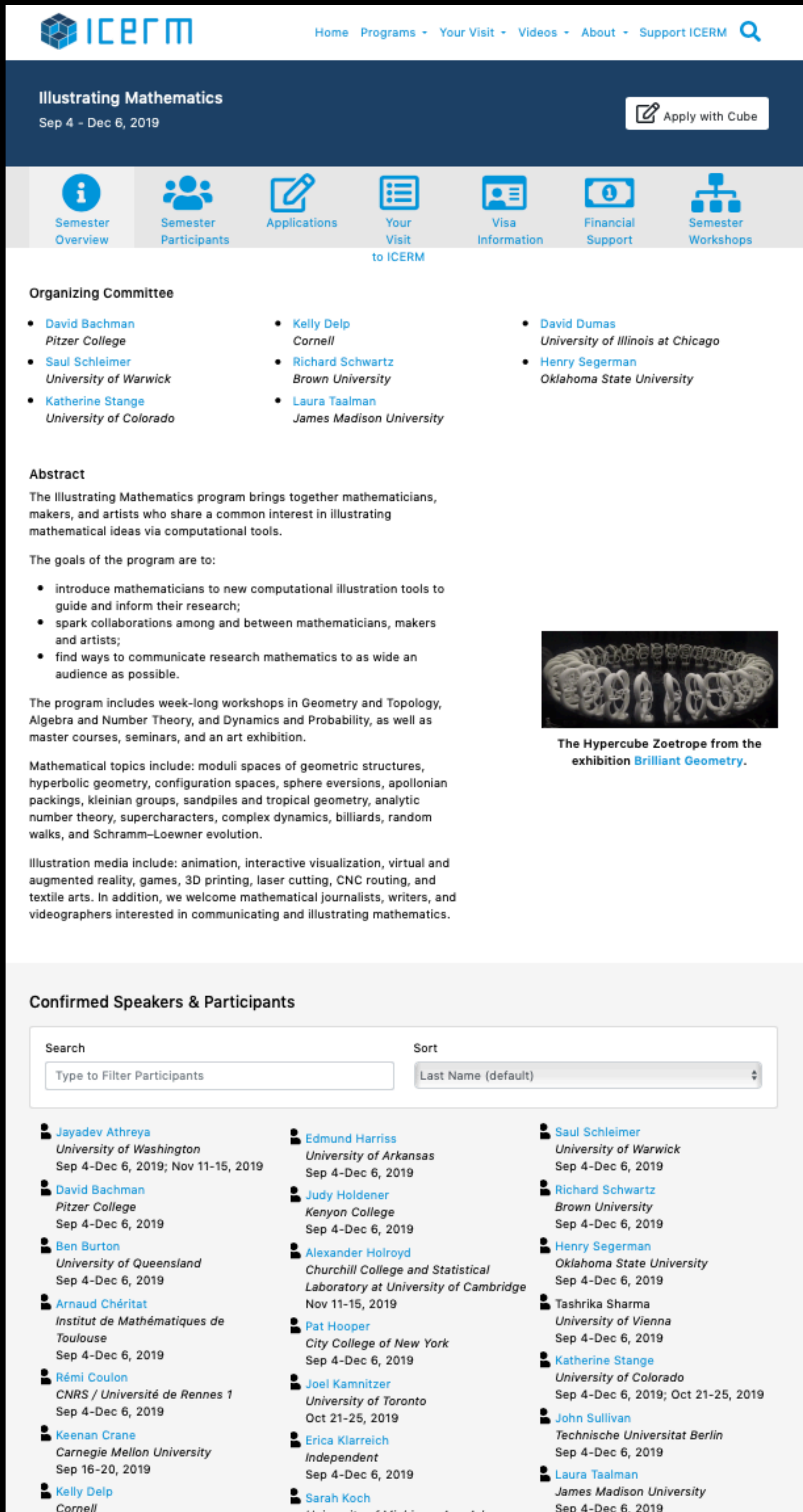
Joint work with Roice Nelson and Michael Woodard
Partially supported by NSF grant DMS-1708239

Illustrating Mathematics at ICERM

Sep 4 - Dec 6, 2019

The Illustrating Mathematics program brings together mathematicians, makers, and artists who share a common interest in illustrating mathematical ideas via computational tools.

https://icerm.brown.edu/programs/sp-f19/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Illustrating Mathematics

Sep 4 - Dec 6, 2019

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Semester Overview | Semester Participants | Applications | Your Visit to ICERM | Visa Information | Financial Support | Semester Workshops

Organizing Committee

- David Bachman, *Pitzer College*
- Saul Schleimer, *University of Warwick*
- Katherine Stange, *University of Colorado*
- Kelly Delp, *Cornell*
- Richard Schwartz, *Brown University*
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The Hypercube Zoetrope from the exhibition Brilliant Geometry.

Confirmed Speakers & Participants

Search: Type to Filter Participants — Sort: Last Name (default)

- Jayadev Athreya, *University of Washington*, Sep 4-Dec 6, 2019; Nov 11-15, 2019
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- Arnaud Chéritat, *Institut de Mathématiques de Toulouse*, Sep 4-Dec 6, 2019
- Rémi Coulon, *CNRS / Université de Rennes 1*, Sep 4-Dec 6, 2019
- Keenan Crane, *Carnegie Mellon University*, Sep 16-20, 2019
- Kelly Delp, *Cornell*
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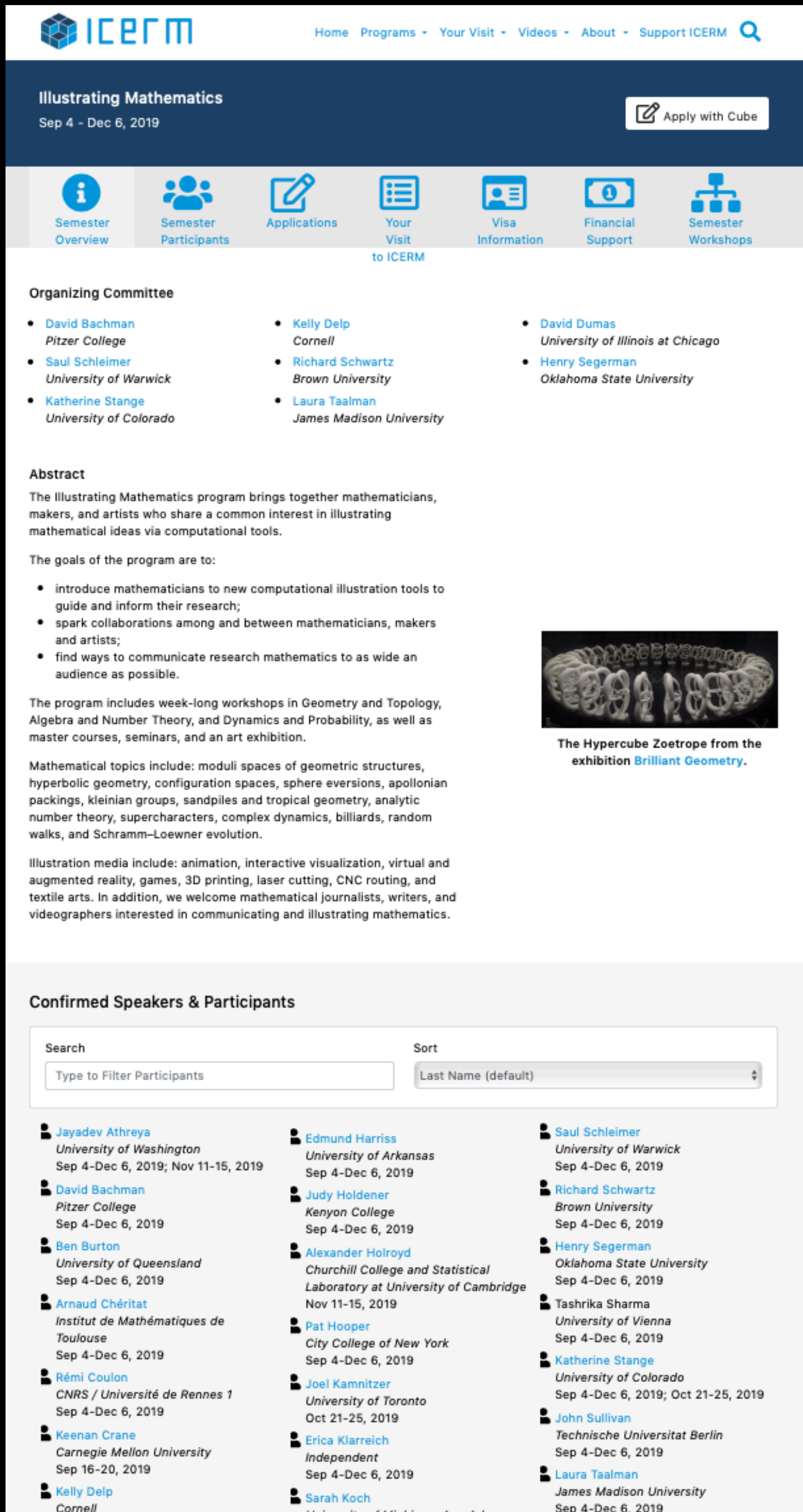

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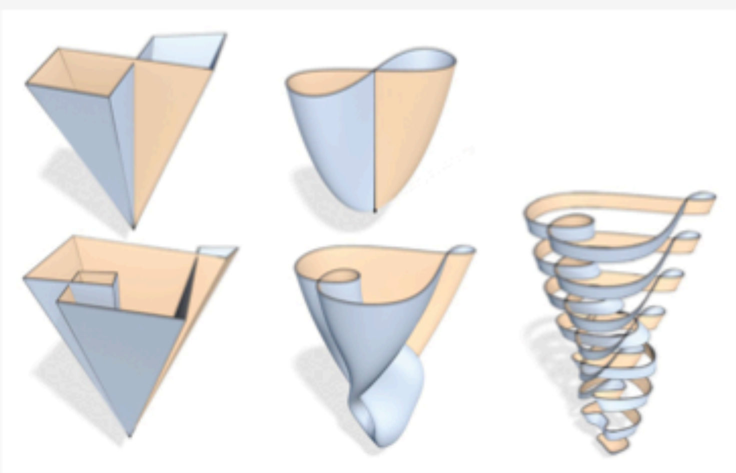
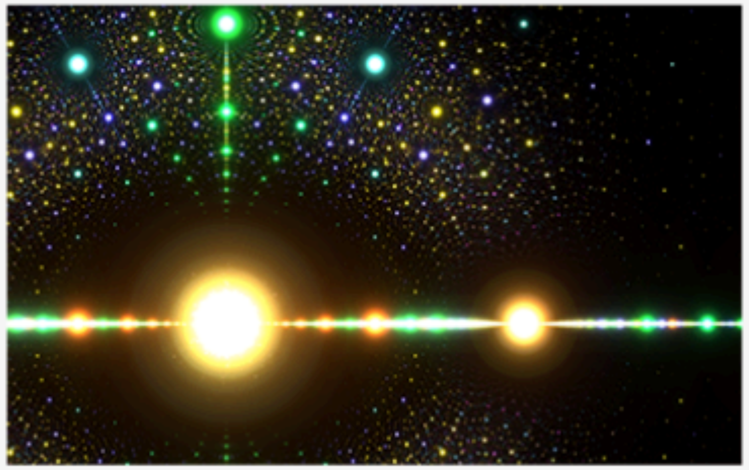
Semester workshops

Illustrating Geometry and Topology



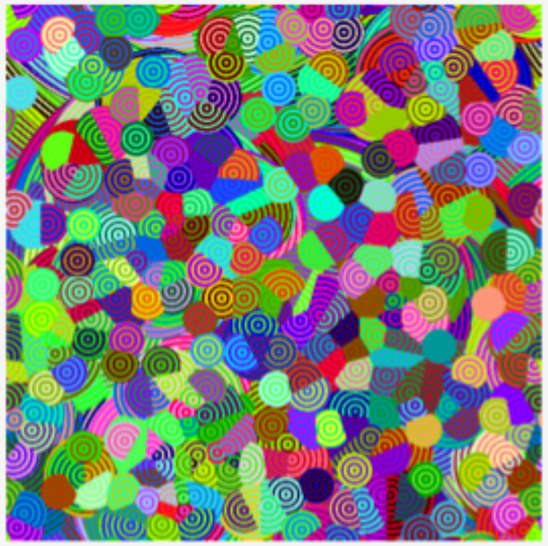
Sep 16 - 20, 2019
Semester Workshop

Illustrating Number Theory and Algebra



Oct 21 - 25, 2019
Semester Workshop

Illustrating Dynamics and Probability



Nov 11 - 15, 2019
Semester Workshop

https://icerm.brown.edu/programs/sp-f19/

Illustrating Geometry and Topology

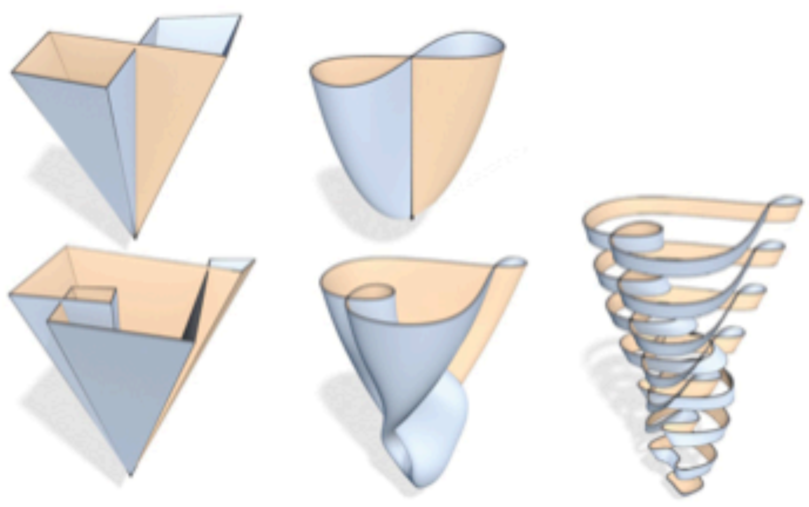
Sep 16 - 20, 2019

Organizing Committee

- Keenan Crane
  Carnegie Mellon University
- David Dumas
  University of Illinois at Chicago

Abstract

This workshop will focus on the interaction between visualization, computer experiment, and theoretical advances in all areas of research in geometry and topology. Fruitful interactions of this type have a long history in the field, with physical models and computer images and animations providing both illustration of existing work and inspiration for new developments. Emerging visualization technologies, such as virtual reality, are poised to further increase the tools available for mathematical illustration and experimentation. By bringing together expert practitioners of mathematical visualization techniques and researchers interested in incorporating such tools into their research, the workshop will give participants a clear picture of the state of the art in this fast-moving field while also fostering new collaborations and innovations in illustrating geometry and topology.



**Obstructions to regular homotopy in smooth and polyhedral surfaces.
Image credit: Albert Chern, Ulrich Pinkall and Peter Schröder.**

https://icerm.brown.edu/programs/sp-f19/

Illustrating Number Theory and Algebra

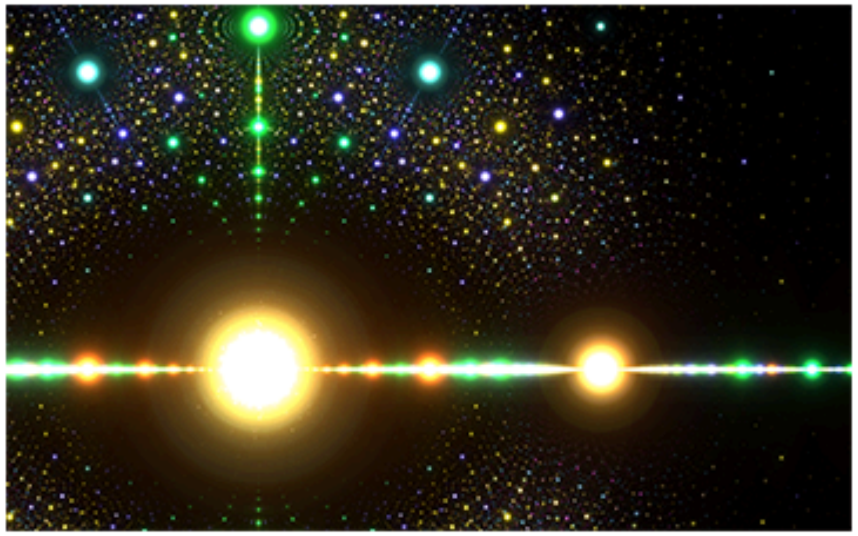
Oct 21 - 25, 2019

Organizing Committee

- Ellen Eischen
  The University of Oregon
- Joel Kamnitzer
  University of Toronto
- Alex Kontorovich
  Rutgers University
- Katherine Stange
  University of Colorado

Abstract

The symbiotic relationship between the illustration of mathematics and mathematical research is now flowering in algebra and number theory. This workshop aims to both showcase and develop these connections, including the development of new visualization tools for algebra and number theory. Topics are wide-ranging, and include Apollonian circle packings and the illustration of the arithmetic of hyperbolic manifolds more generally, the visual exploration of the statistics of integer sequences, and the illustrative geometry of such objects as Gaussian periods and Fourier coefficients of modular forms. Other topics may include expander graphs, abelian sandpiles, and Diophantine approximation on varieties. We will also focus on diagrammatic algebras and categories such as Khovanov-Lauda-Rouquier algebras, Soergel bimodule categories, spider categories, and foam categories. The ability to visualize complicated relations diagrammatically has led to important advances in representation theory and knot theory in recent years.



Algebraic numbers in the complex plane.
Image credit: David Moore, based on earlier work by Stephen J. Brooks

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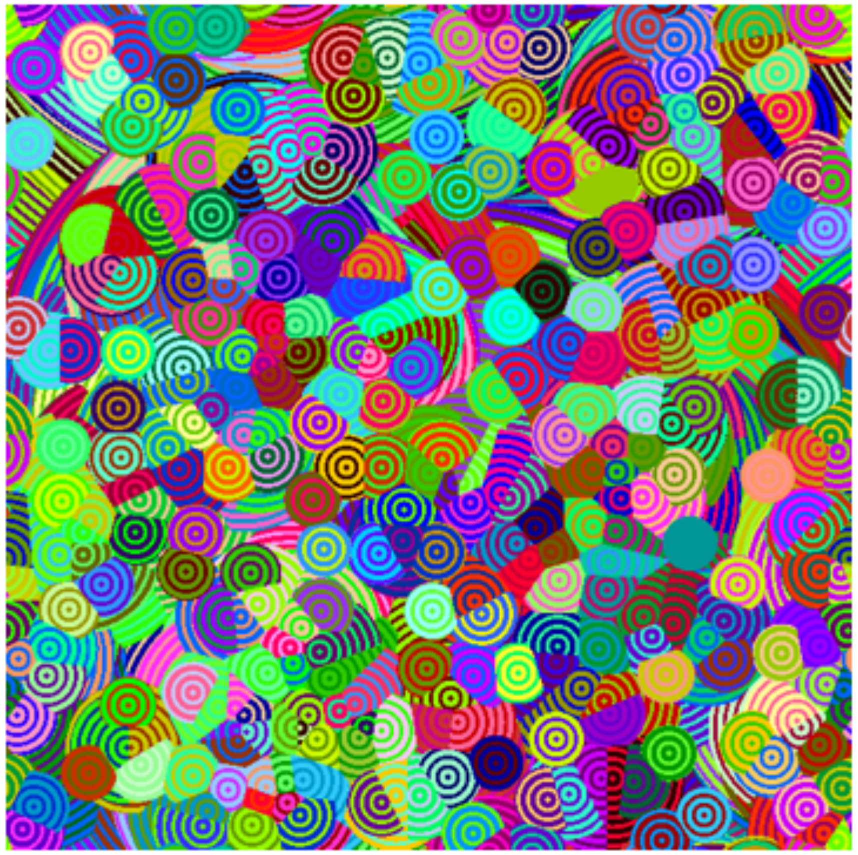

A stable matching in the plane.
Image credit: Alexander E. Holroyd.
Picture based on research by Christopher Hoffman, Alexander Holroyd and Yuval Peres.

Abstract

This workshop will focus on the theoretical insights developed via illustration, visualization, and computational experiment in dynamical systems and probability theory. Some topics from complex dynamics include: dynamical moduli spaces and their dynamically-defined subvarieties, degenerations of dynamical systems as one moves toward the boundary of moduli space, and the structure of algebraic data coming from a family of dynamical systems. In classical dynamical systems, some topics include: flows on hyperbolic spaces and Lorentz attractors, simple physical systems like billiards in two and three dimensional domains, and flows on moduli spaces. In probability theory, the workshop features: random walks and continuous time random processes like Brownian motion, SLE, and scaling limits of discrete systems.

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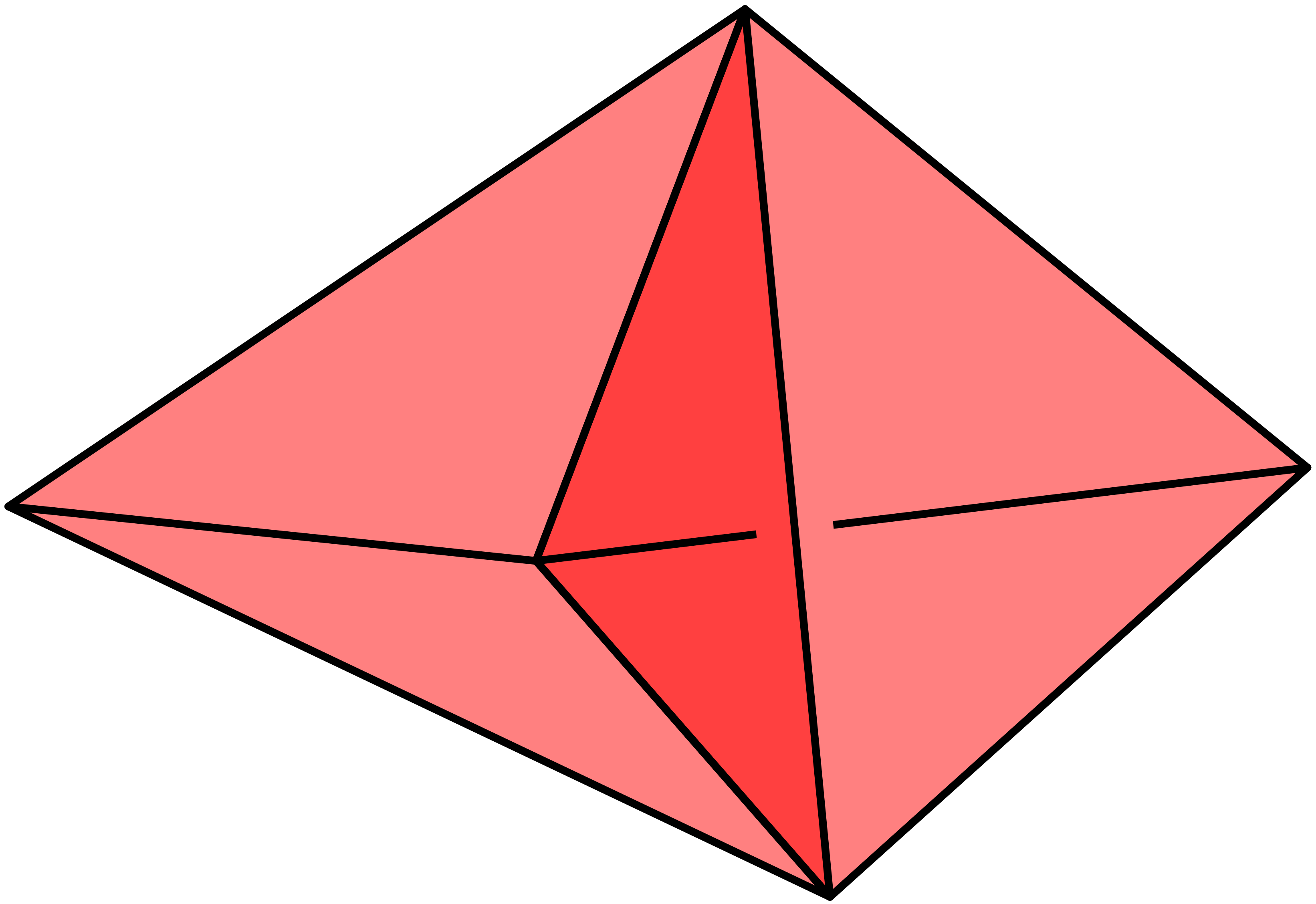
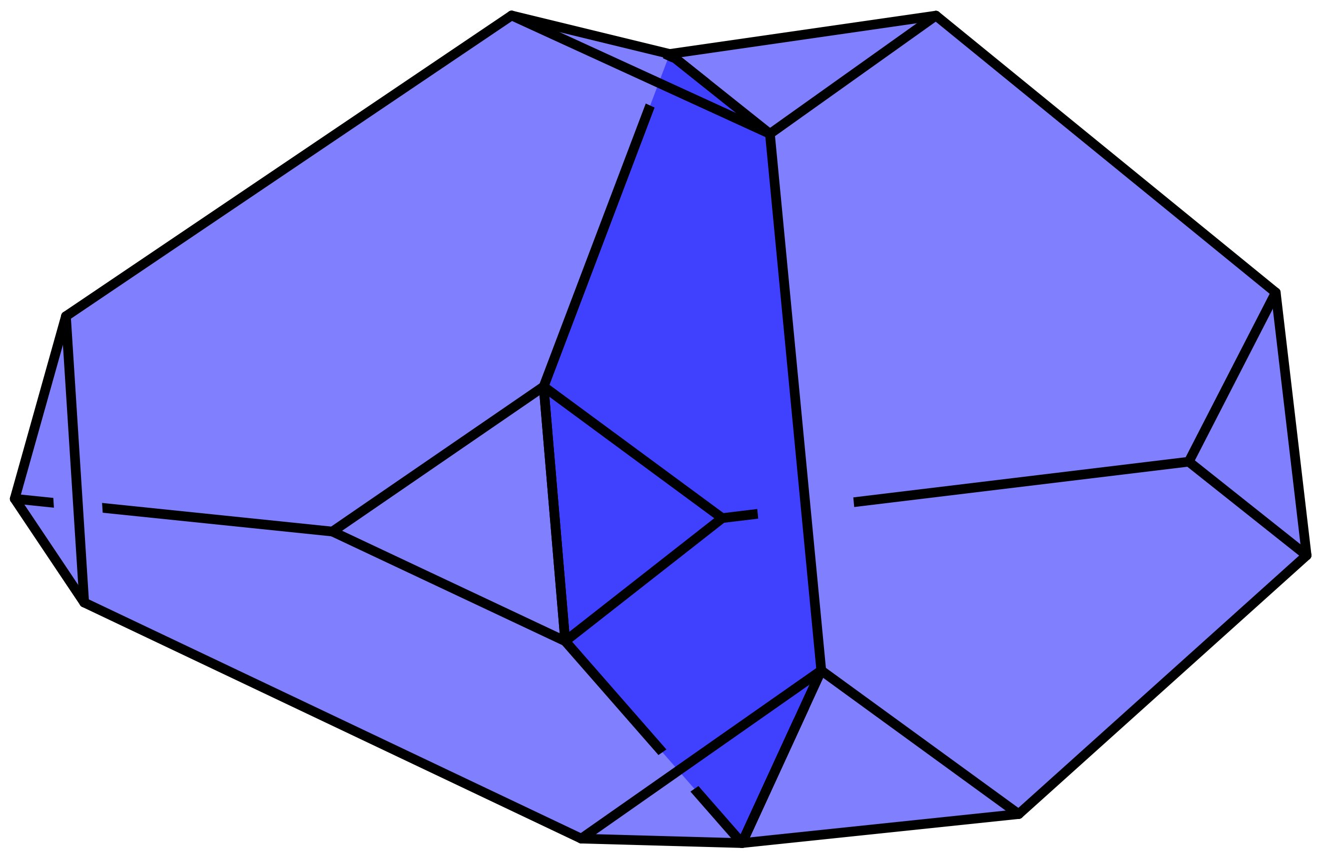
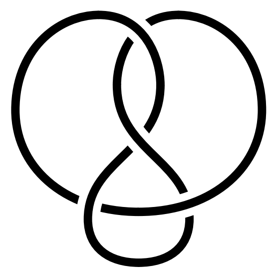
Triangulations

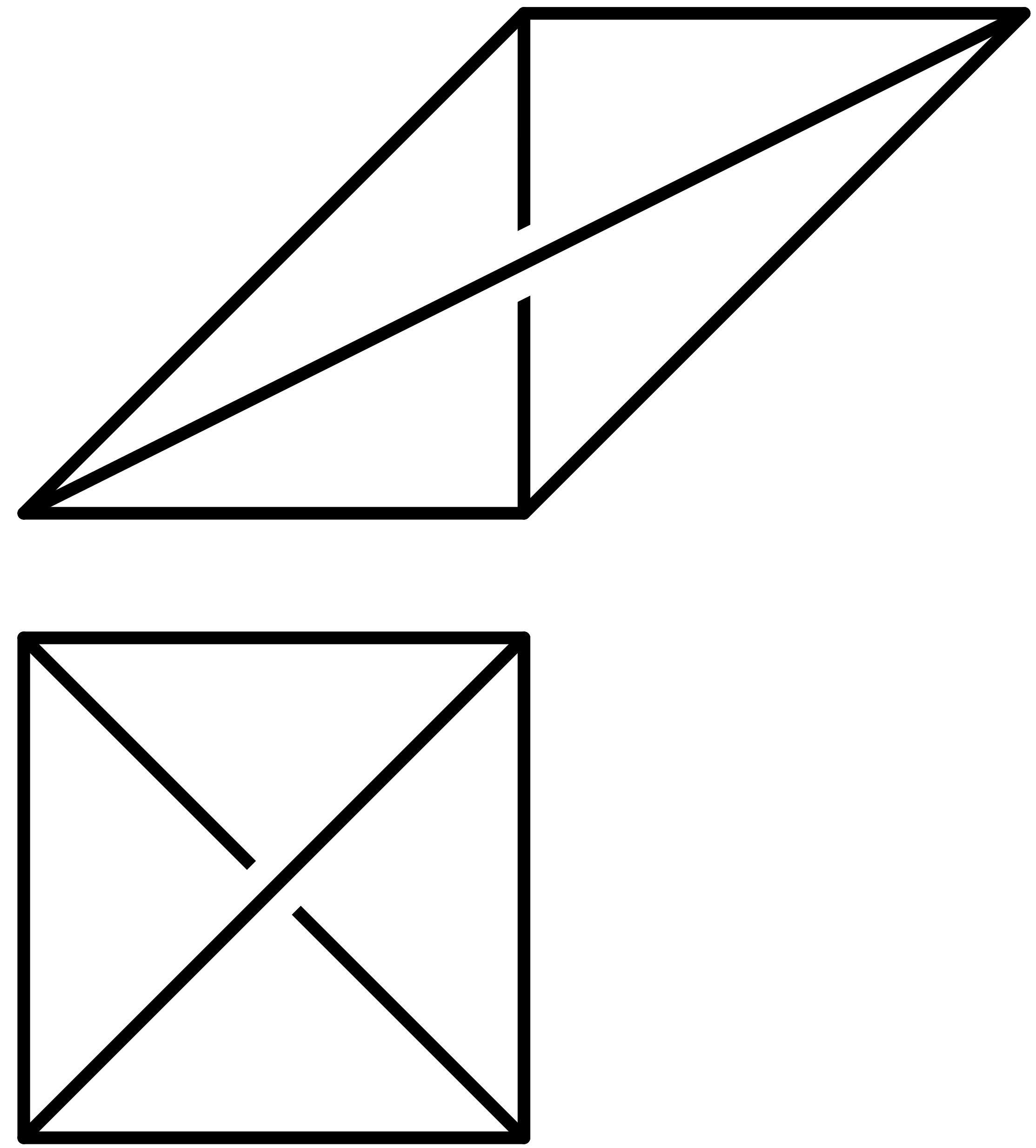


Ideal Triangulations



Ex: the figure 8 knot complement

$M_8 = S^3 -$ 



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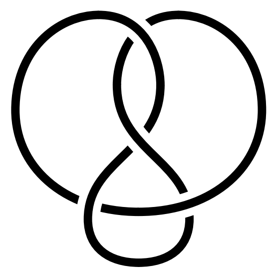
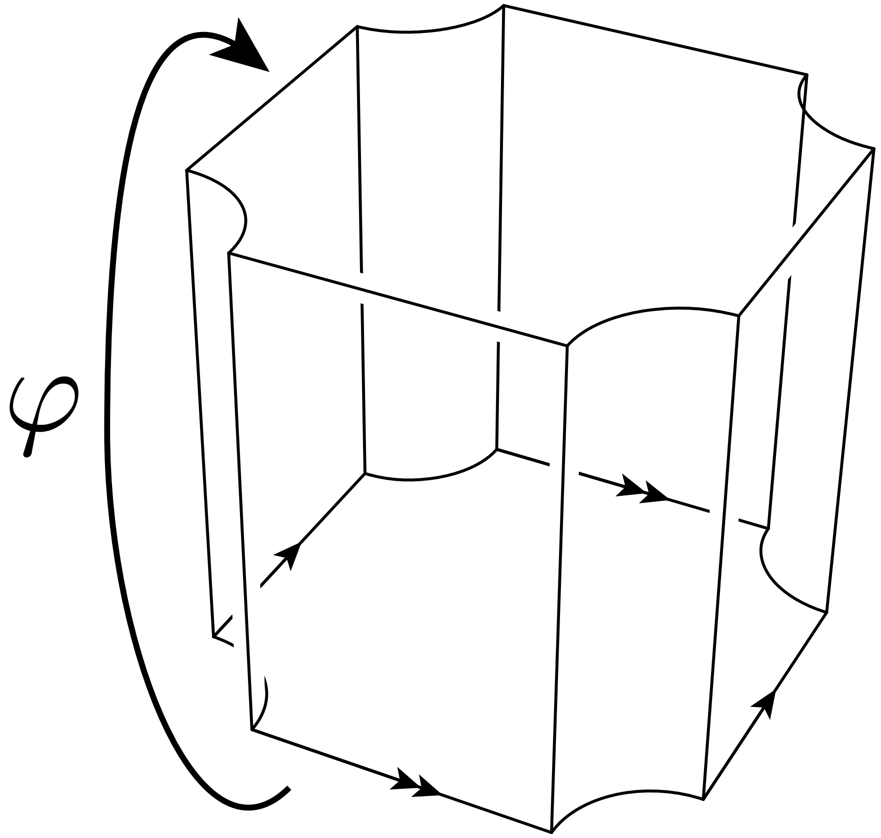
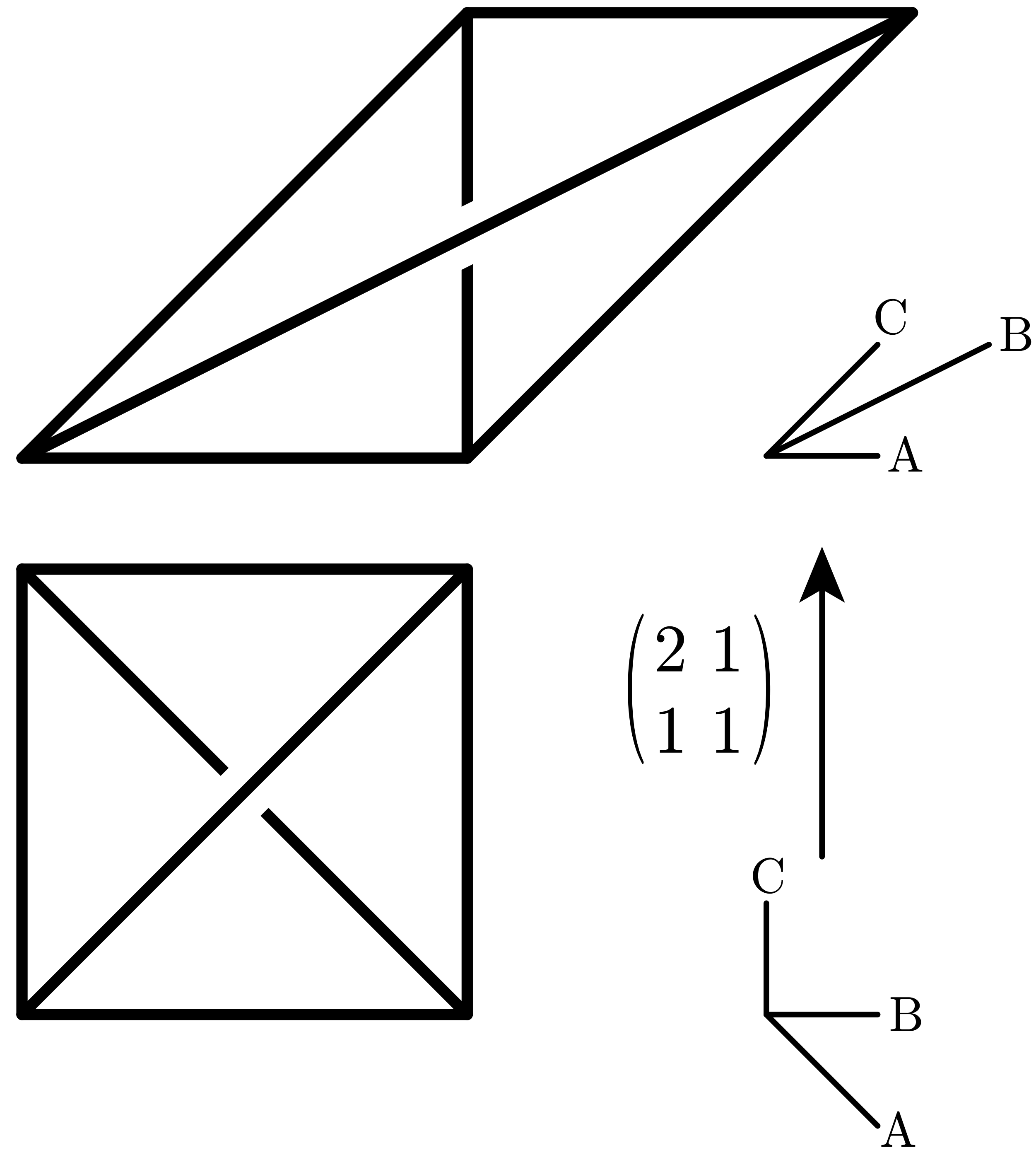
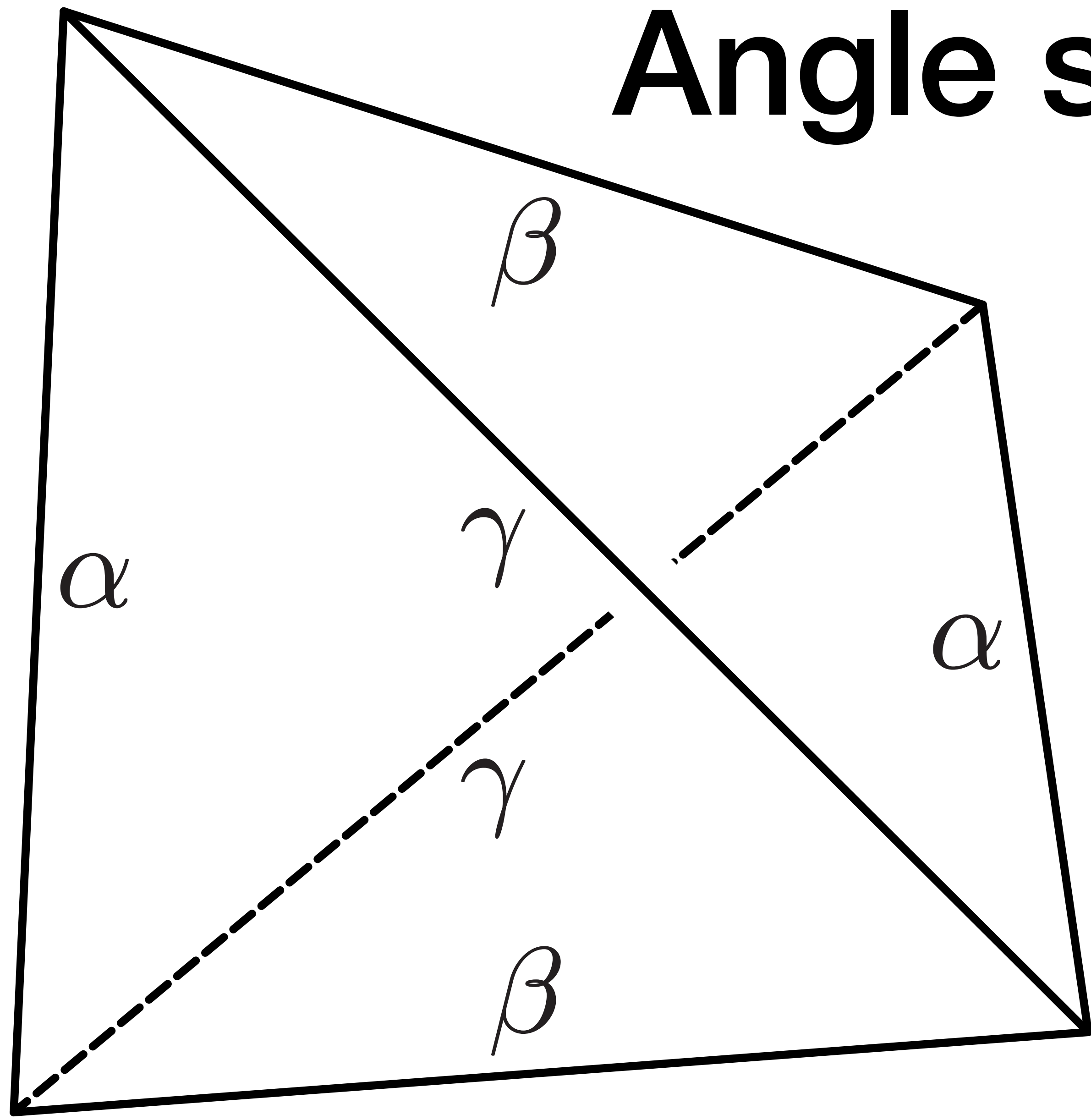
$$M_8 = S^3 -$$ 





$$M_8 \cong M_\varphi = (T^2_* \times I)/(x,0) \sim (\varphi(x),1)$$

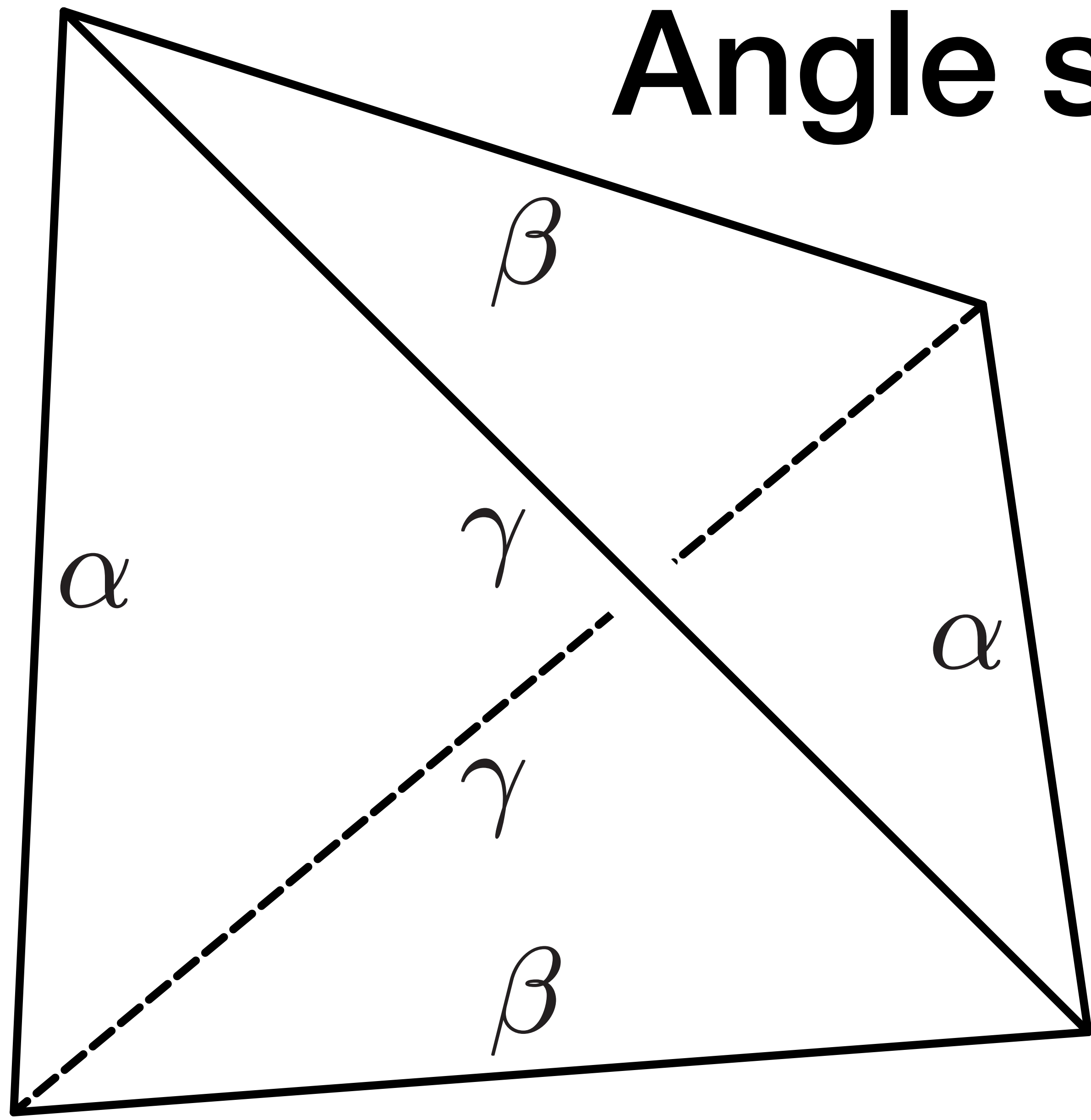
Angle structure



$$\alpha + \beta + \gamma = \pi$$

$$\sum_{\text{around edge}} \alpha_i = 2\pi$$

Angle structure



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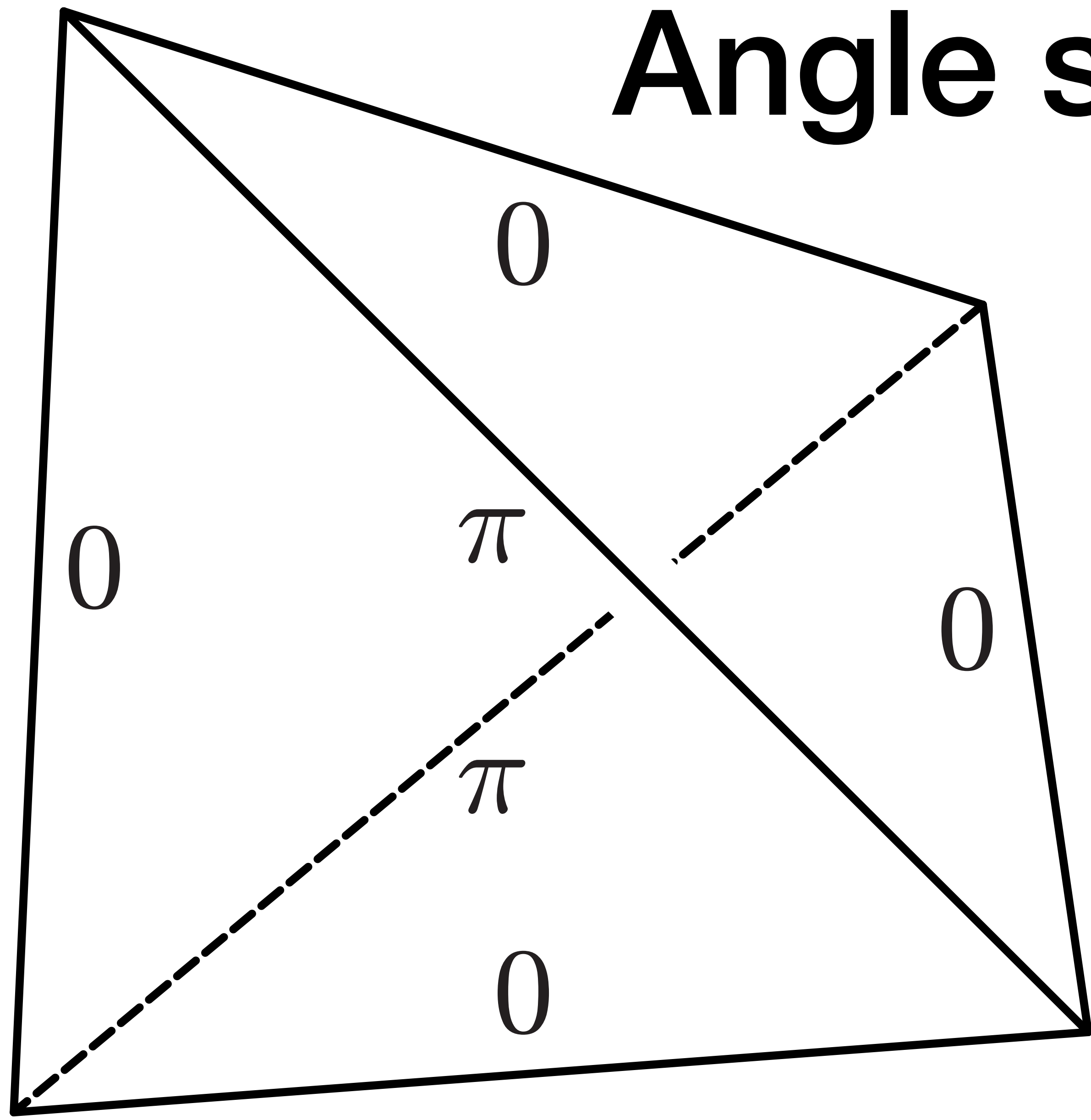
strict angle structure if:

$$\alpha_i \in (0, \pi)$$

Angle structure



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taut angle structure if:

$$\alpha_i \in \{0, \pi\}$$

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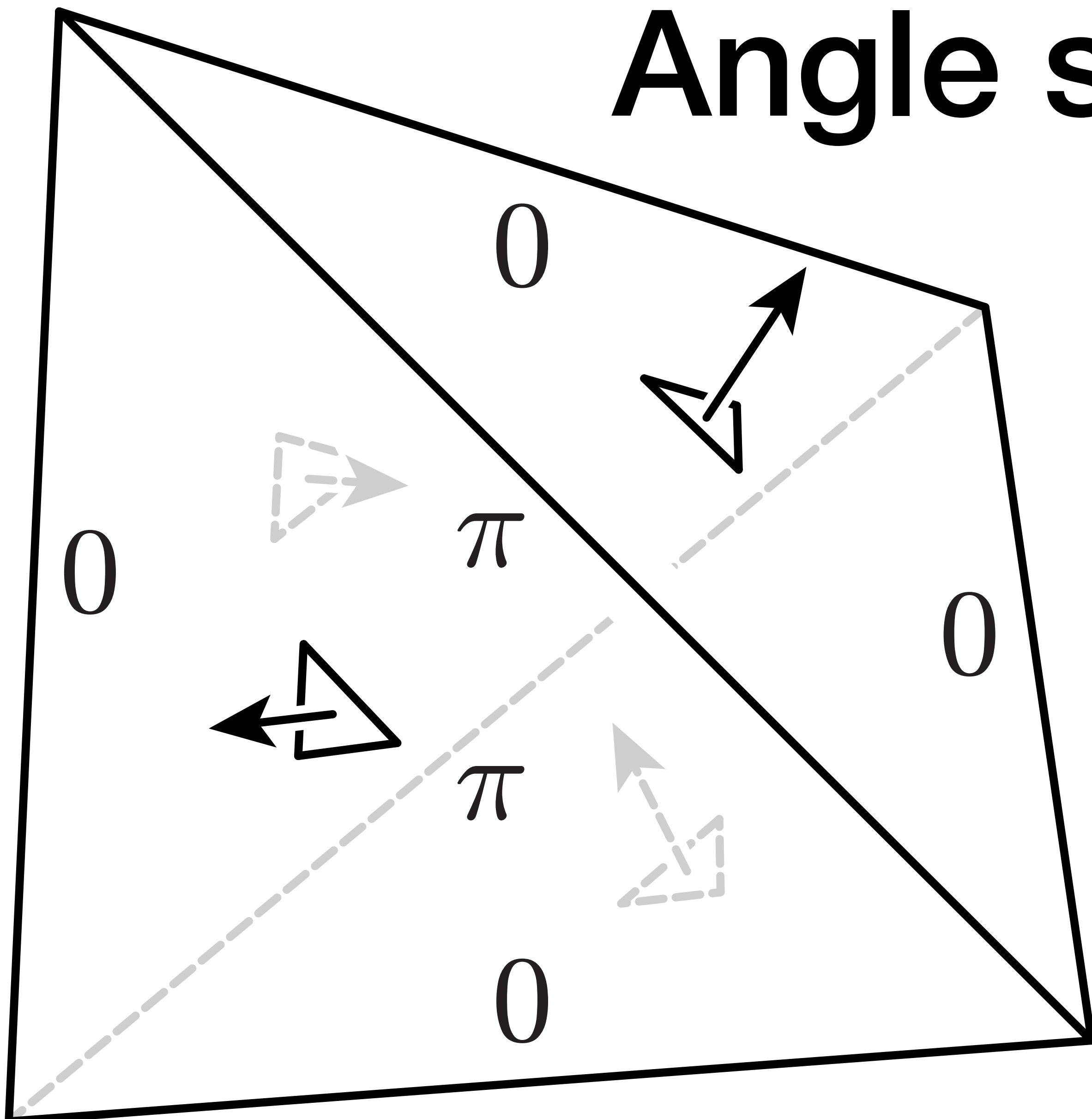

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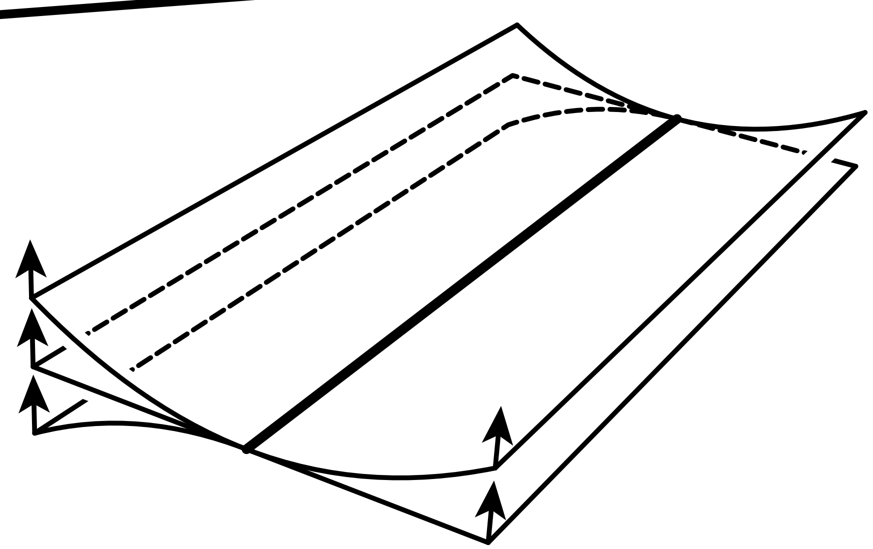
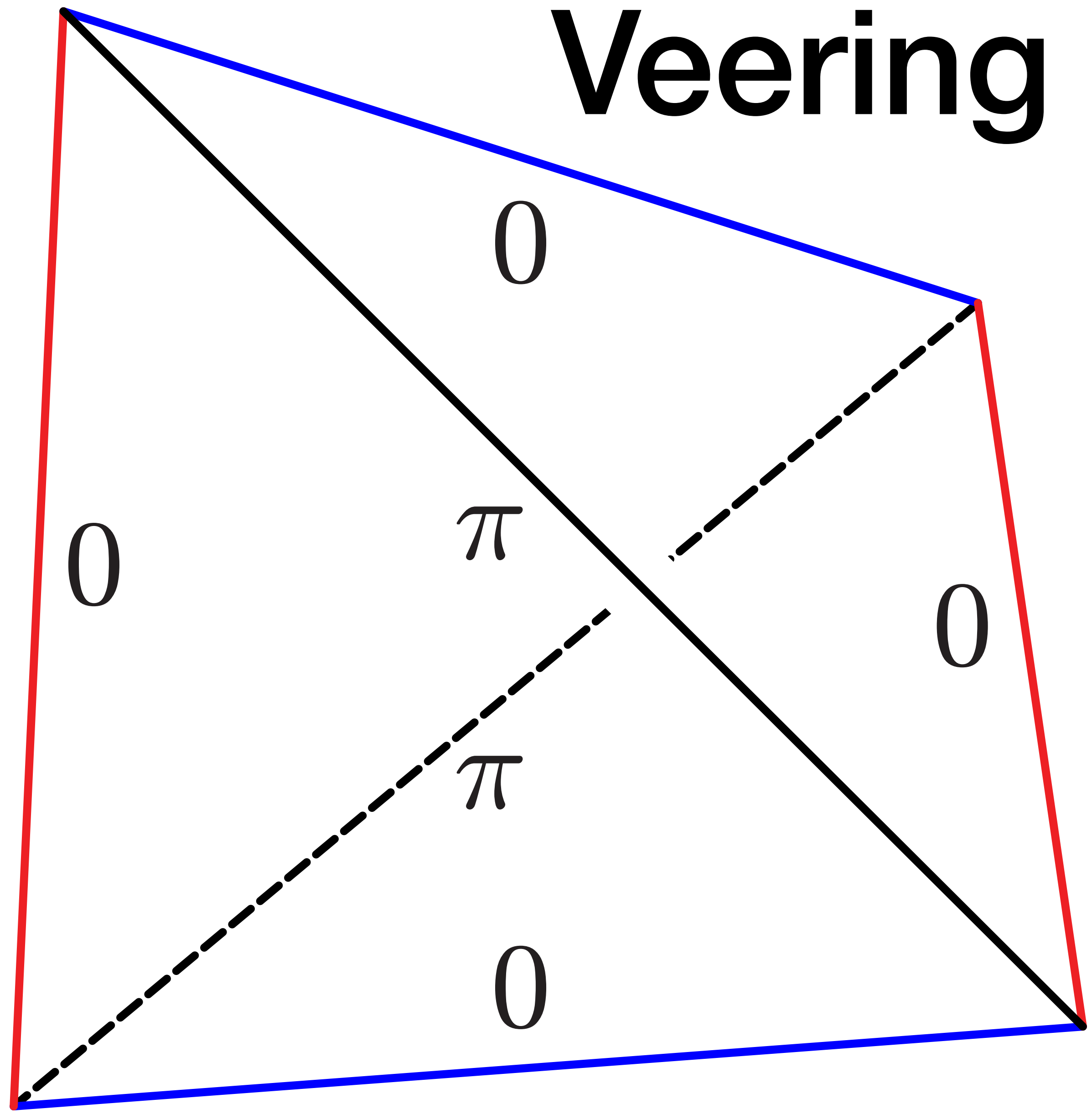


transverse taut angle structure if:

coorientations on faces and $\alpha_i \in \{0, \pi\}$

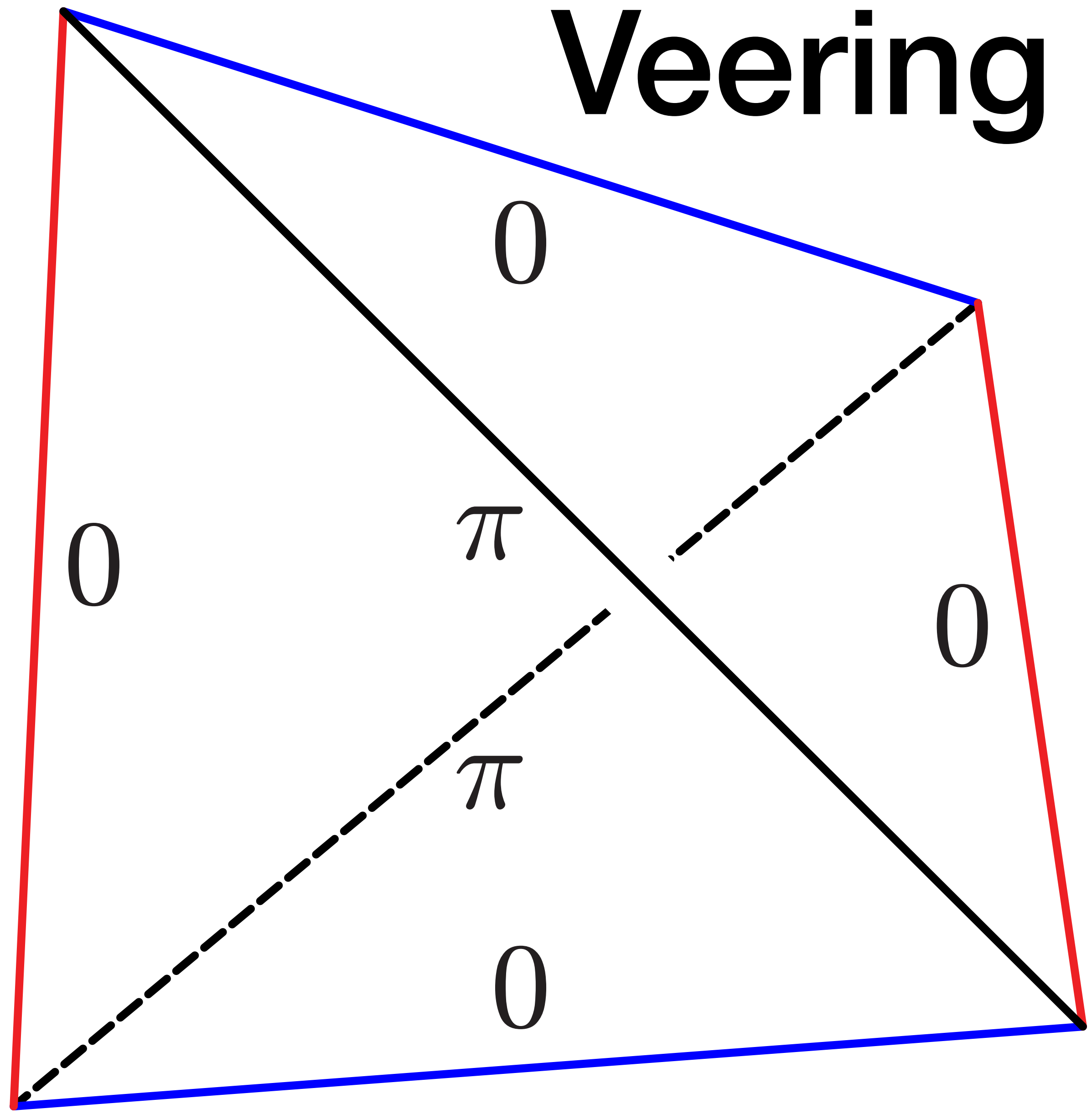
Veering structure



taut angle structure, and:

Each tetrahedron colours its 0 angle edges red or blue.

Veering structure



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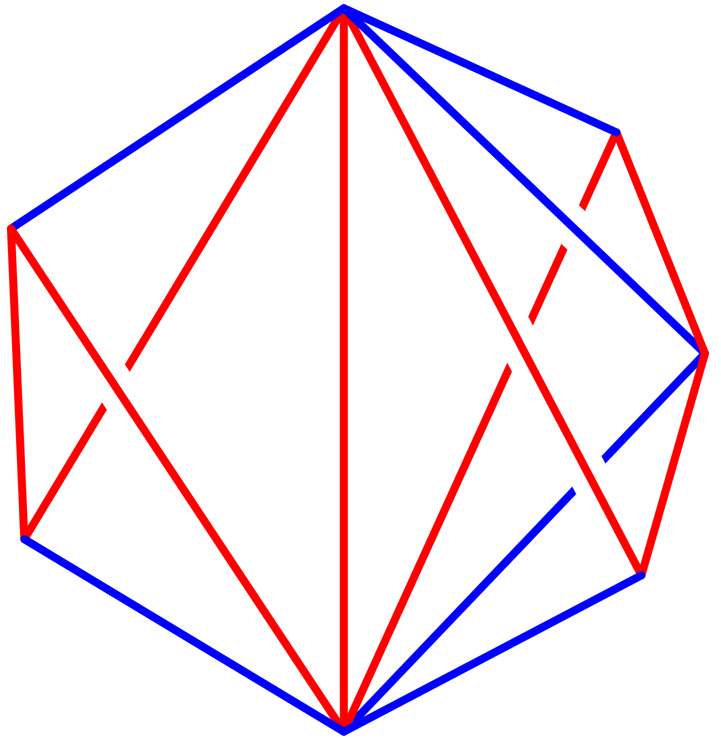
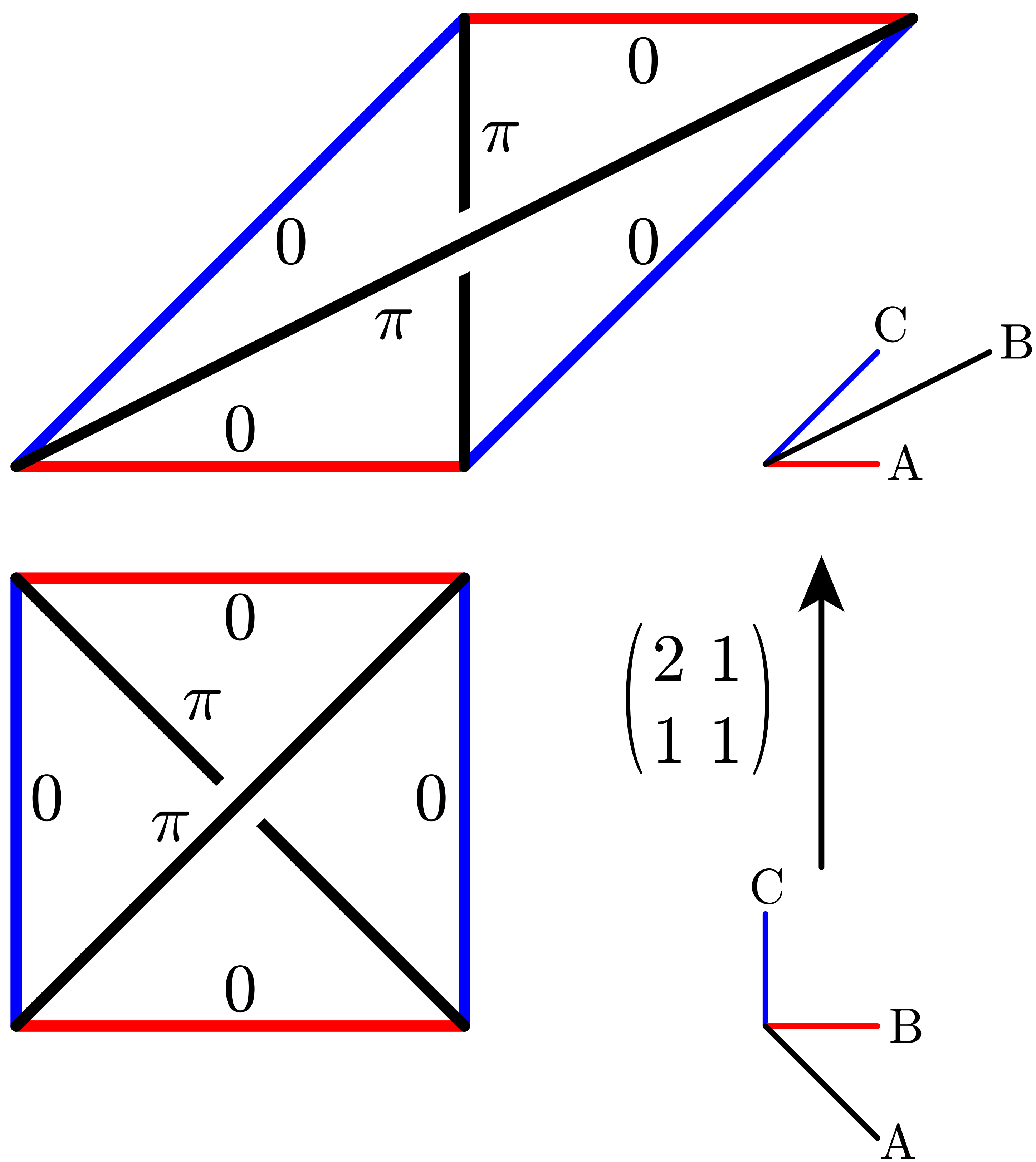
Each tetrahedron colours its 0 angle edges red or blue.

These colours must be consistent for all tetrahedra incident to the edge.



Ex: the figure 8 knot complement



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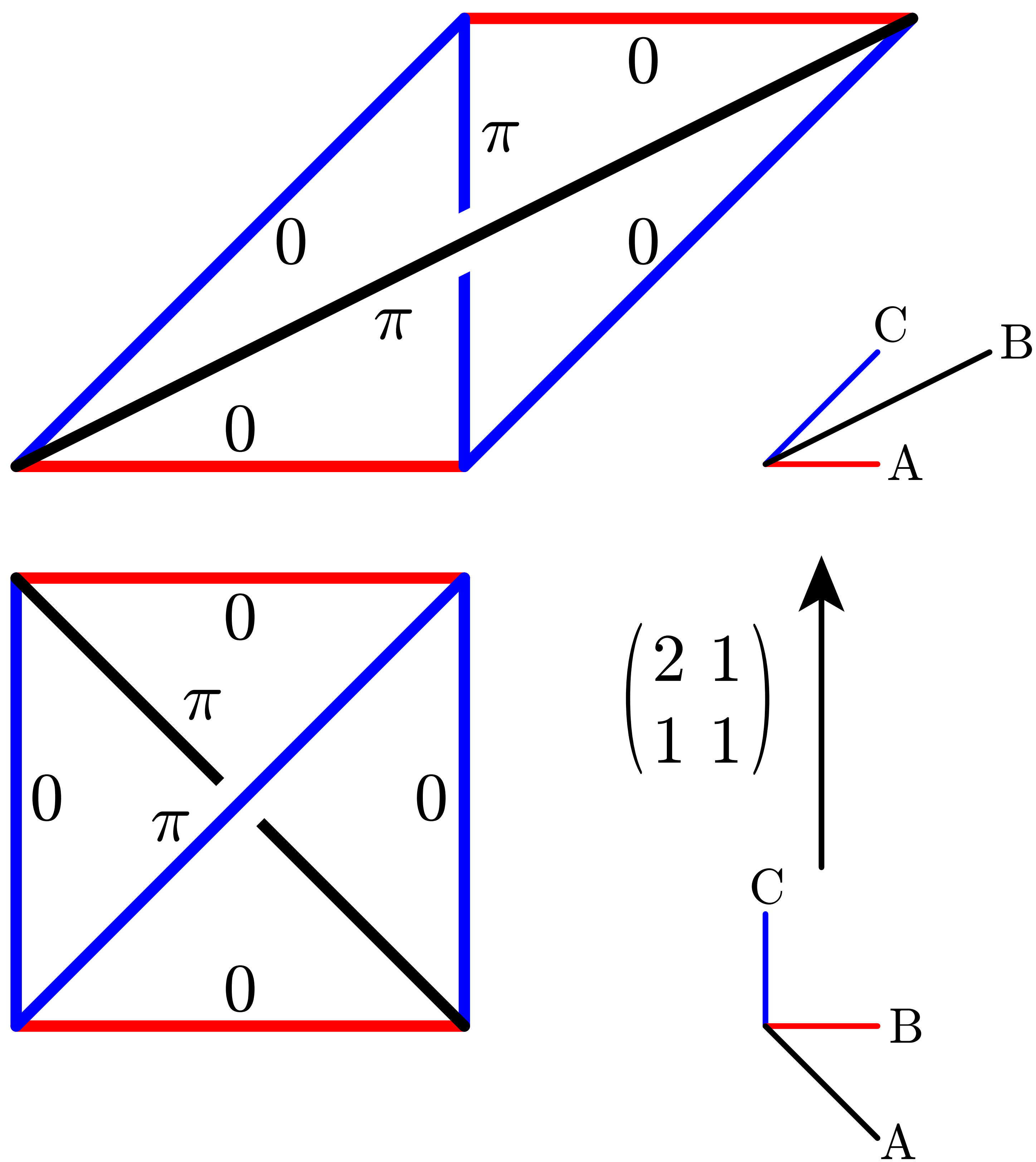

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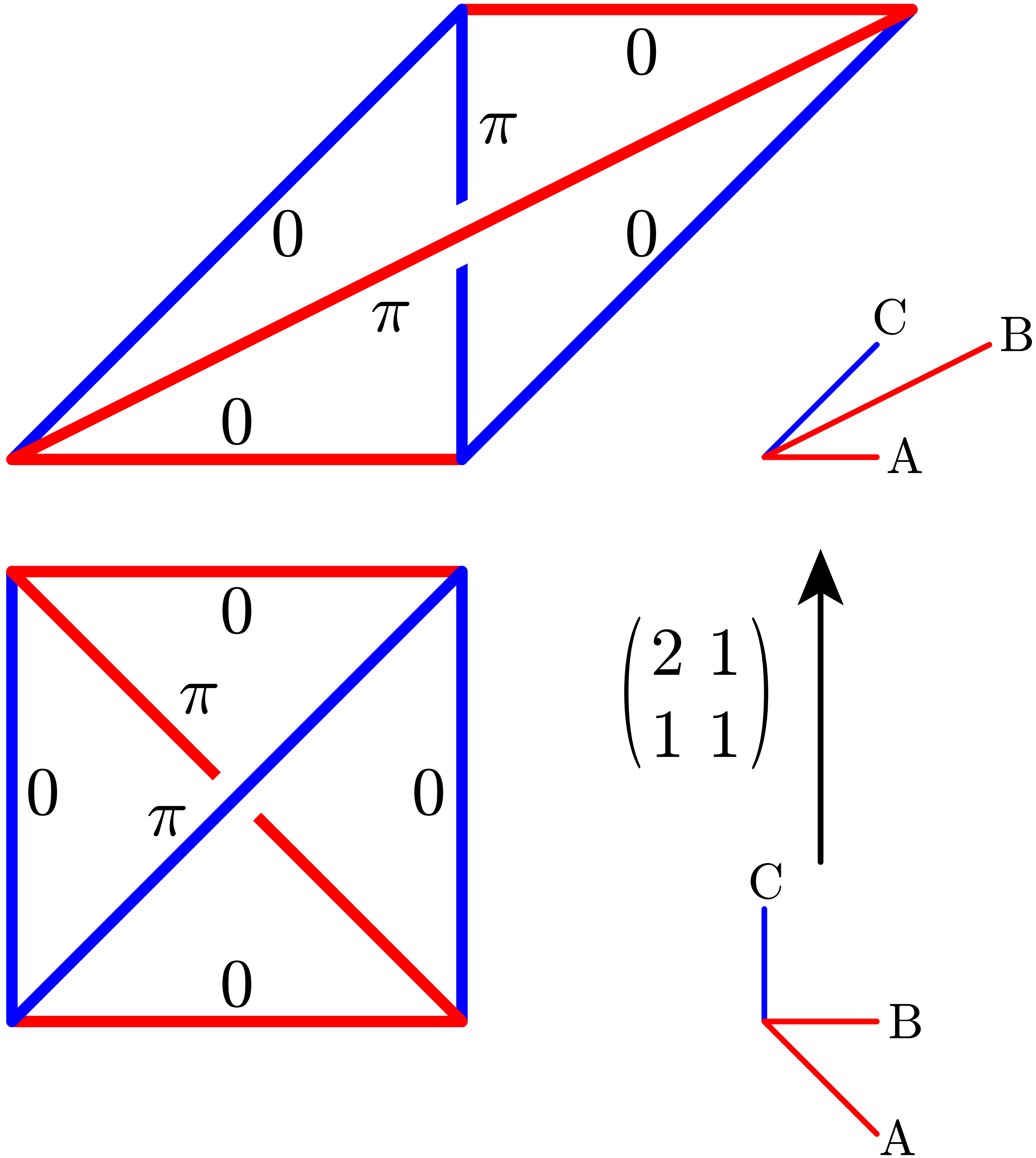

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

(Agol 2010) Let M be a surface bundle with pseudo-Anosov monodromy φ. The result of drilling out singular orbits admits a veering triangulation canonically associated to φ.

$$\begin{array}{c}\text{pseudo-Anosov} \\ \text{surface bundle}\end{array} \xrightarrow[\exists\ \text{canonical}]{\text{(Agol)}} \begin{array}{c}\text{triangulation} \\ \text{w/ veering} \\ \text{structure}\end{array}$$

(Hodgson-Rubinstein-S-Tillmann 2011) Triangulations with veering structures admit strict angle structures. (Also found non-fibered examples by computer search.)

(Futer-Guéritaud 2013) Give explicit strict angle structures for triangulations admitting veering structures.

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$$\begin{tikzcd}[column sep=huge]
\text{pseudo-Anosov surface bundle} \arrow[r, "\text{(Agol)}", "\exists\text{ canonical}"'] & \text{triangulation w/ veering structure} \arrow[r, "\text{(Hodgson-Rubinstein-S-Tillmann)}", "\text{(Futer-Guéritaud)}"'] & \text{strict angle structure} \arrow[d, "\text{(Casson, Rivin)}"] \\
& & \text{manifold is hyperbolic}
\end{tikzcd}$$

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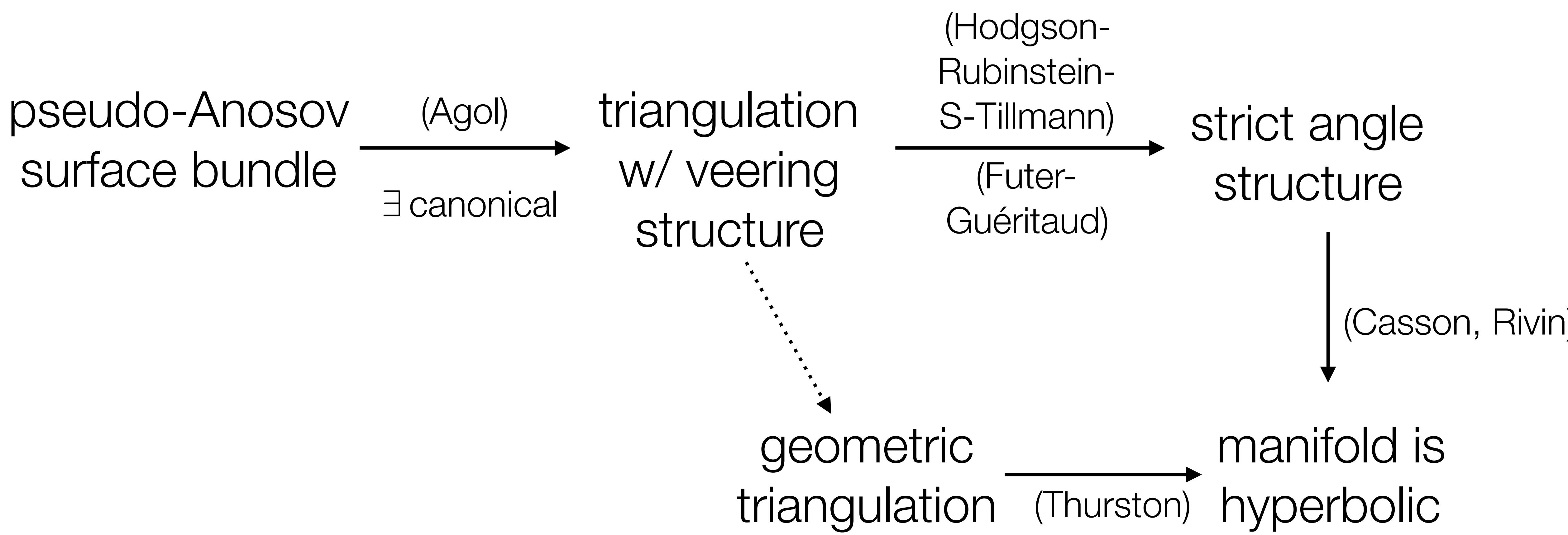
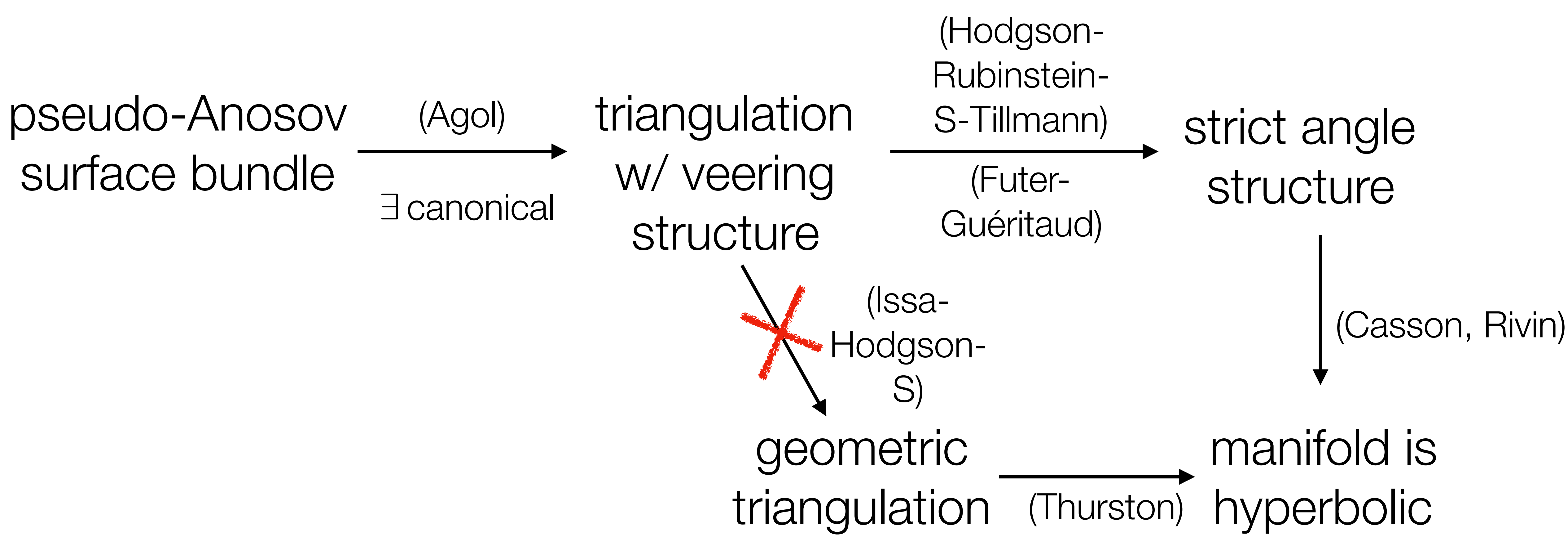
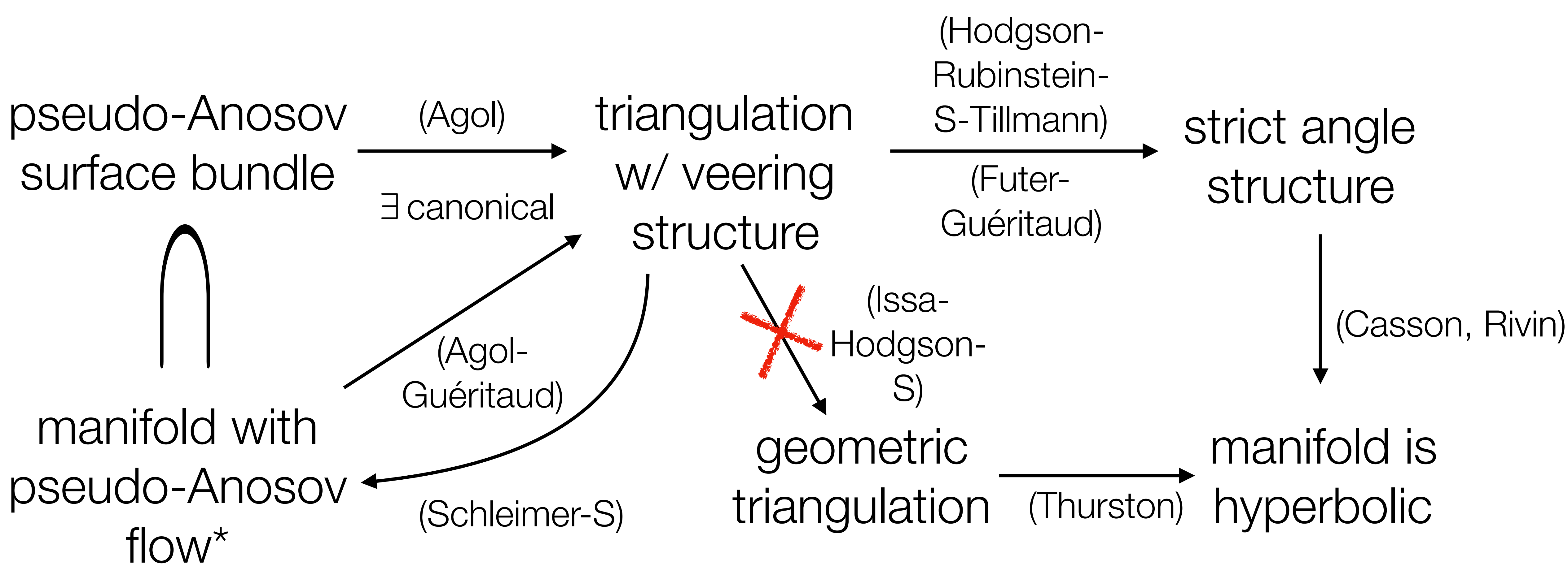


(Issa-Hodgson-S 2016) There are non-geometric triangulations with veering structures.



(Agol-Guéritaud, unpublished) Extend the construction to manifolds with pseudo-Anosov flows.

(Schleimer-S, work in progress) Prove the converse.



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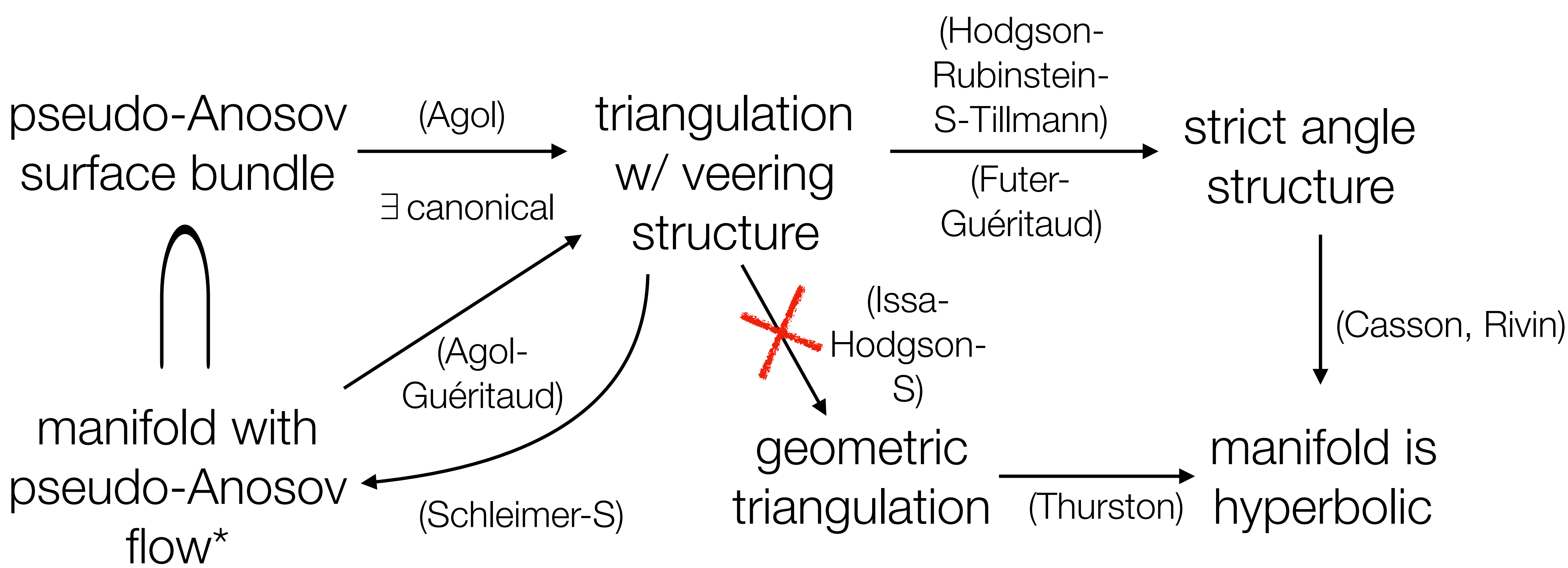

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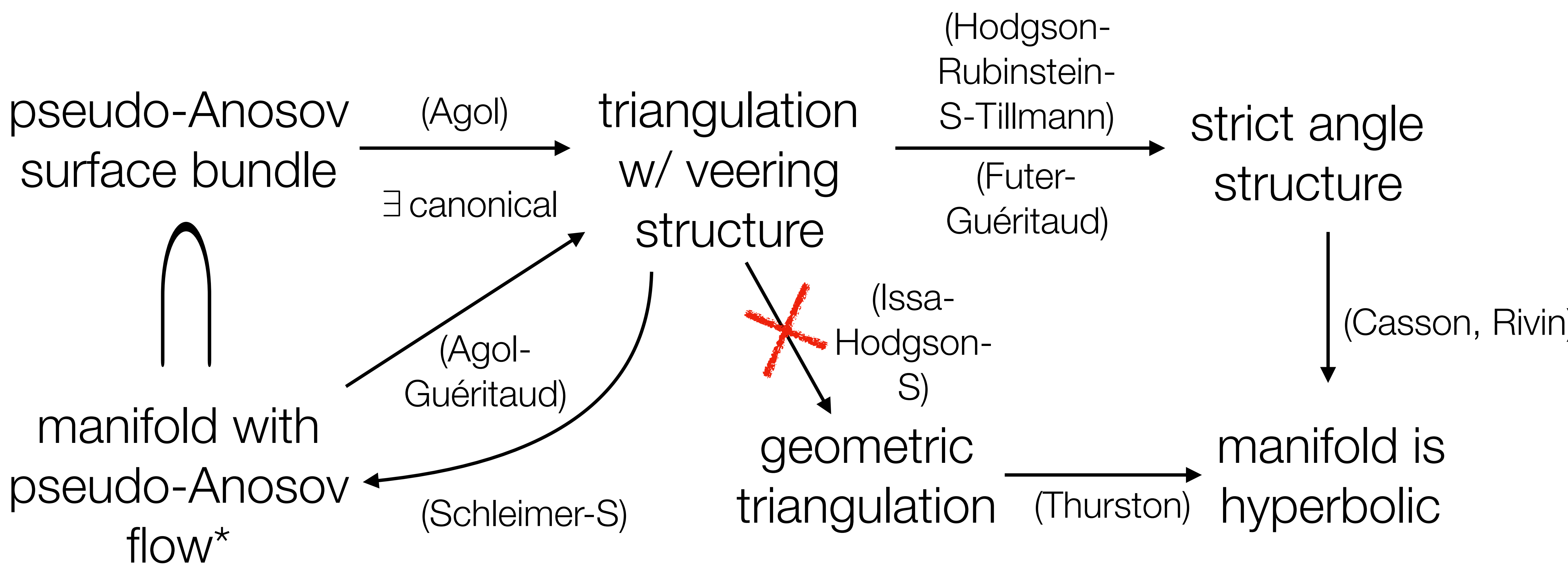

(Other work on veering structures by Landry, Minksy, Sakata, Taylor, Worden…)

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How common are veering structures?

Of the 4,815 orientable triangulations in the `SnapPea` census (up to 7 tetrahedra):

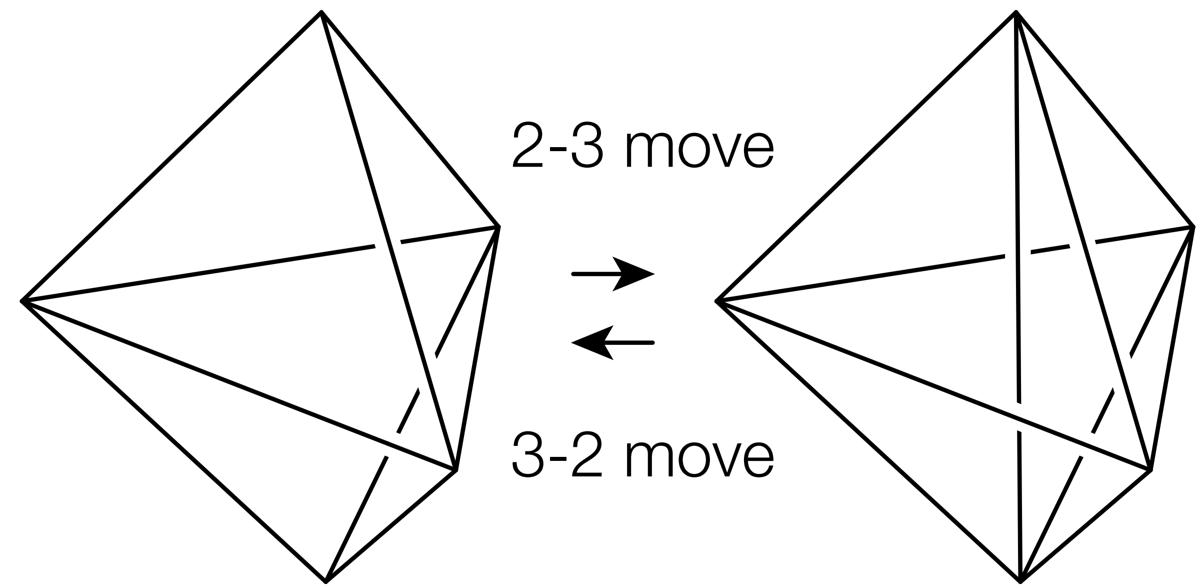
All are geometric so all have strict angle structures
There are 13,599 taut angle structures
There are 158 veering structures (on 151 triangulations)

So on this non-random sample, approx 1.1% of triangulations admit veering structures.

How common are veering structures?

Another way to sample triangulations: explore the *Pachner graph* of triangulations of a manifold.

(Matveev (1987), Piergallini (1988)) The Pachner graph is connected under 2-3 and 3-2 moves.



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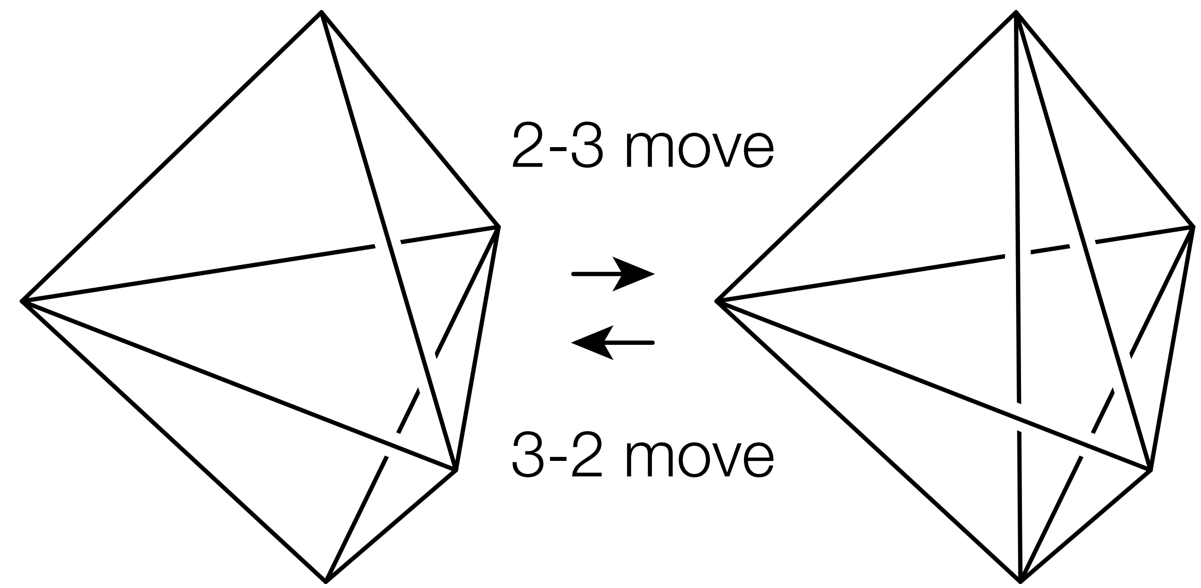

In the "ceiling 10" subgraph of the Pachner graph for the figure 8 knot complement:

triangulations	19,470,660	100%
admit a taut angle structure		
admit a strict angle structure		
admit a veering structure		

How common are veering structures?

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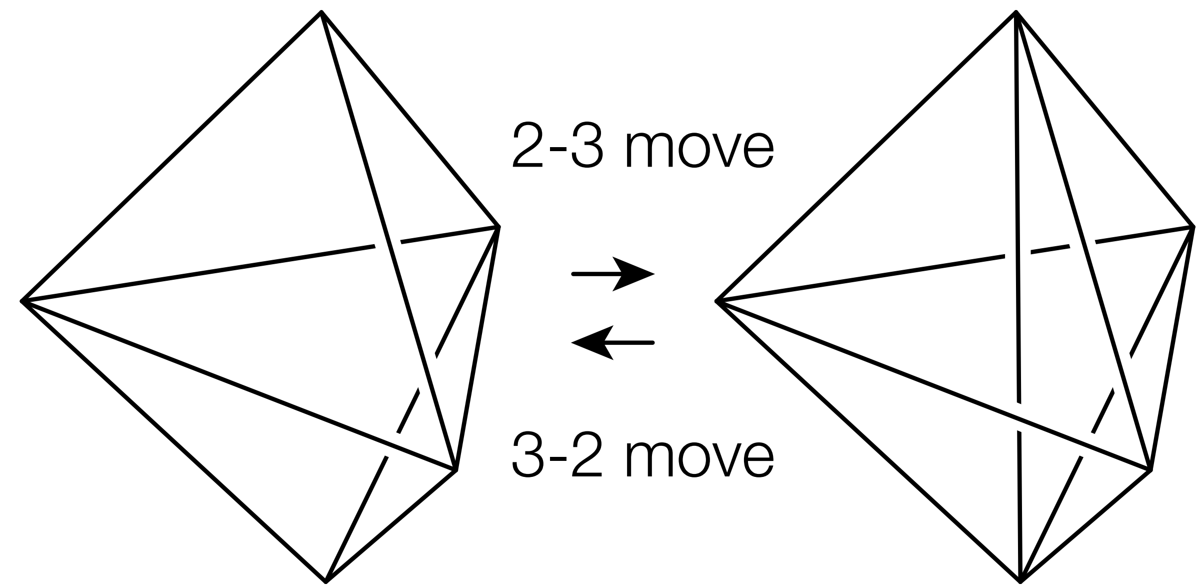

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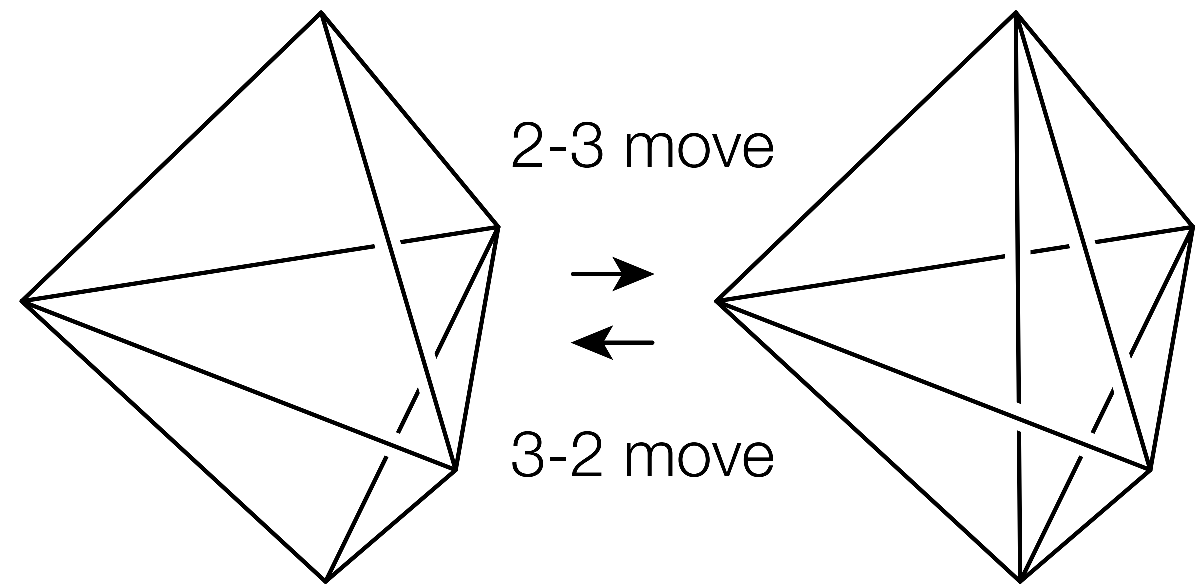

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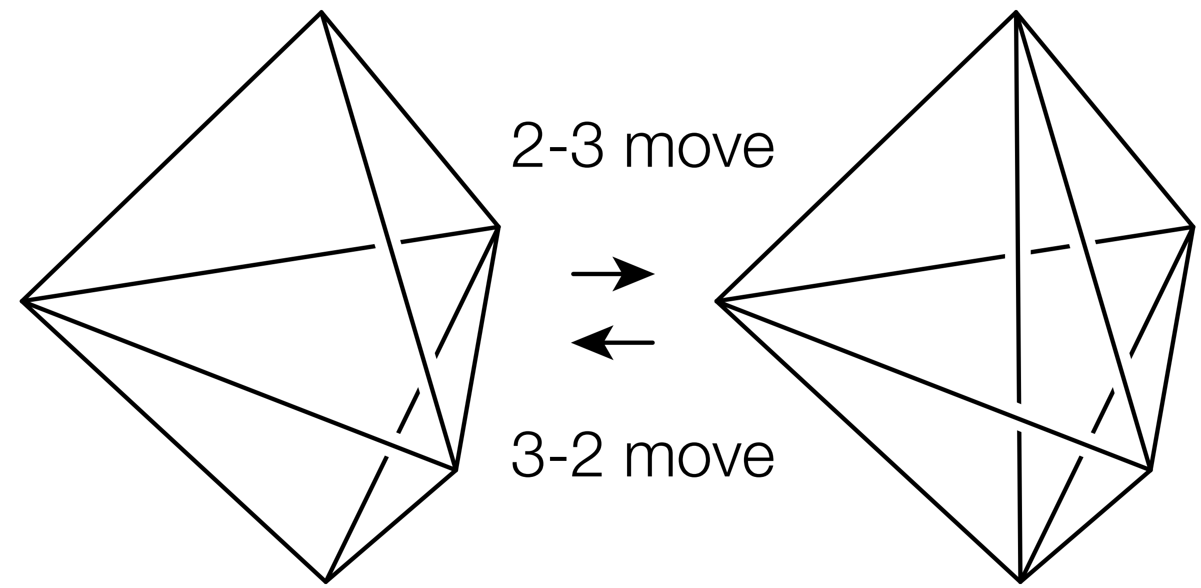

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admit a strict angle structure	2,621	0.013%
admit a veering structure	1	0.0000051%

Conjecture:
Each manifold admits a finite number of veering structures.

Another approach to finding veering structures

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Generate *all* (transverse) veering triangulations with up to n tetrahedra directly. (Work with Masters student Andreas Giannopolous.)

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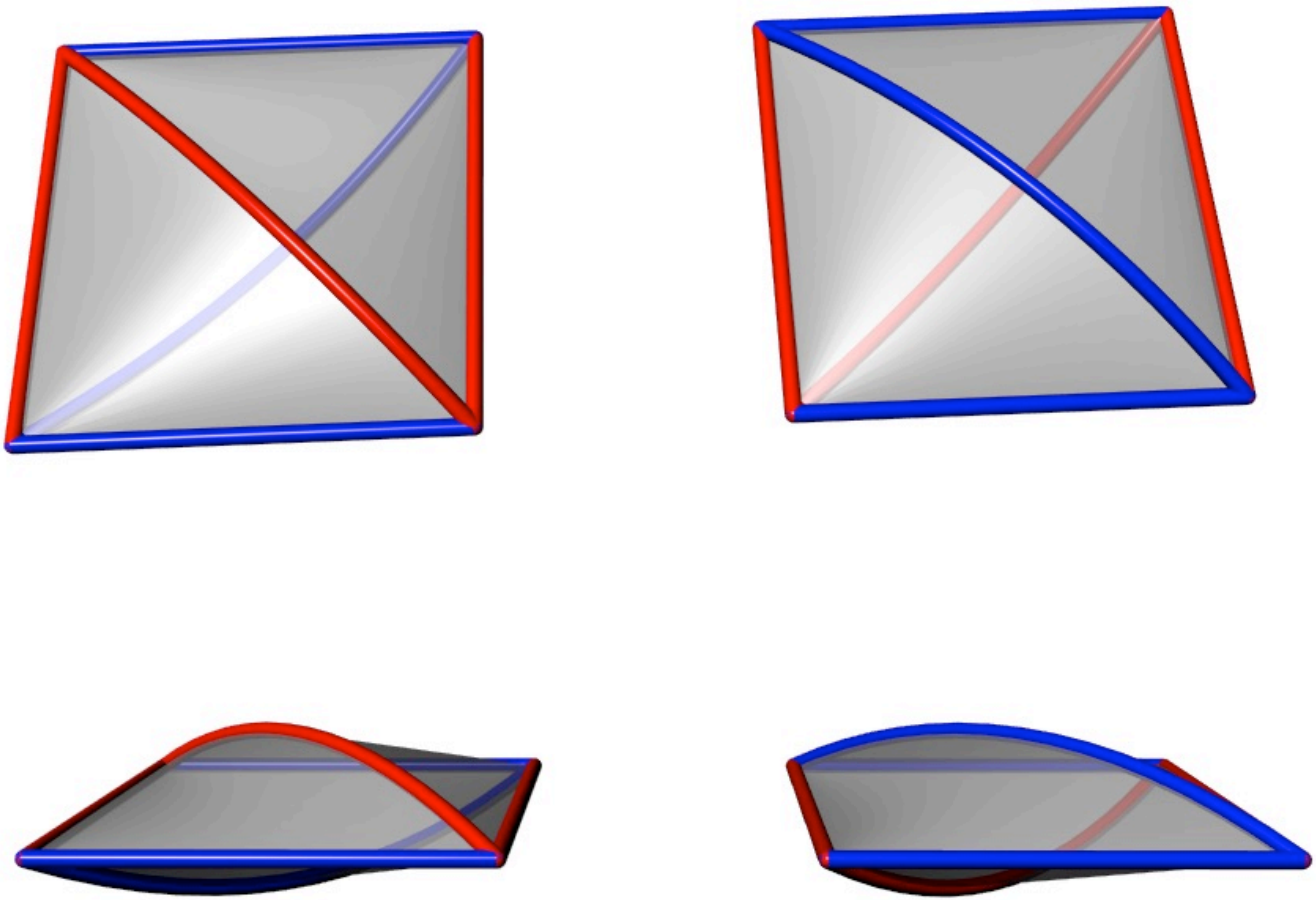
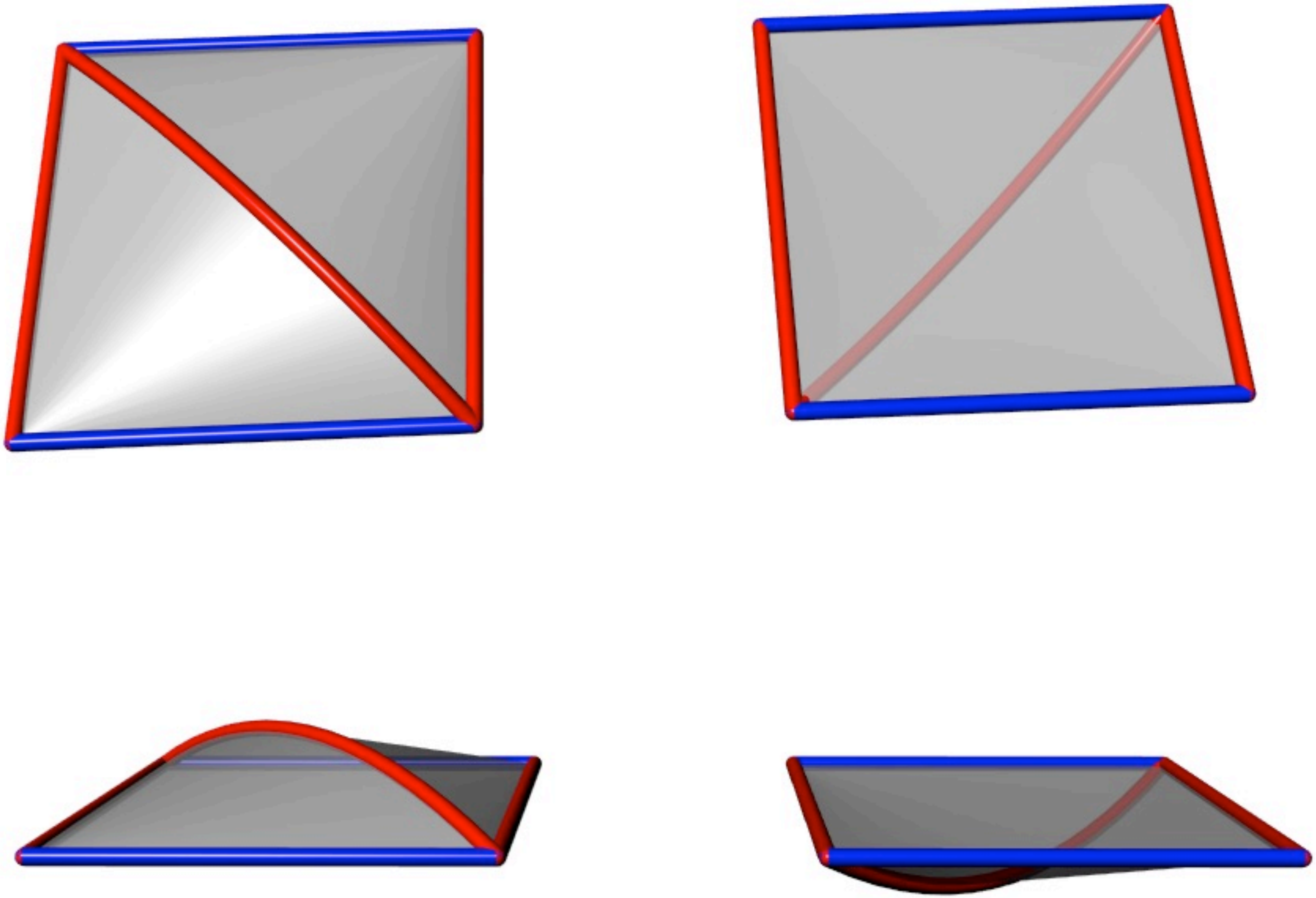
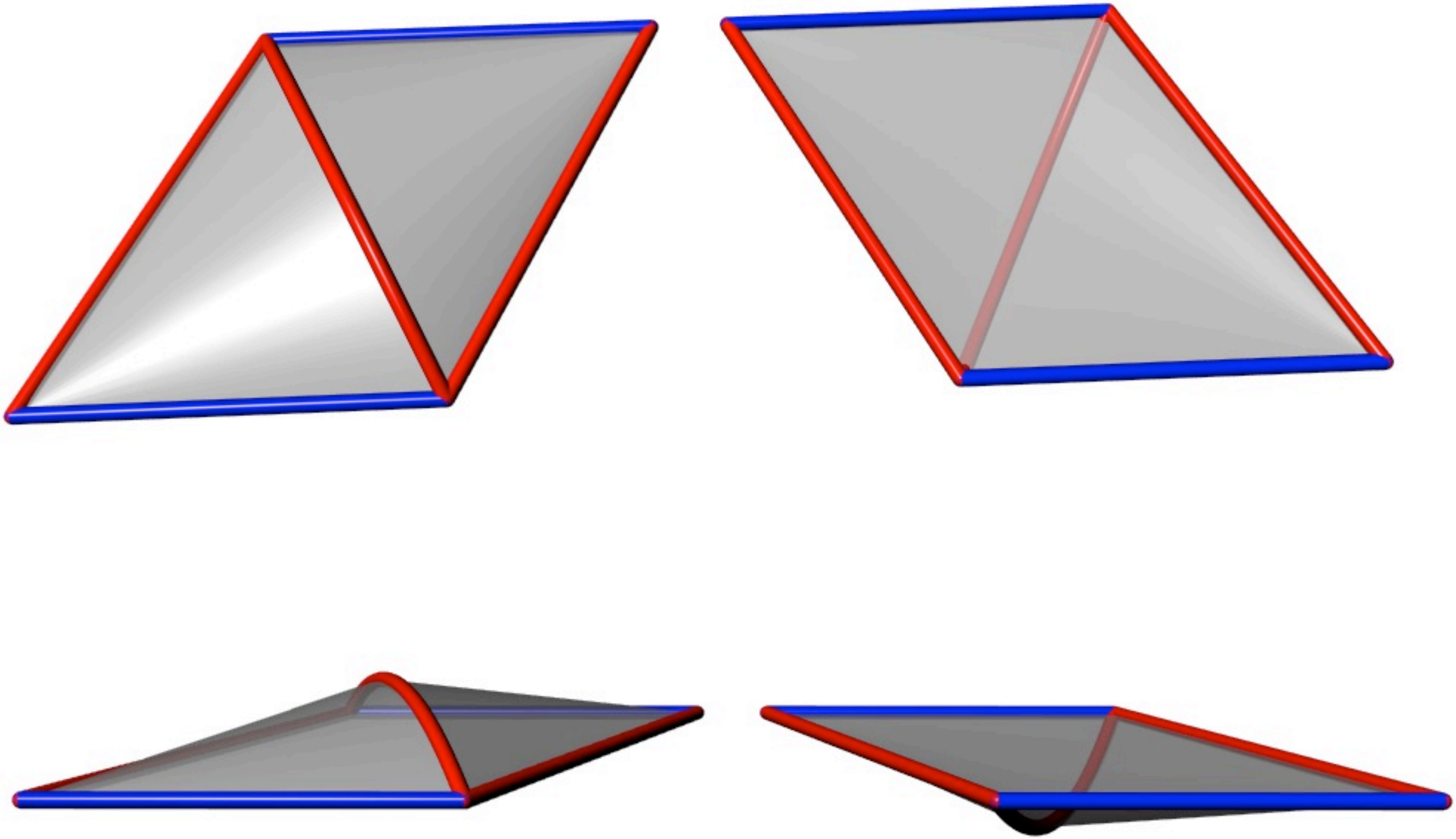
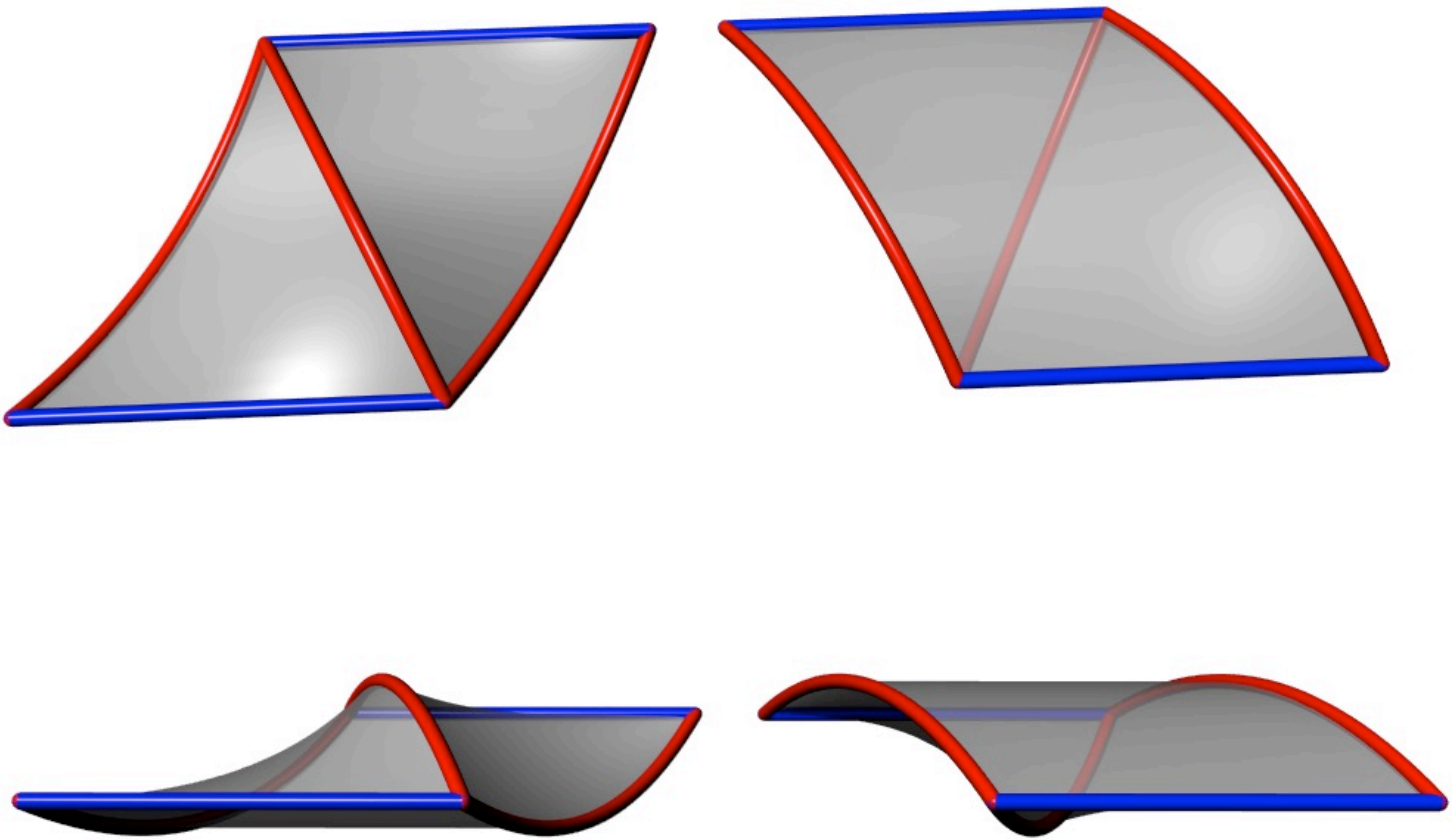
We use the following result to reduce the search space:

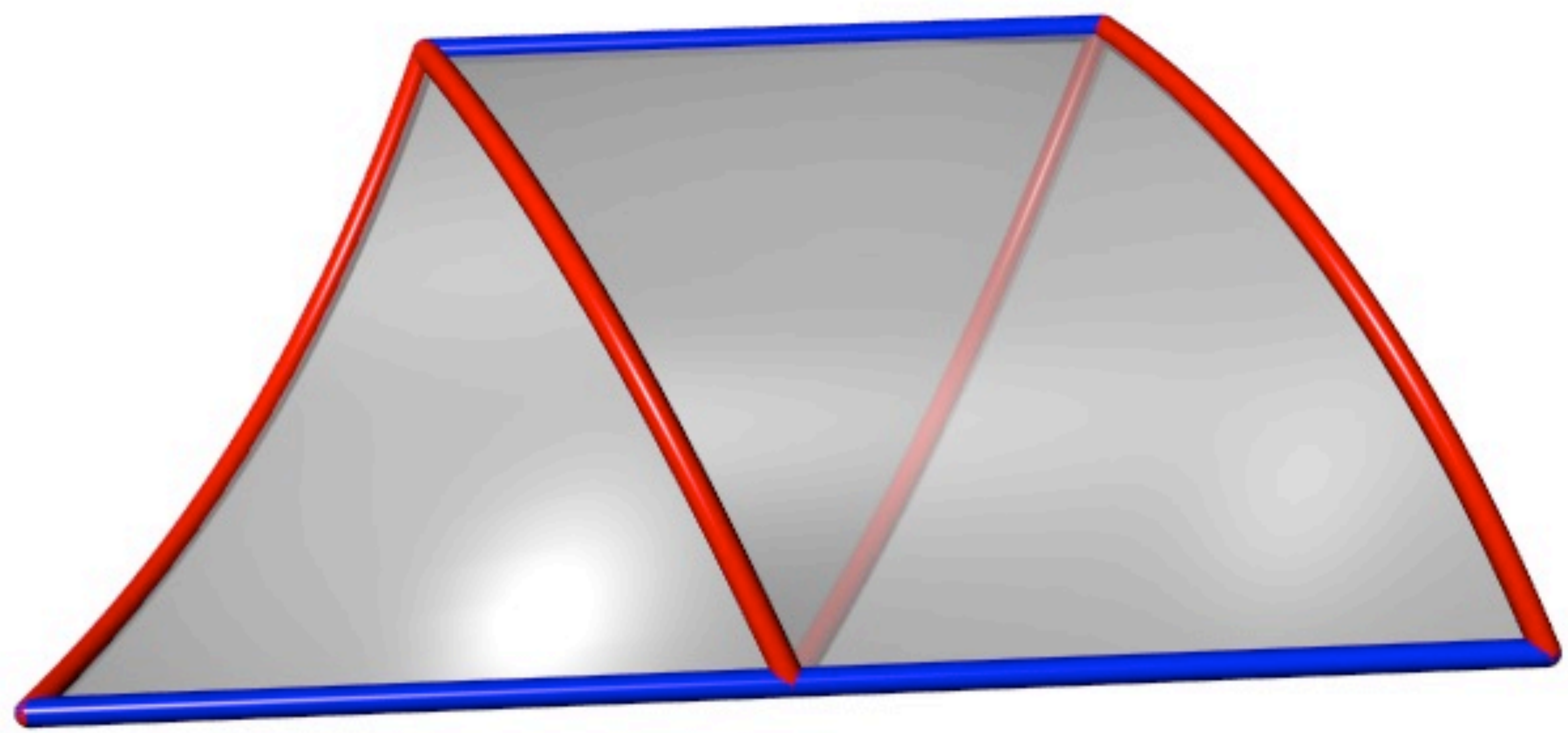
Theorem (Schleimer, S): A manifold with a veering triangulation admits a canonical decomposition into *veering ideal solid tori*.

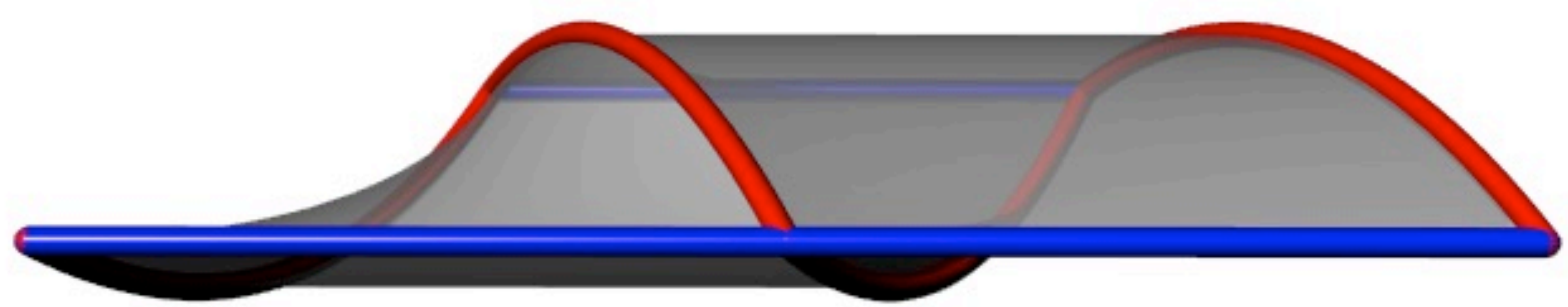


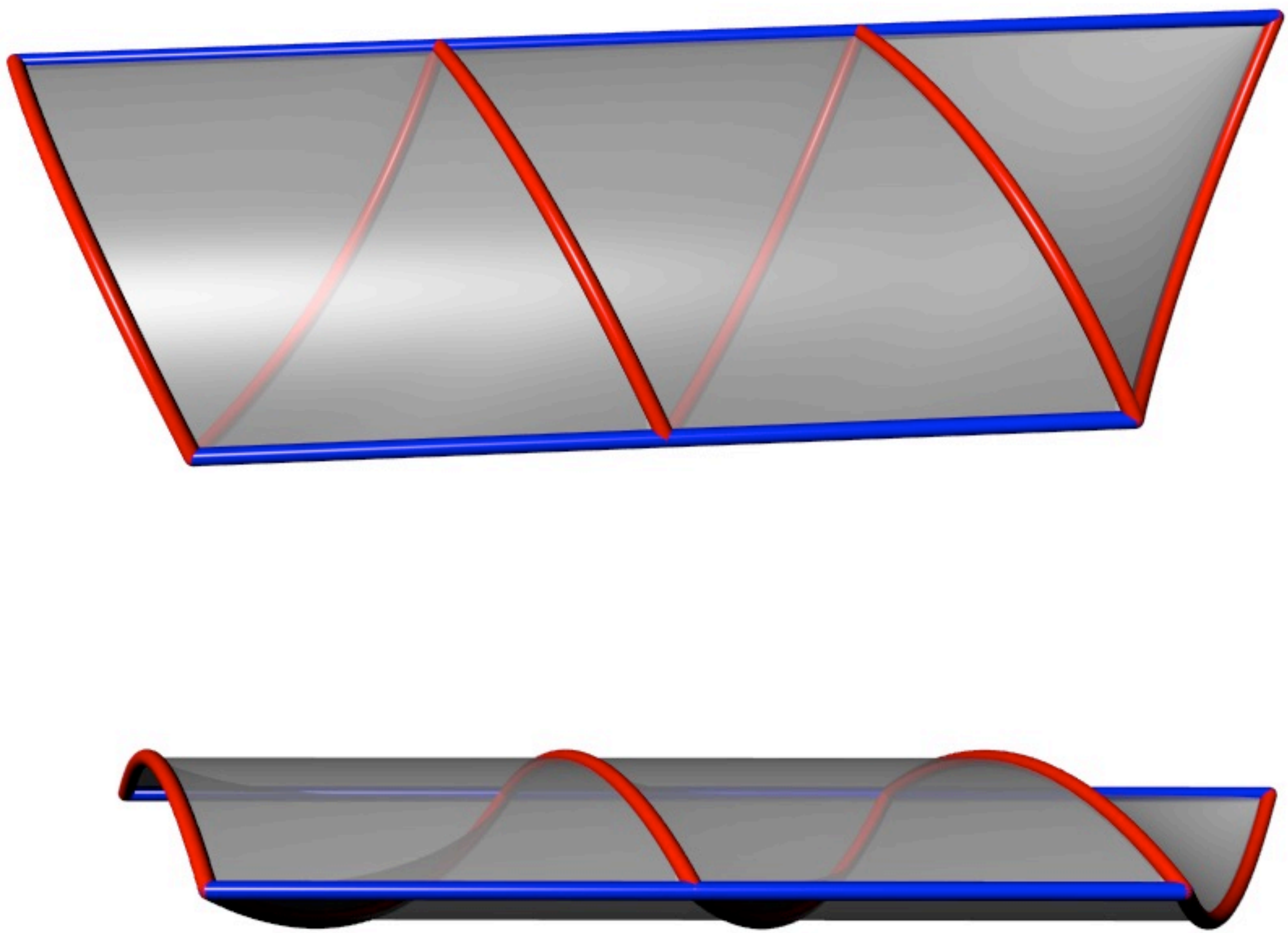




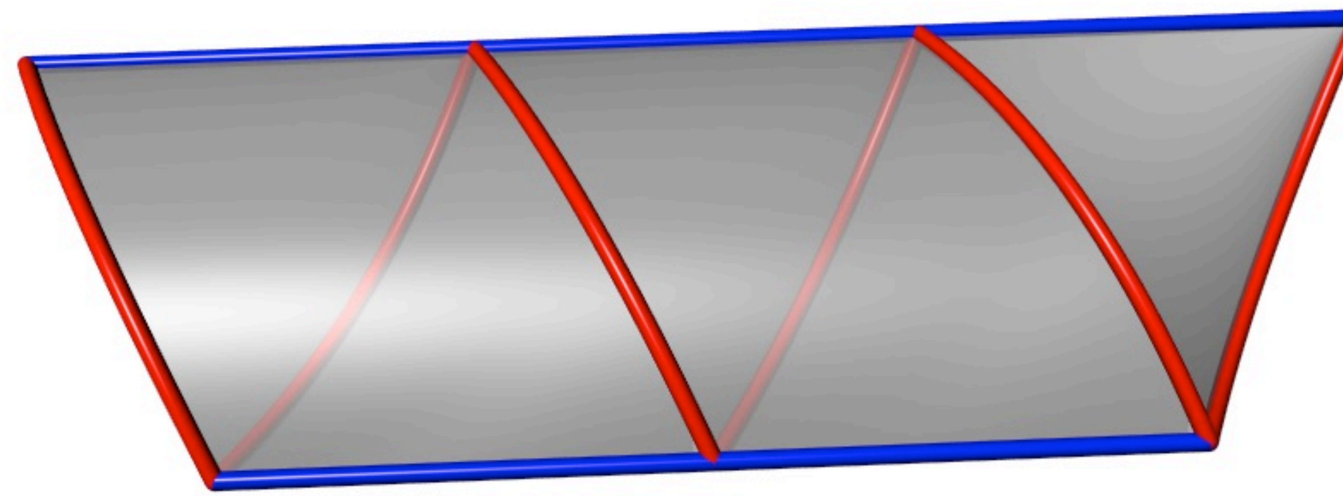




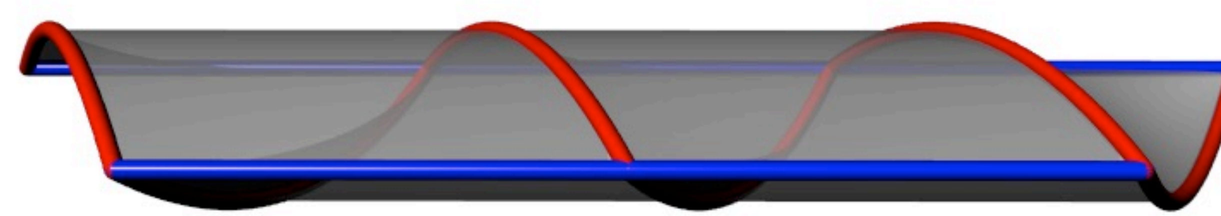




Solid tori glue to each other along rhombuses on their boundaries, matching edge colours.



To build our census of transverse veering structures, we try all such gluings.

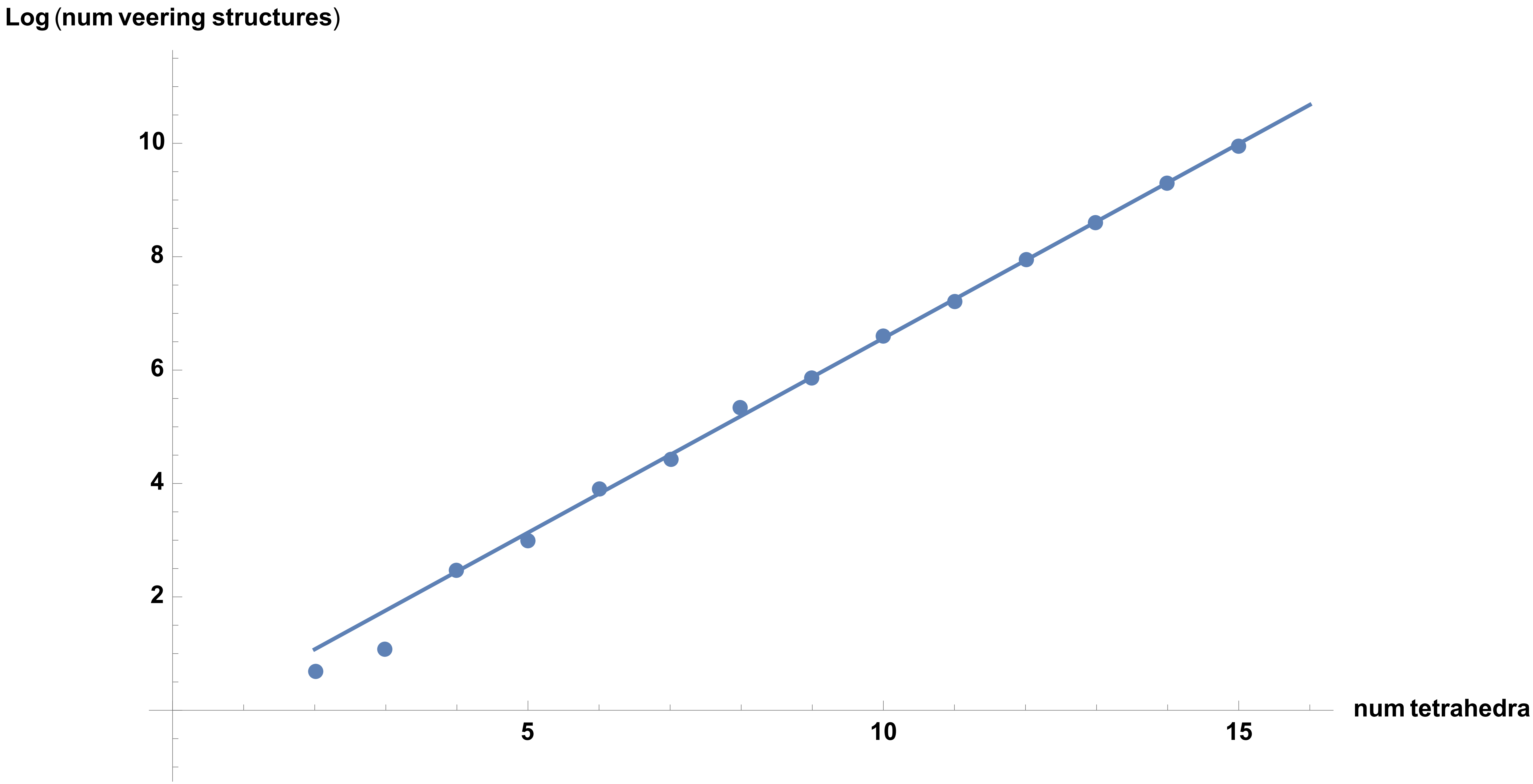


We get a transverse veering structure if the total angle at each edge is 2π.

Transverse taut veering structures

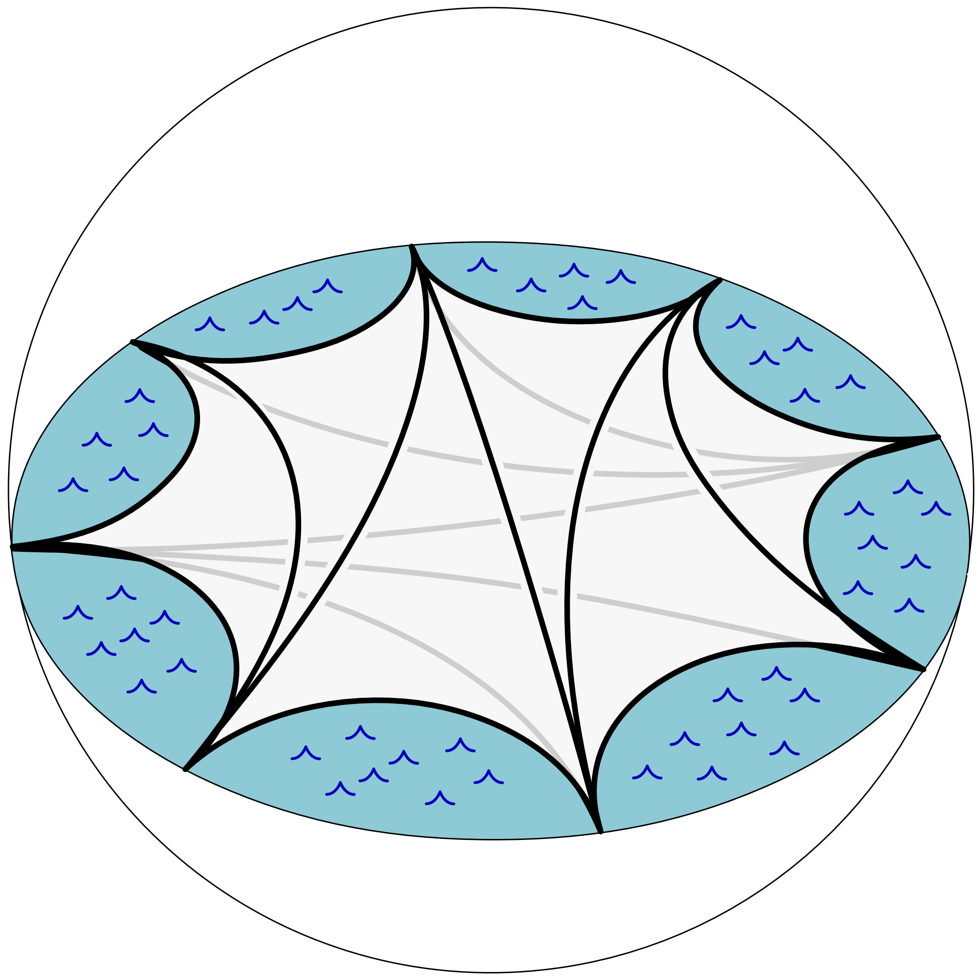
#tetrahedra	#veering structs	#duplicate triangs	#non geom triangs	#non layered structs	fraction non layered
2	2	0	0	0	0.000
3	3	0	0	0	0.000
4	12	0	0	0	0.000
5	20	0	0	4	0.200
6	50	0	0	13	0.260
7	85	1	0	24	0.282
8	205	6	0	61	0.298
9	356	2	1	120	0.337
10	750	10	3	255	0.340
11	1358	2	9	492	0.362
12	2871	12	22	1035	0.361
13	5332	10	52	2075	0.389
14	10986	35	110	4269	0.389
15	21290	32	234	8788	0.413

Transverse taut veering structures



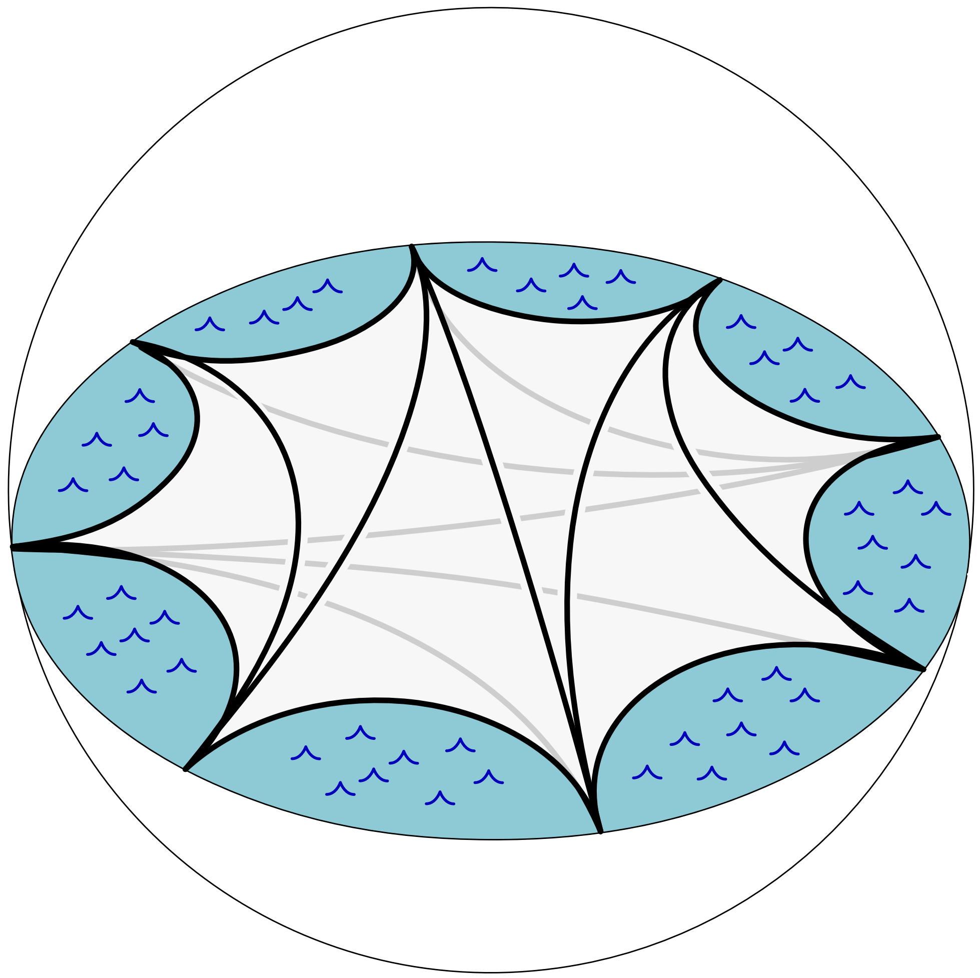
The number of veering structures approximately doubles every time we increase the number of tetrahedra by one.

Layers and continents in the universal cover



Taut ideal tetrahedra layer to make larger taut ideal polyhedra: *continents*.

Layers and continents in the universal cover



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This gives a circular order to the vertices of the tetrahedra.

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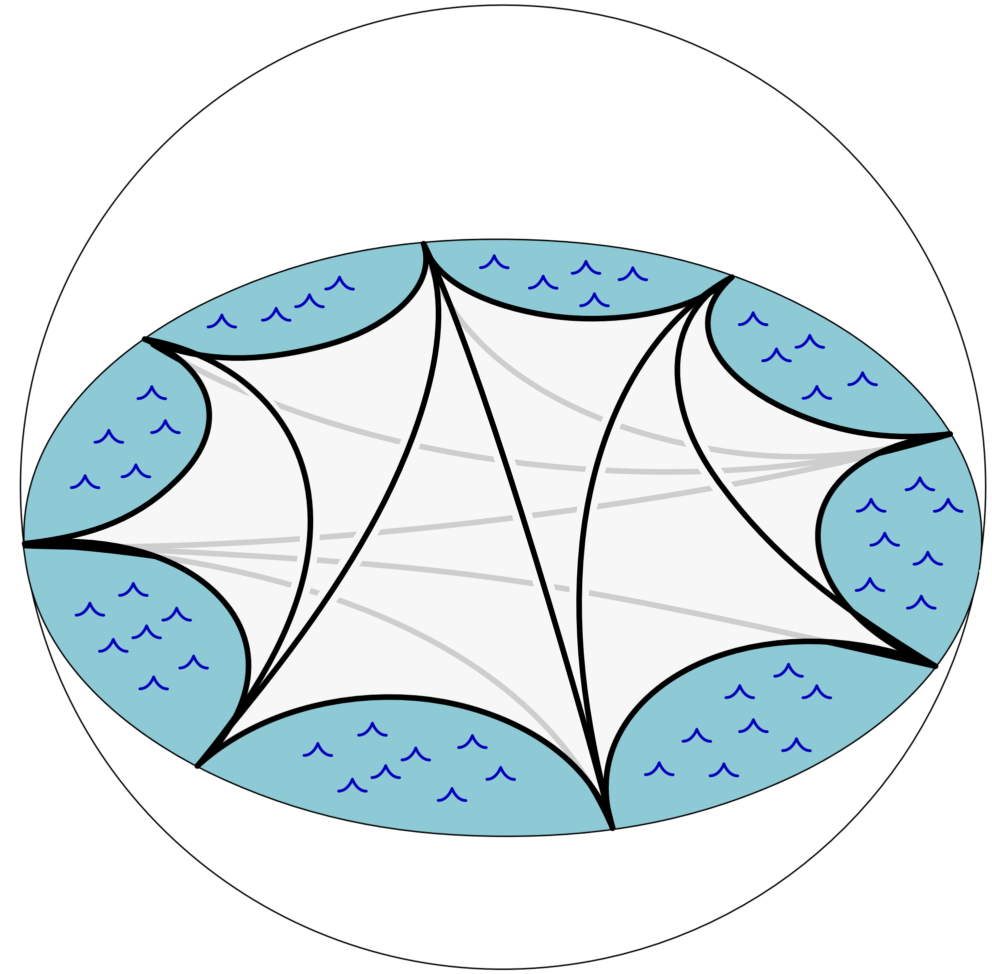

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Theorem (Schleimer, S): A veering triangulation admits a *unique* circular order on the vertices of the universal cover.

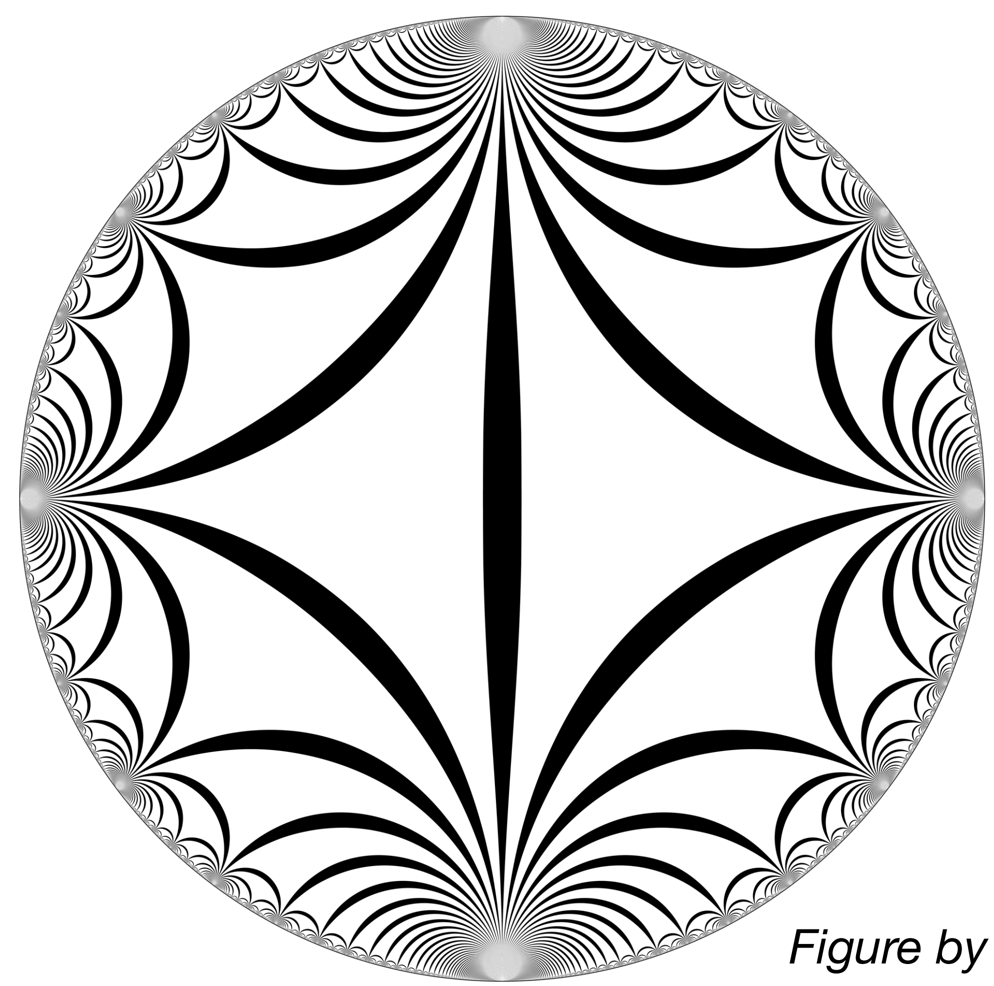

Figure by Roice Nelson

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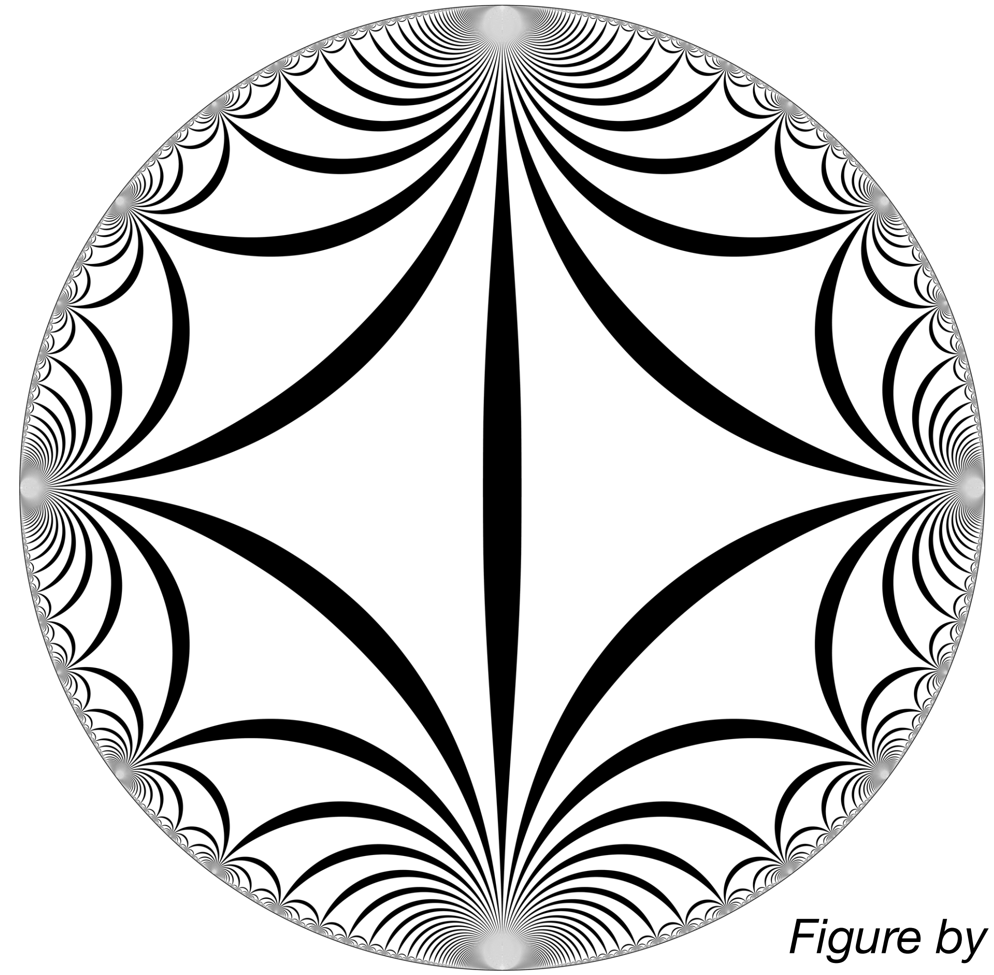
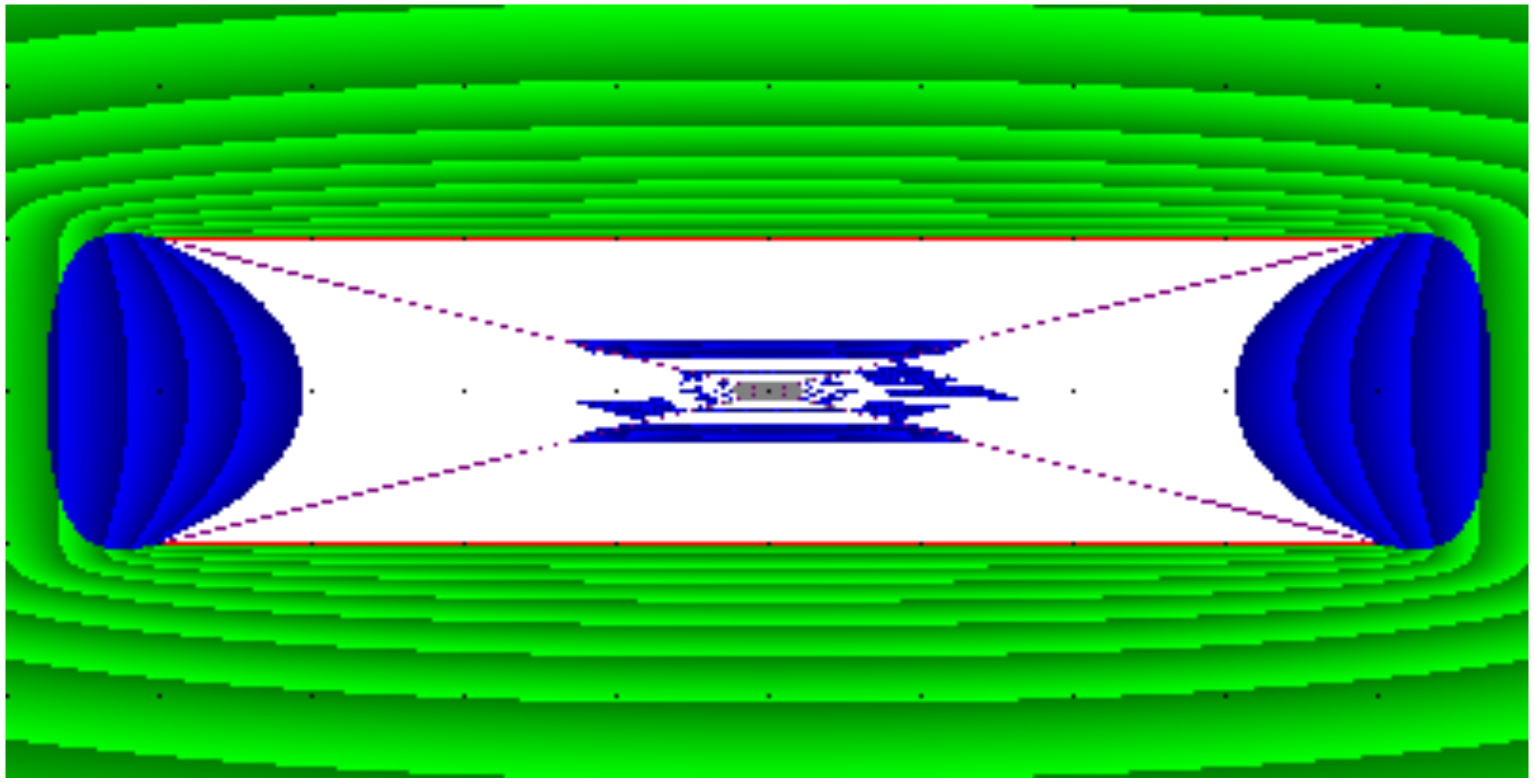


Figure by Roice Nelson

Example There are two taut angle structures on the canonical triangulation of the figure 8 knot complement that admit *uncountably many* circular orders on the vertices of the universal cover.



Picture of Dehn surgery space (generated with `SnapPy`)

Dynamic pairs and topological pseudo-Anosov flows

veering structure $\xrightarrow[\text{(Schleimer, S)}]{\textbf{canonical}}$ dynamic pair

Dynamic pairs and topological pseudo-Anosov flows

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filling slope (avoiding finite number of lines)

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Dynamic pairs and topological pseudo-Anosov flows

veering structure $\xrightarrow[\text{(Schleimer, S)}]{\textbf{canonical}}$ dynamic pair

+

filling slope (avoiding finite number of lines) $\xrightarrow[\text{(Mosher)}]{}$ topological pseudo-Anosov flow

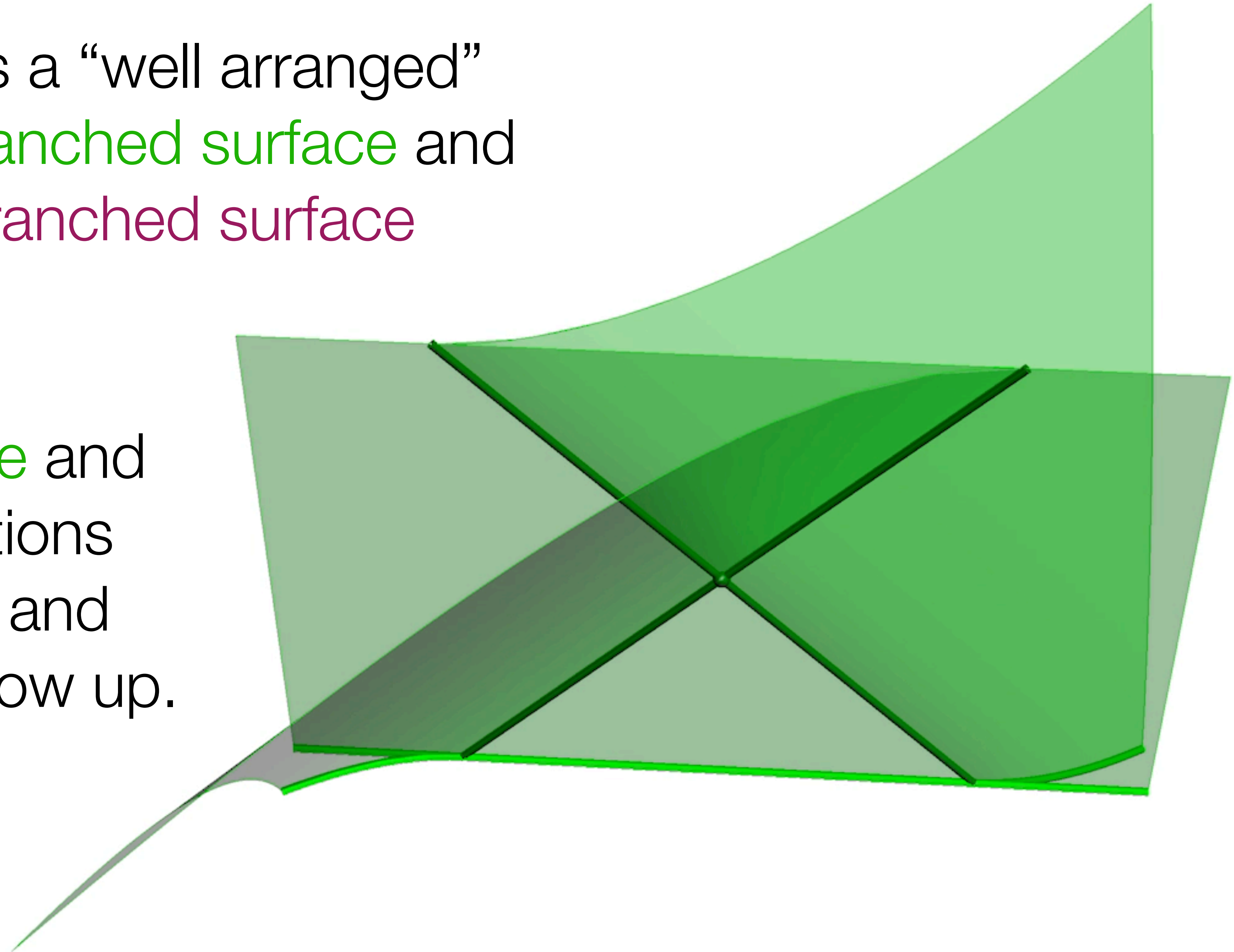
with

+

(avoiding a few more lines) $\xrightarrow[\text{(Schleimer, S)}]{}$ no perfect fits

A dynamic pair is a "well arranged" pair of a stable branched surface and an unstable branched surface

These carry stable and unstable laminations which "expand" and "contract" as we flow up.



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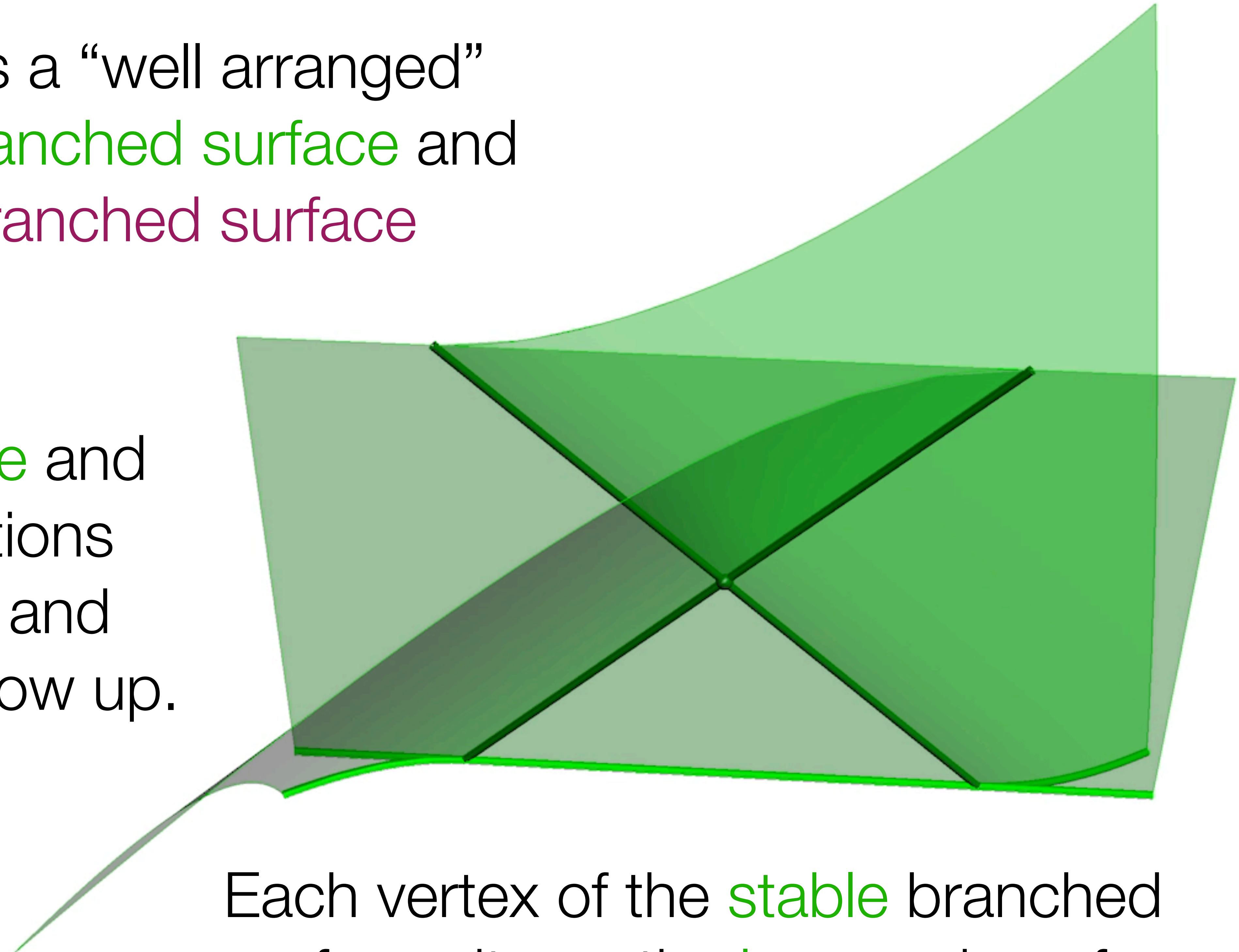

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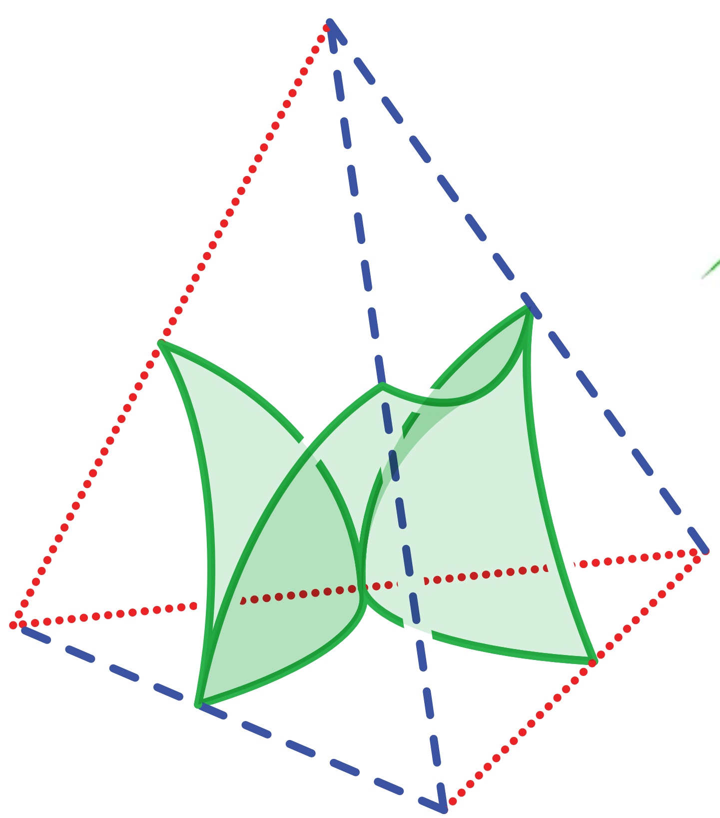
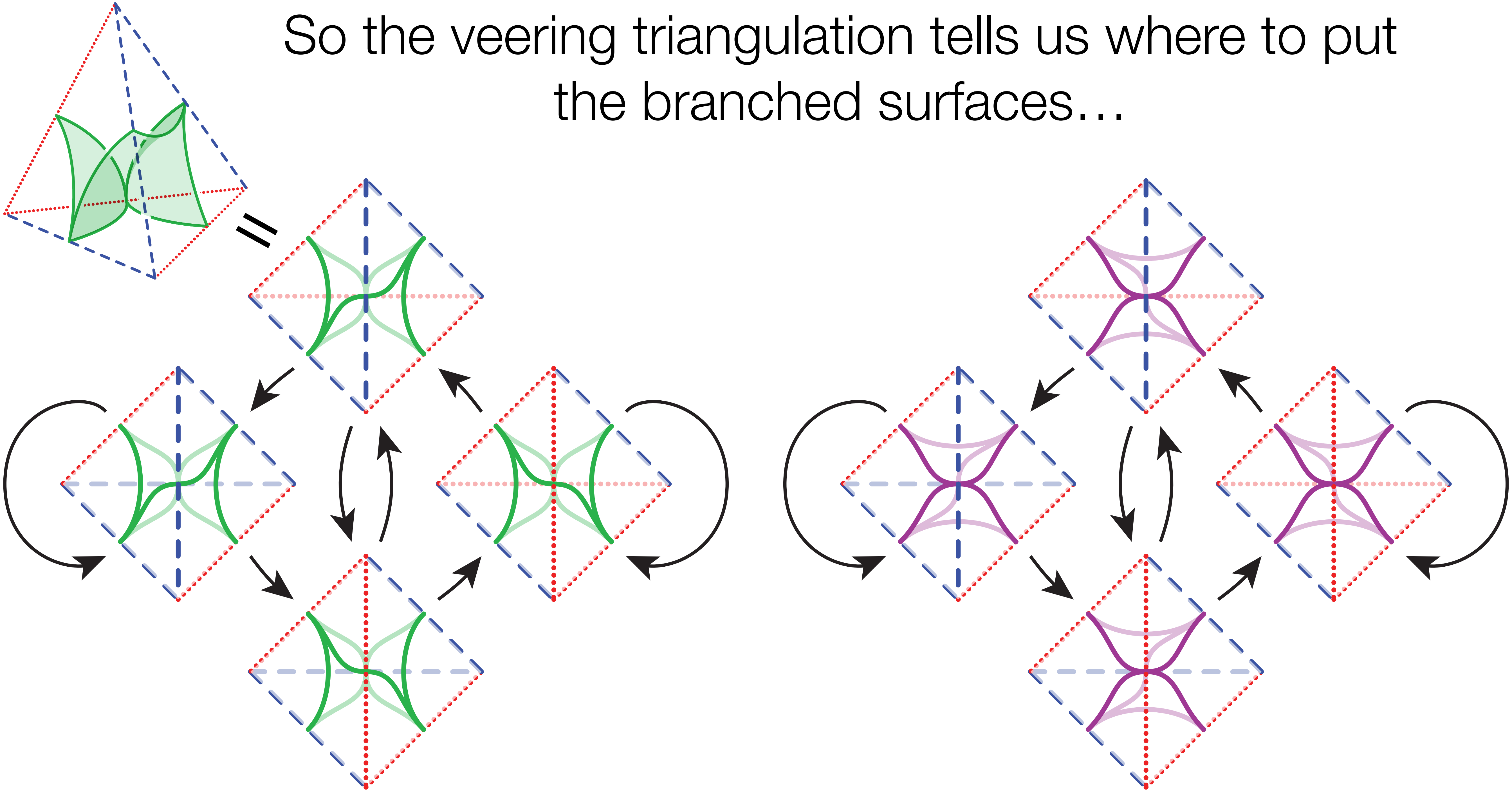


Each vertex of the stable branched surface sits on the lower edge of a veering tetrahedron.

Each vertex of the unstable branched surface sits on the upper edge of a veering tetrahedron.

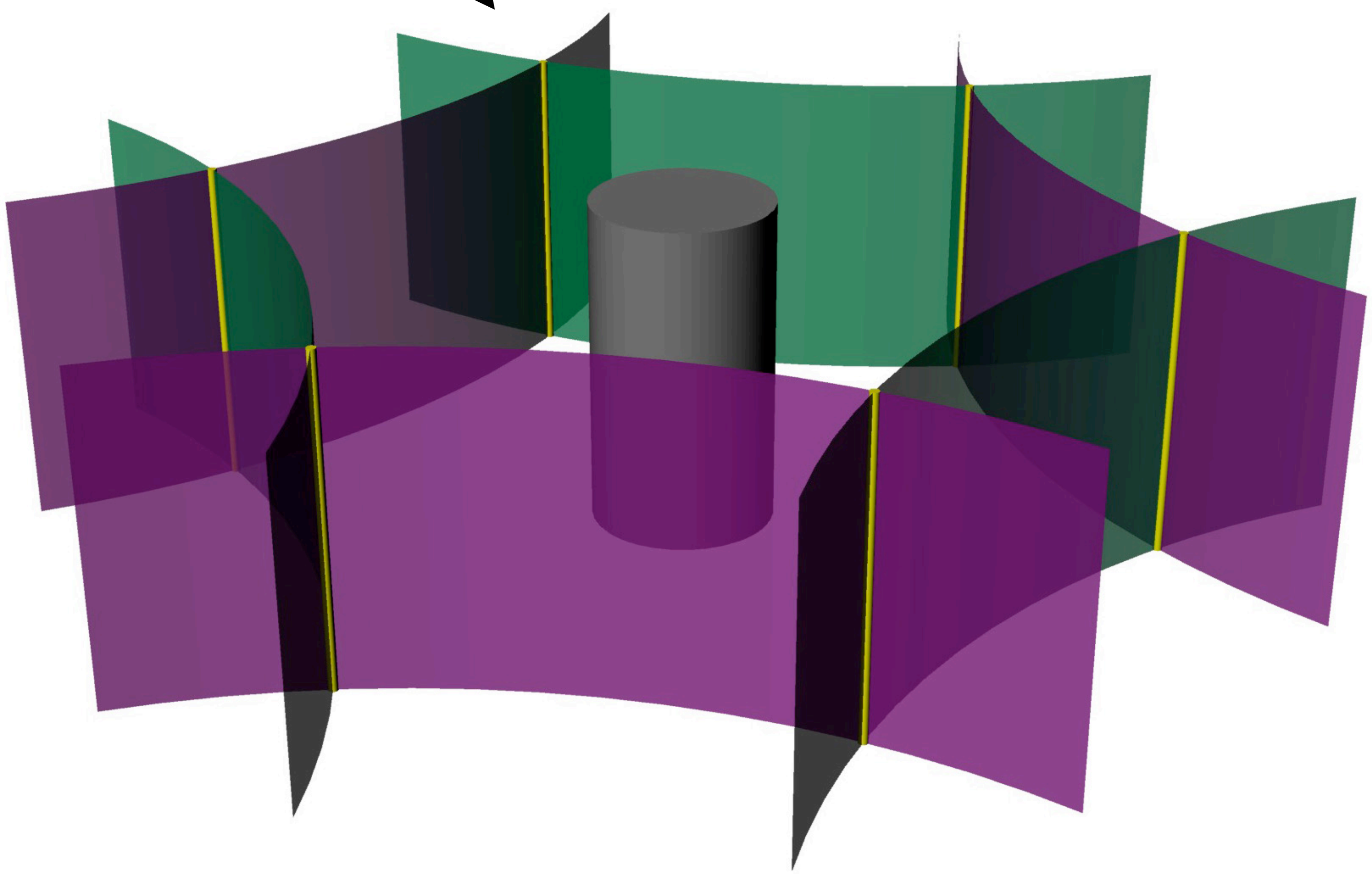
So the veering triangulation tells us where to put the branched surfaces…

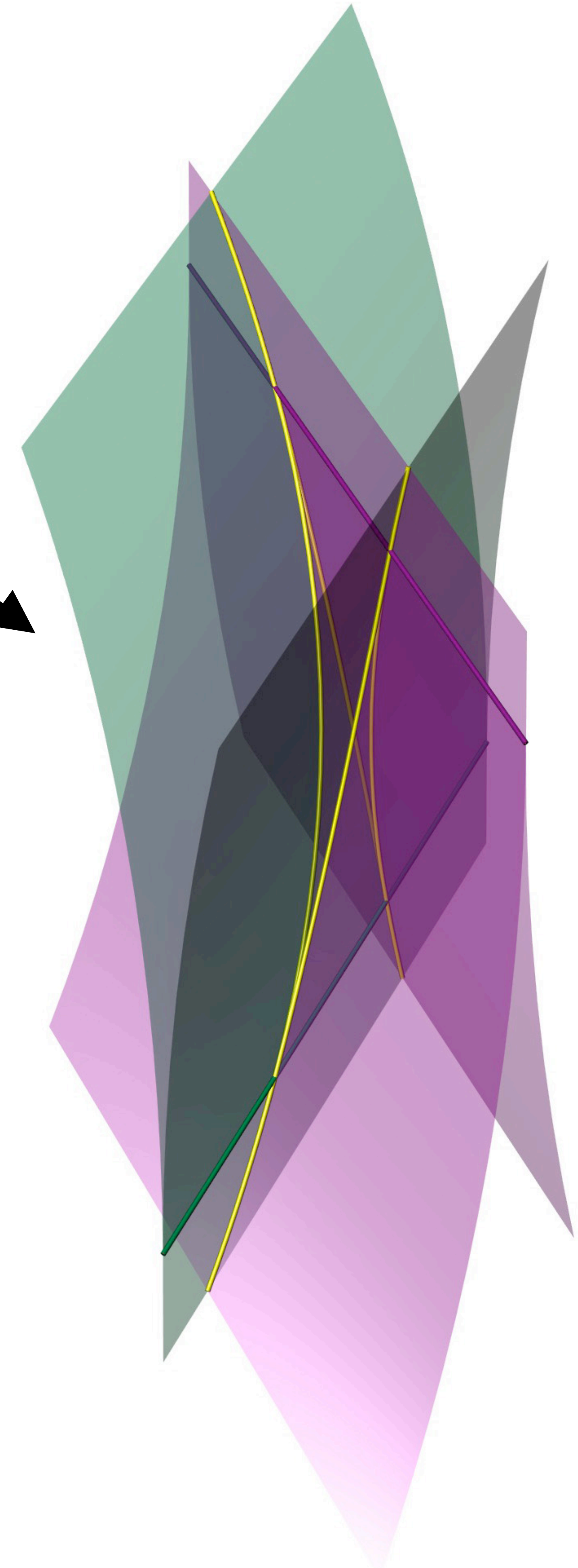


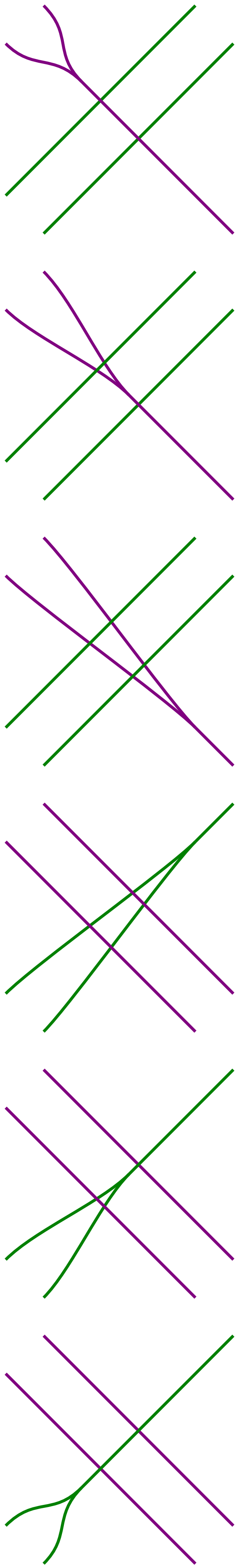
But we need more for a dynamic pair:

1) The branched surfaces should be transverse.
(At the moment they coincide in some tetrahedra!)

2) The complementary components to the stable and unstable branched surfaces should be *dynamic torus shells* and *pinched tetrahedra.*







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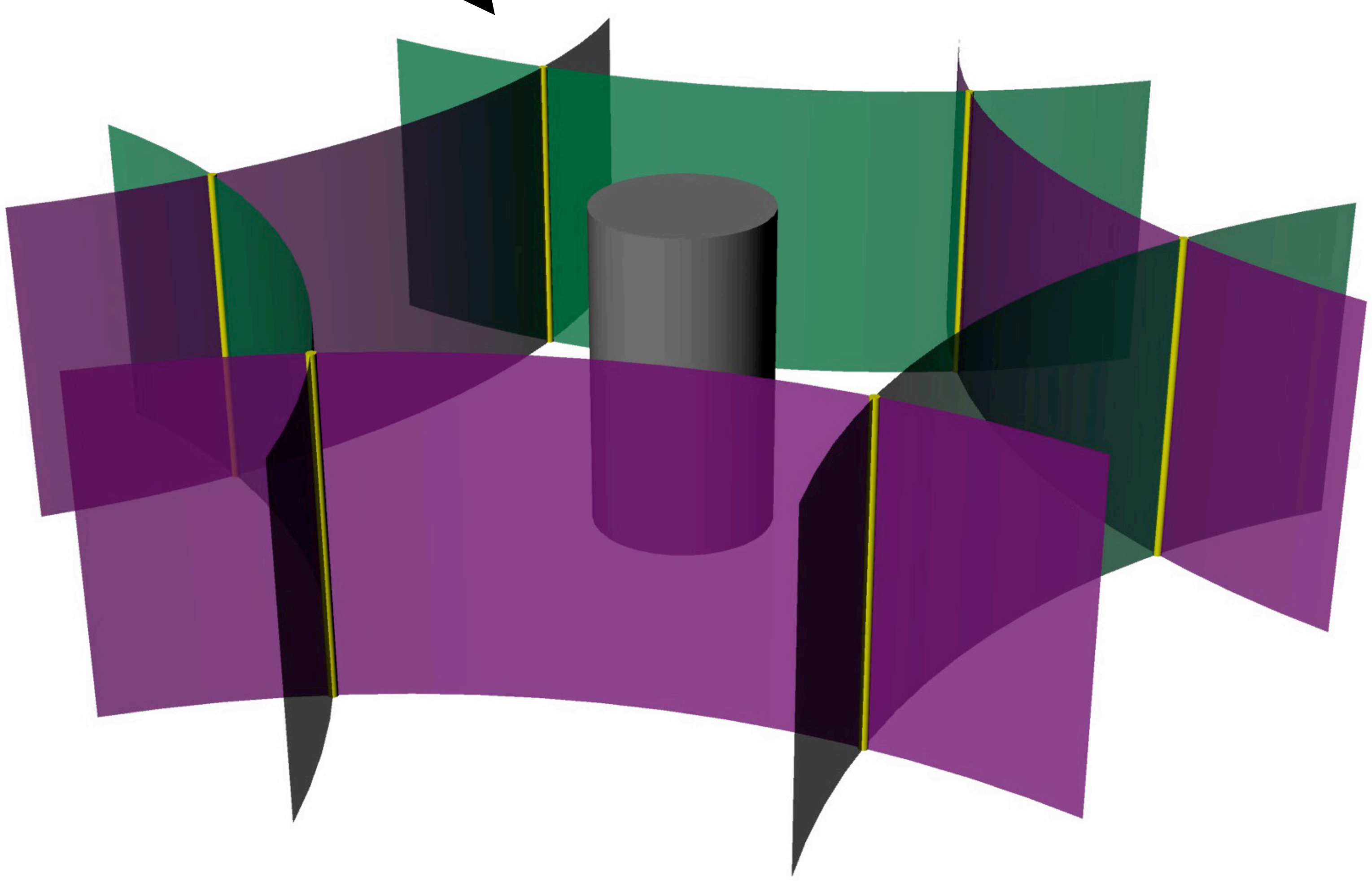
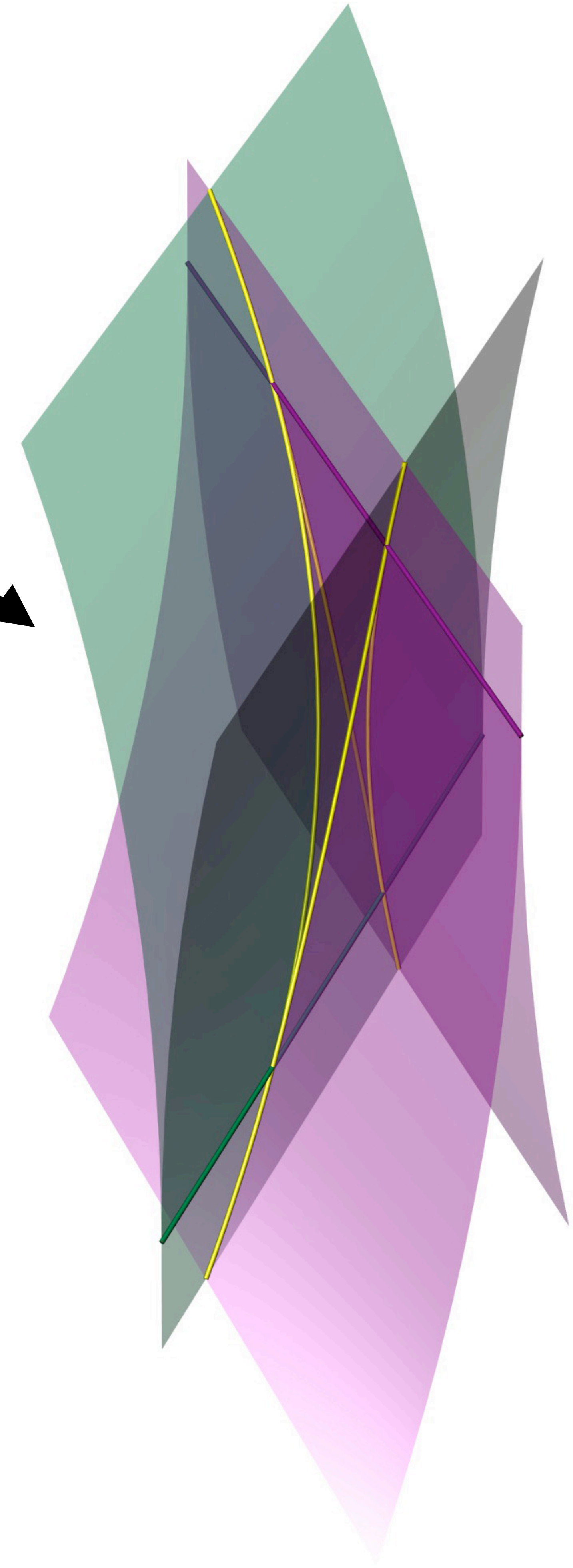
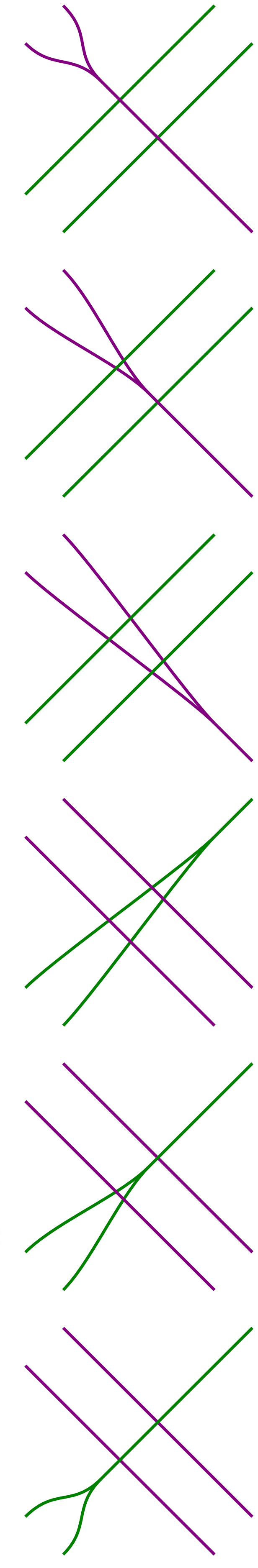
3) …

4) …

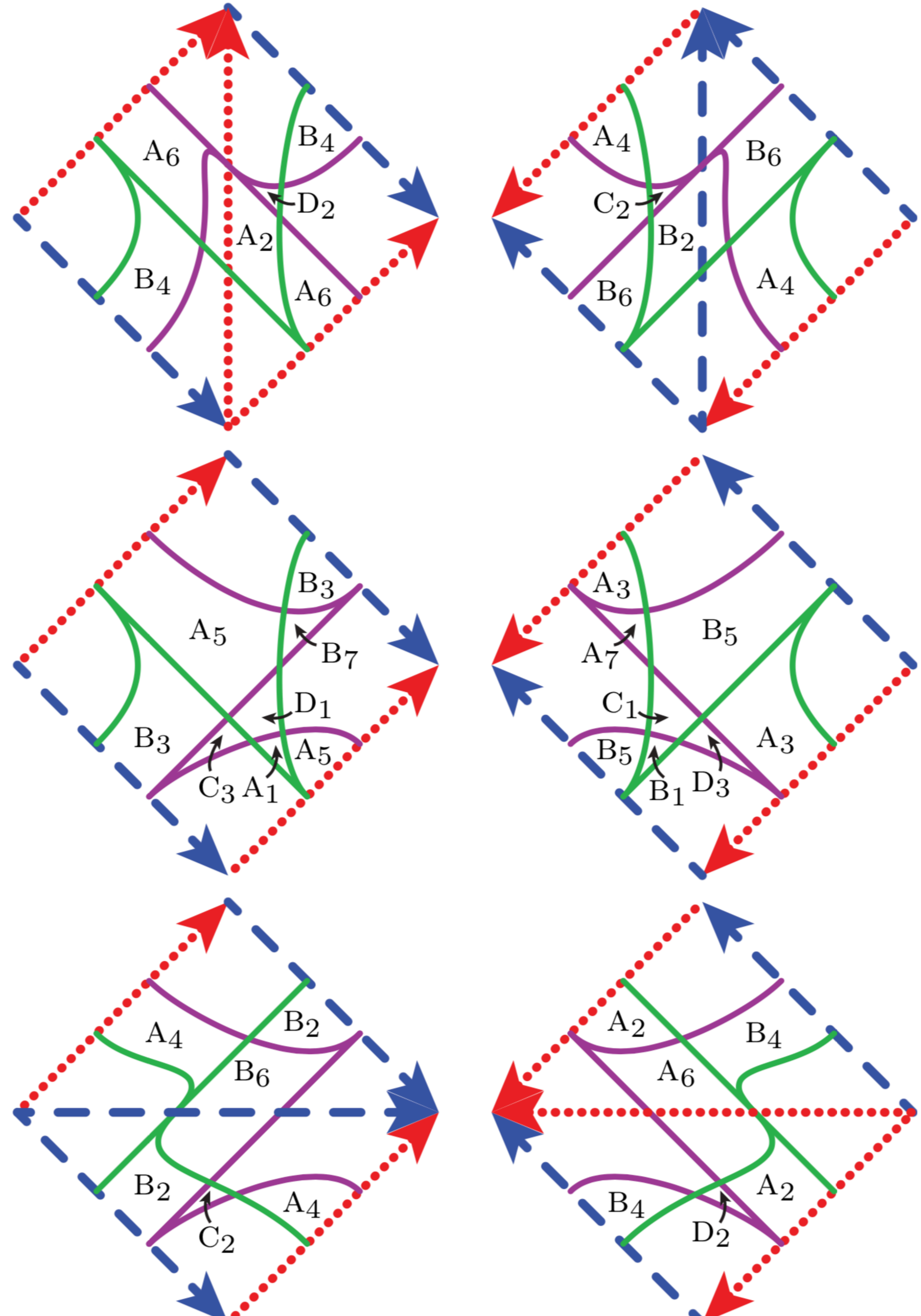






Ex: the figure 8 knot complement

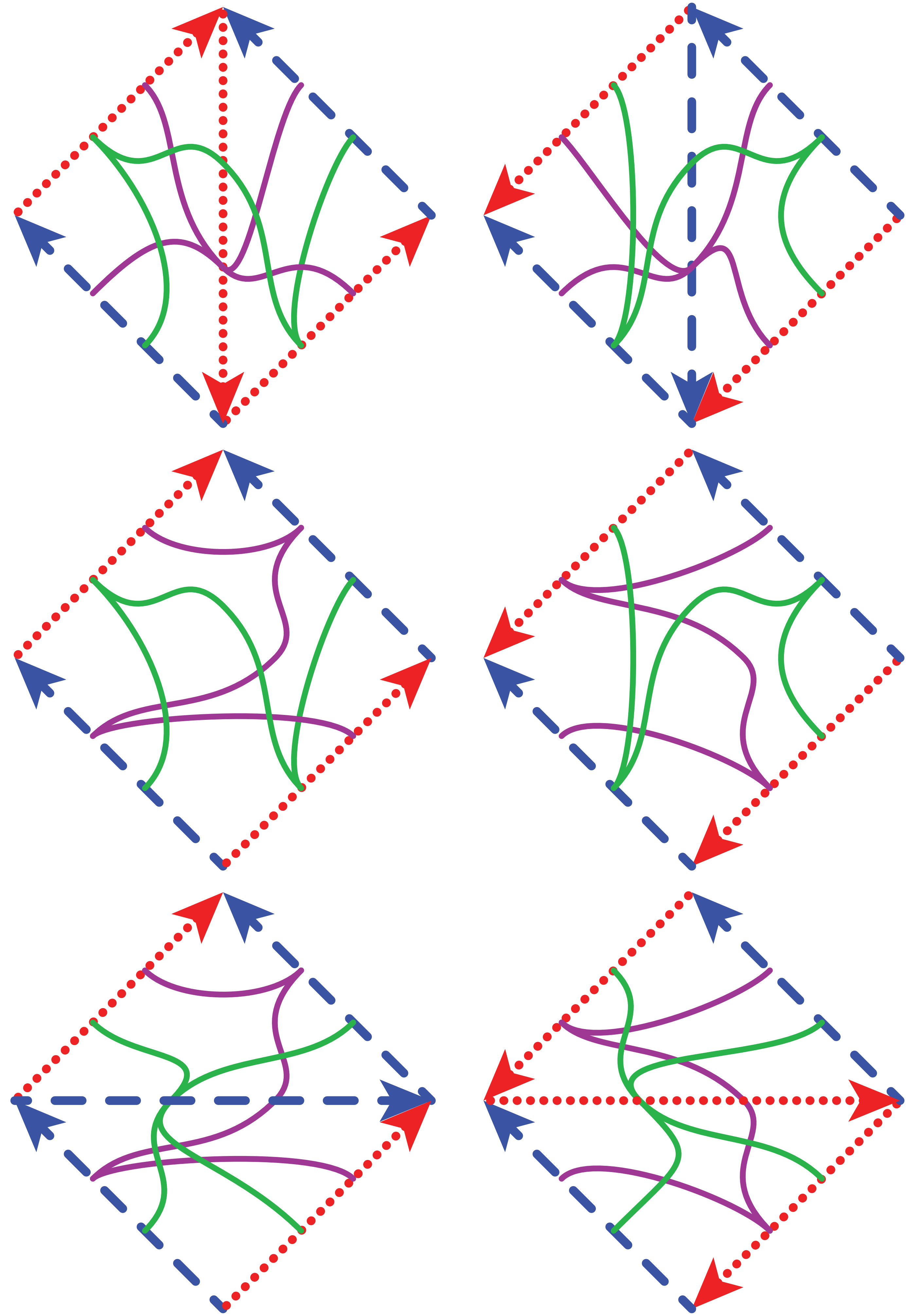
Pushing the branched surfaces off of each other works in some cases…

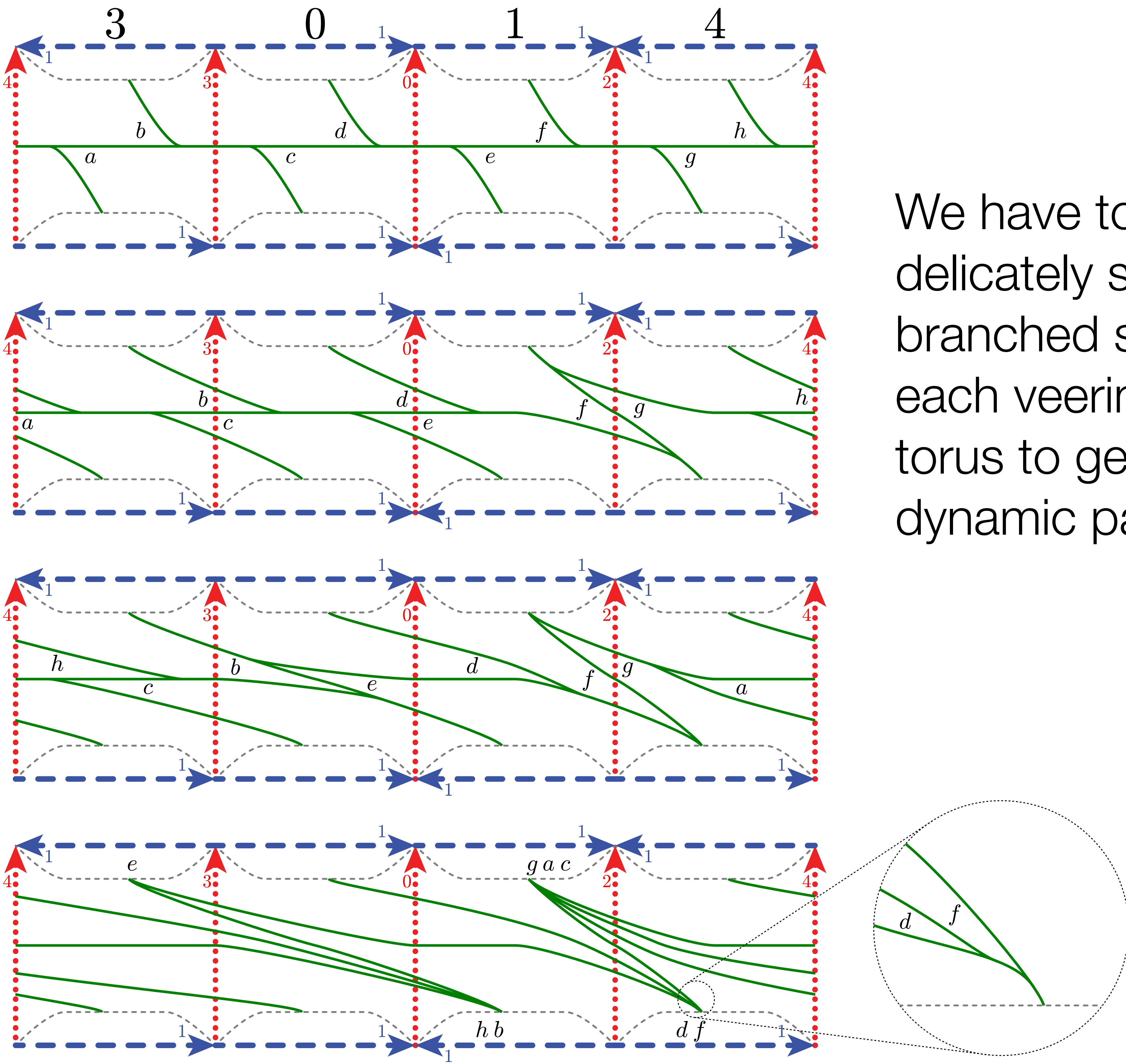


Ex: the figure 8 knot sister

Pushing the branched surfaces off of each other works in some cases…

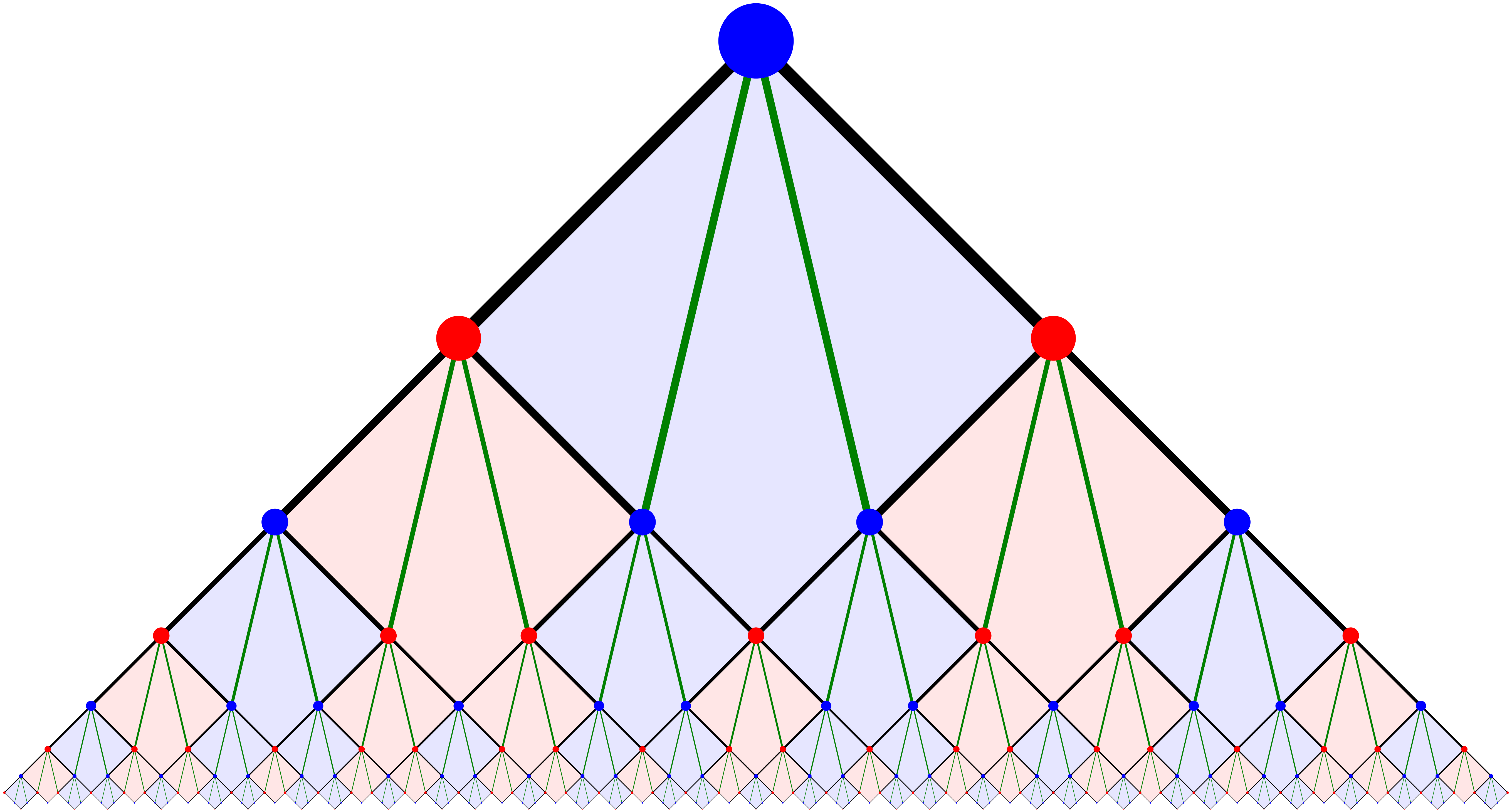
…but fails catastrophically in general.





We have to very delicately split the branched surfaces in each veering solid torus to get a dynamic pair.

Thanks!



A leaf carried by the stable branched surface for the veering triangulation of the figure 8 knot complement. The leaf is decomposed into sectors, and then into normal disks.