Avoiding inessential edges

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Oklahoma State University

joint work with
Tejas Kalelkar and
Saul Schleimer

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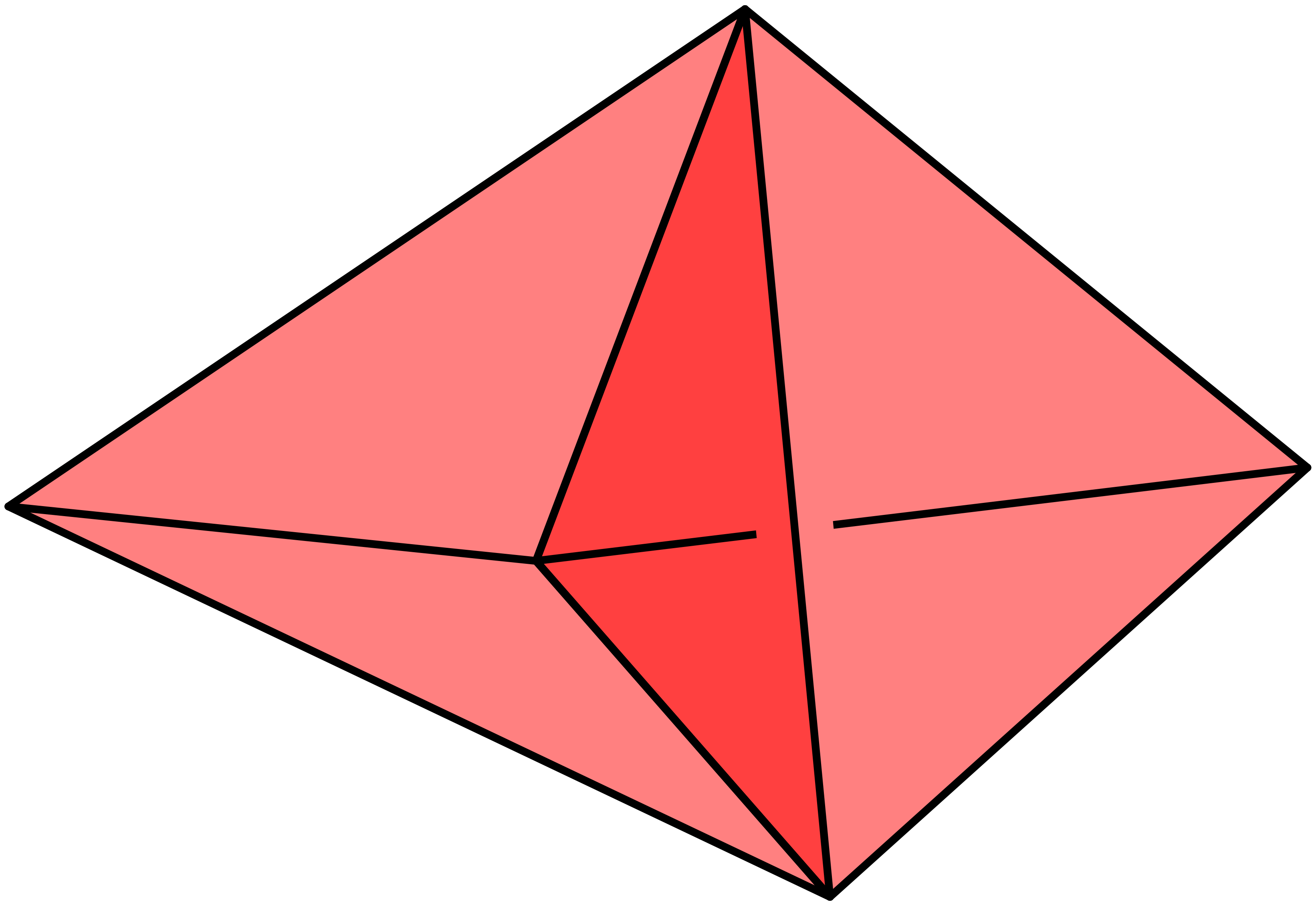
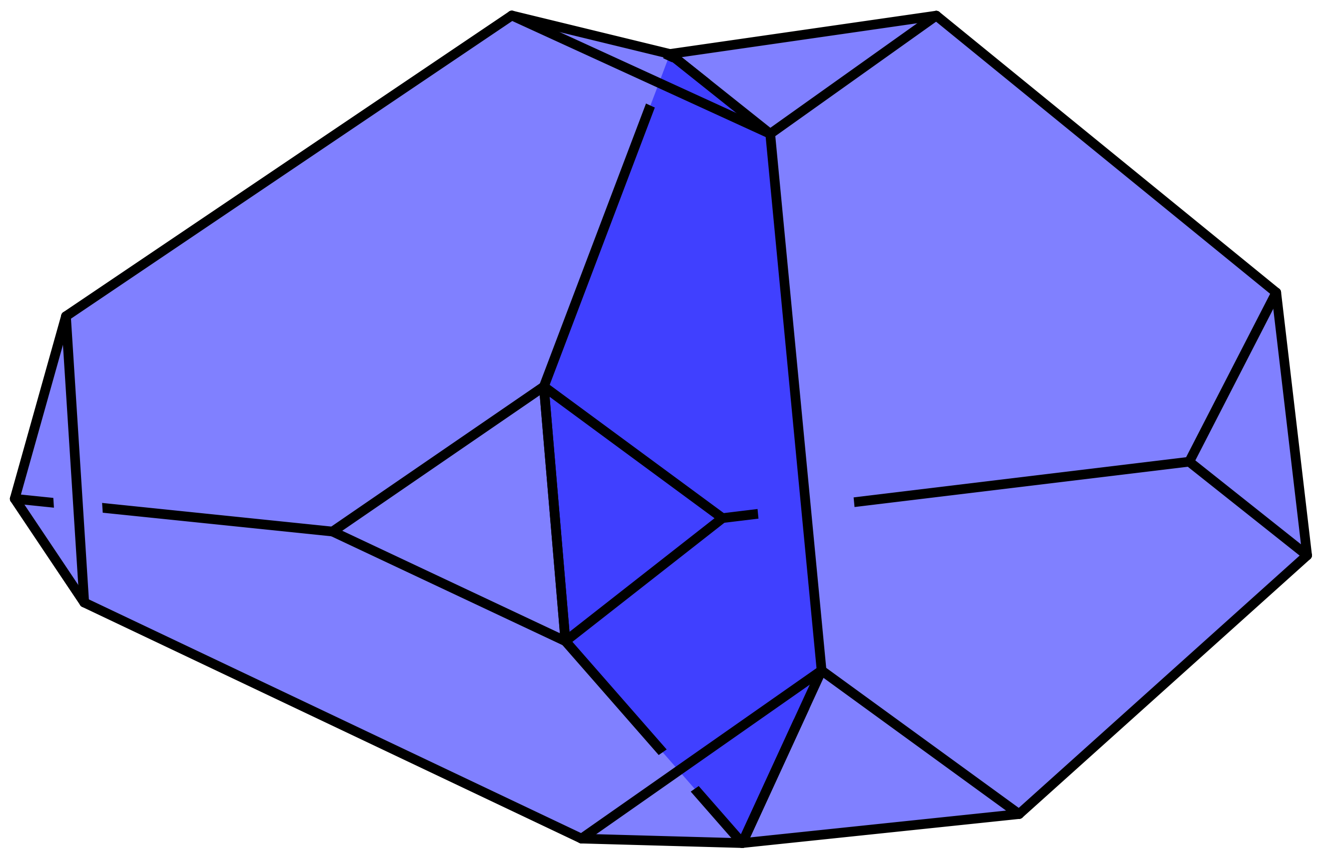
Triangulations



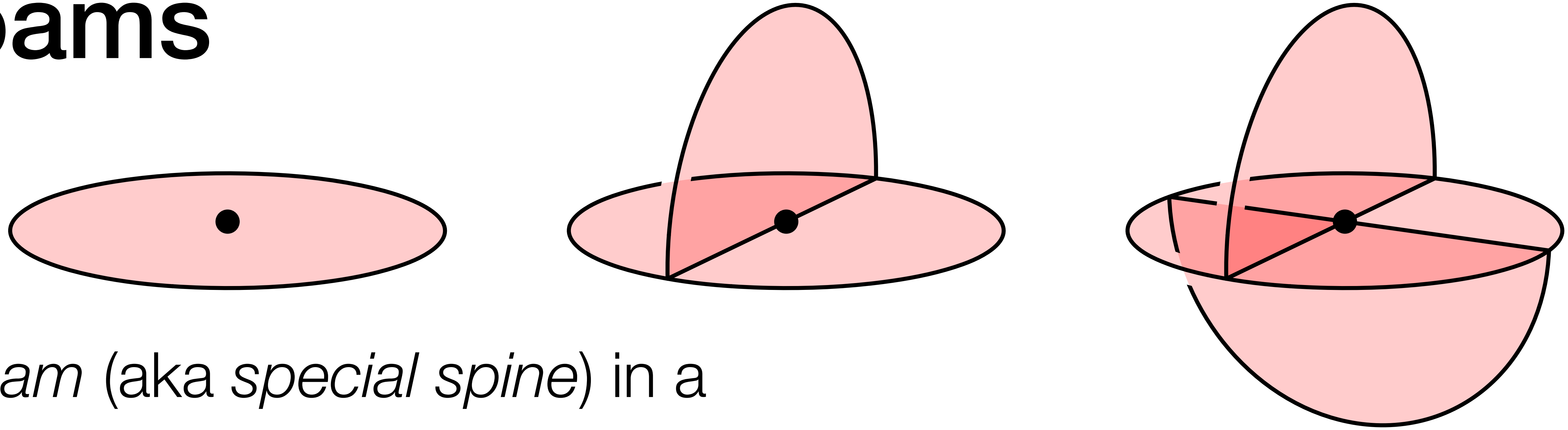
(material vertices)

Ideal Triangulations

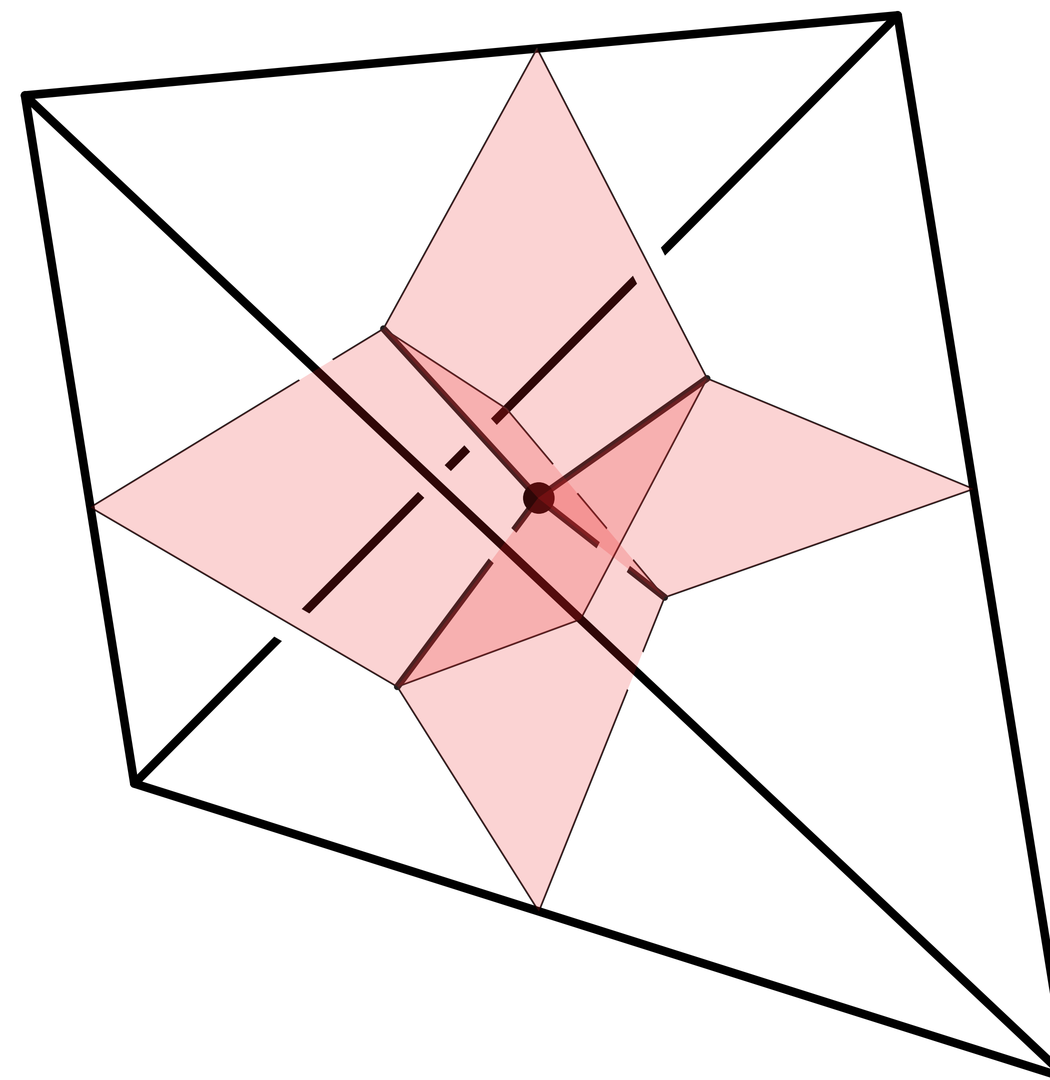


(ideal vertices)

Foams



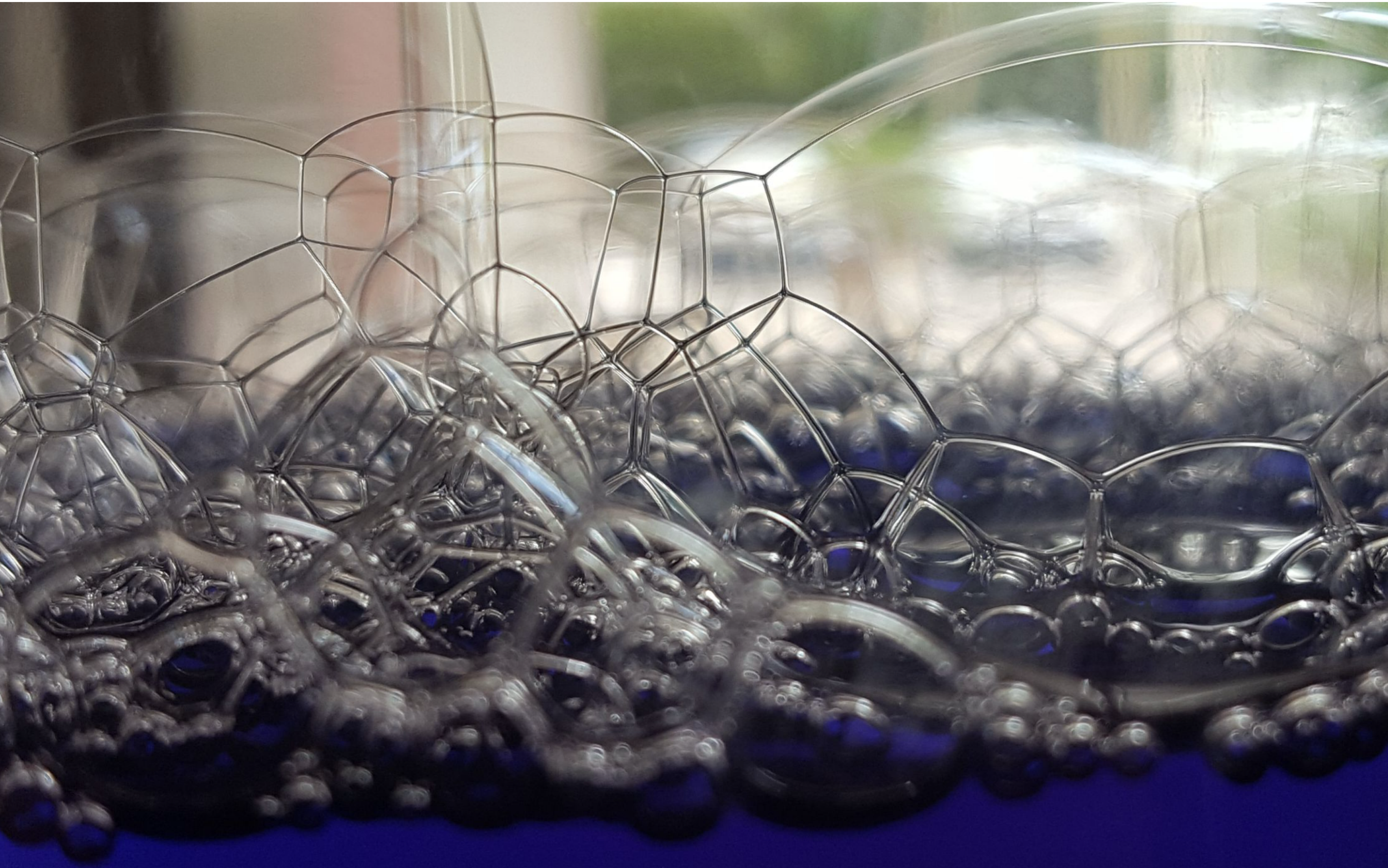
A *foam* (aka *special spine*) in a manifold M is a two-dimensional complex such that each point has one of the three neighbourhoods above.



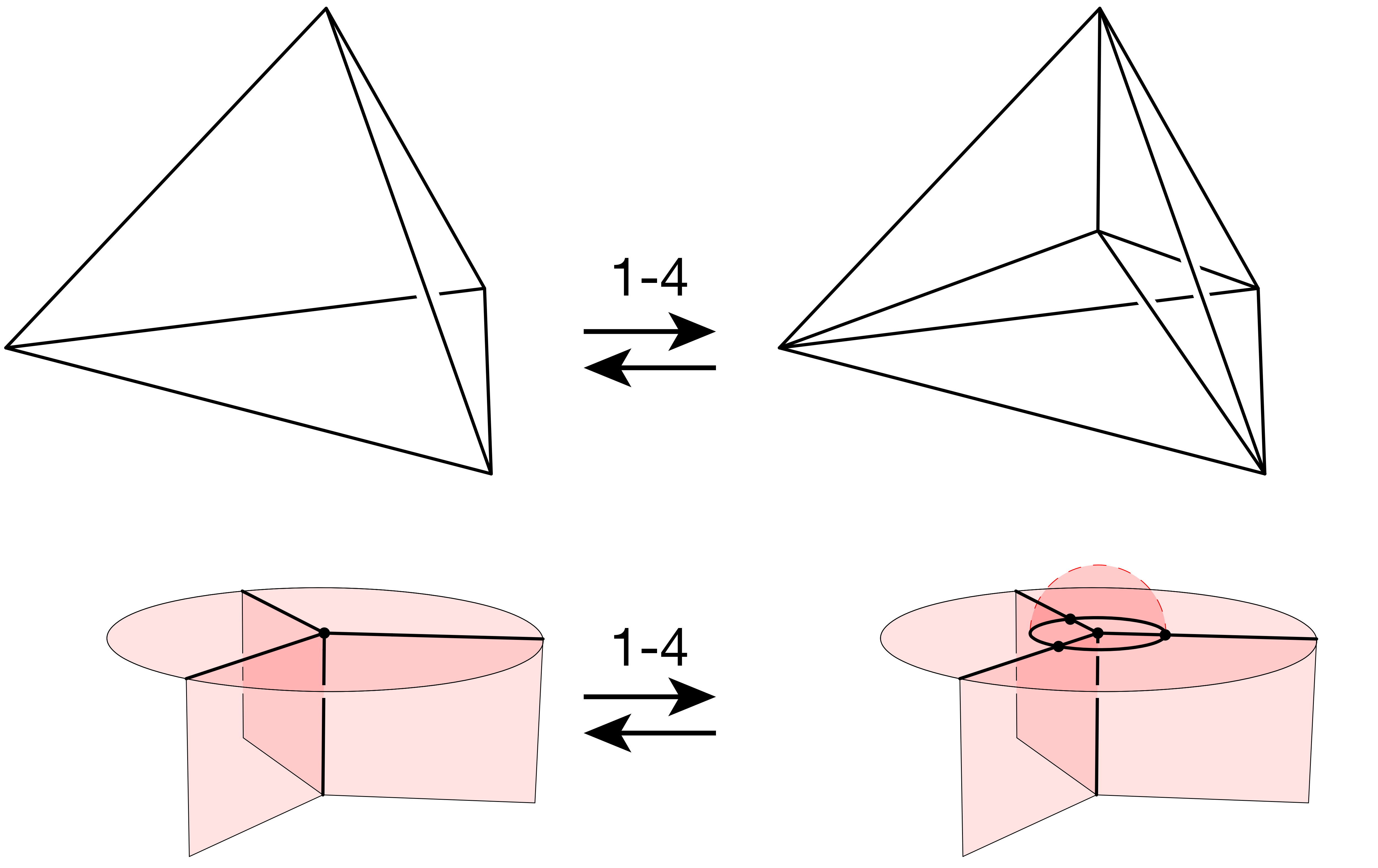
Edges are intervals, faces are disks, complementary regions are balls or surface $\times$ interval (for components of ∂M).

Foams are dual to triangulations.

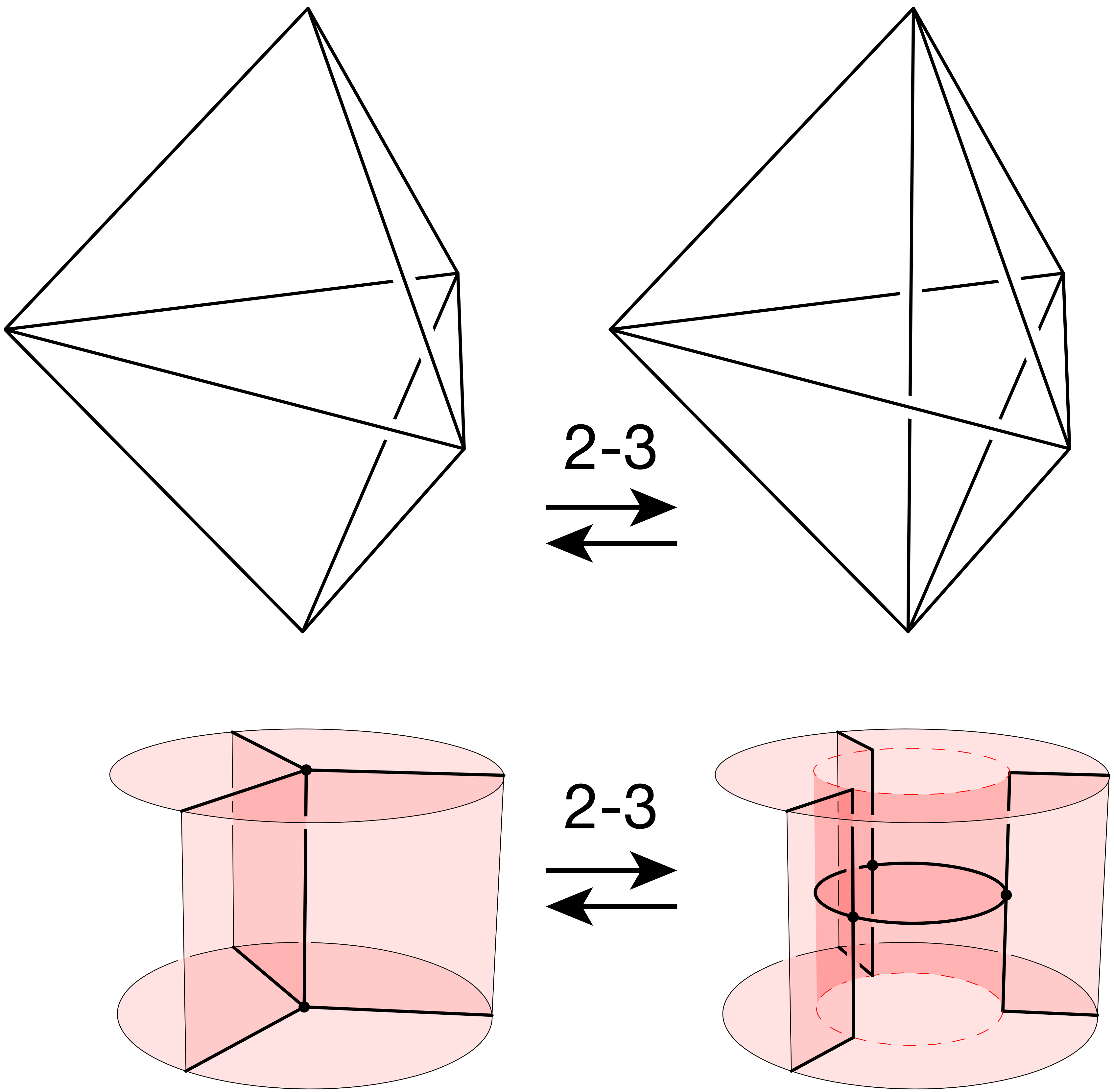
Foams



1-4 move



2-3 move



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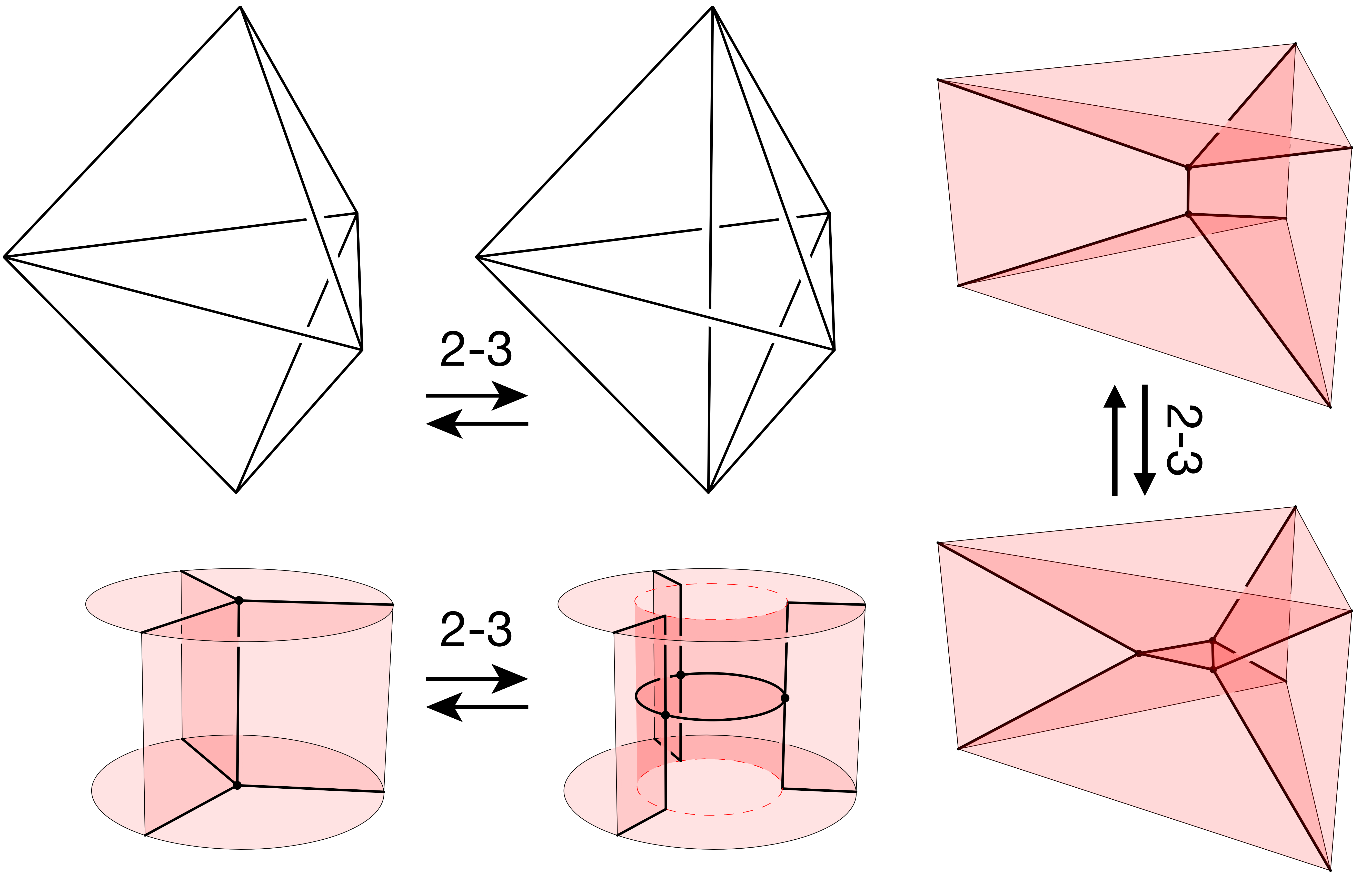

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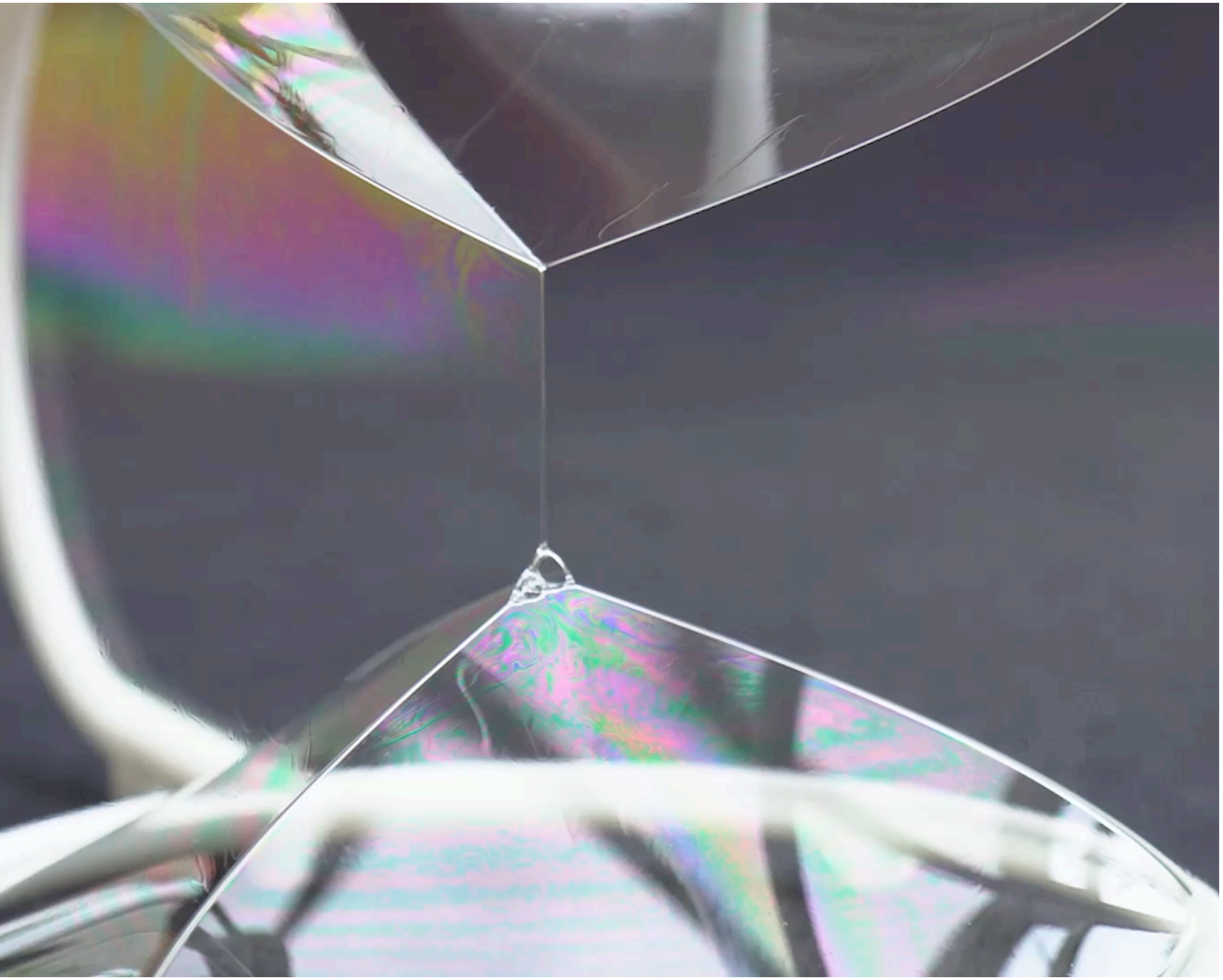

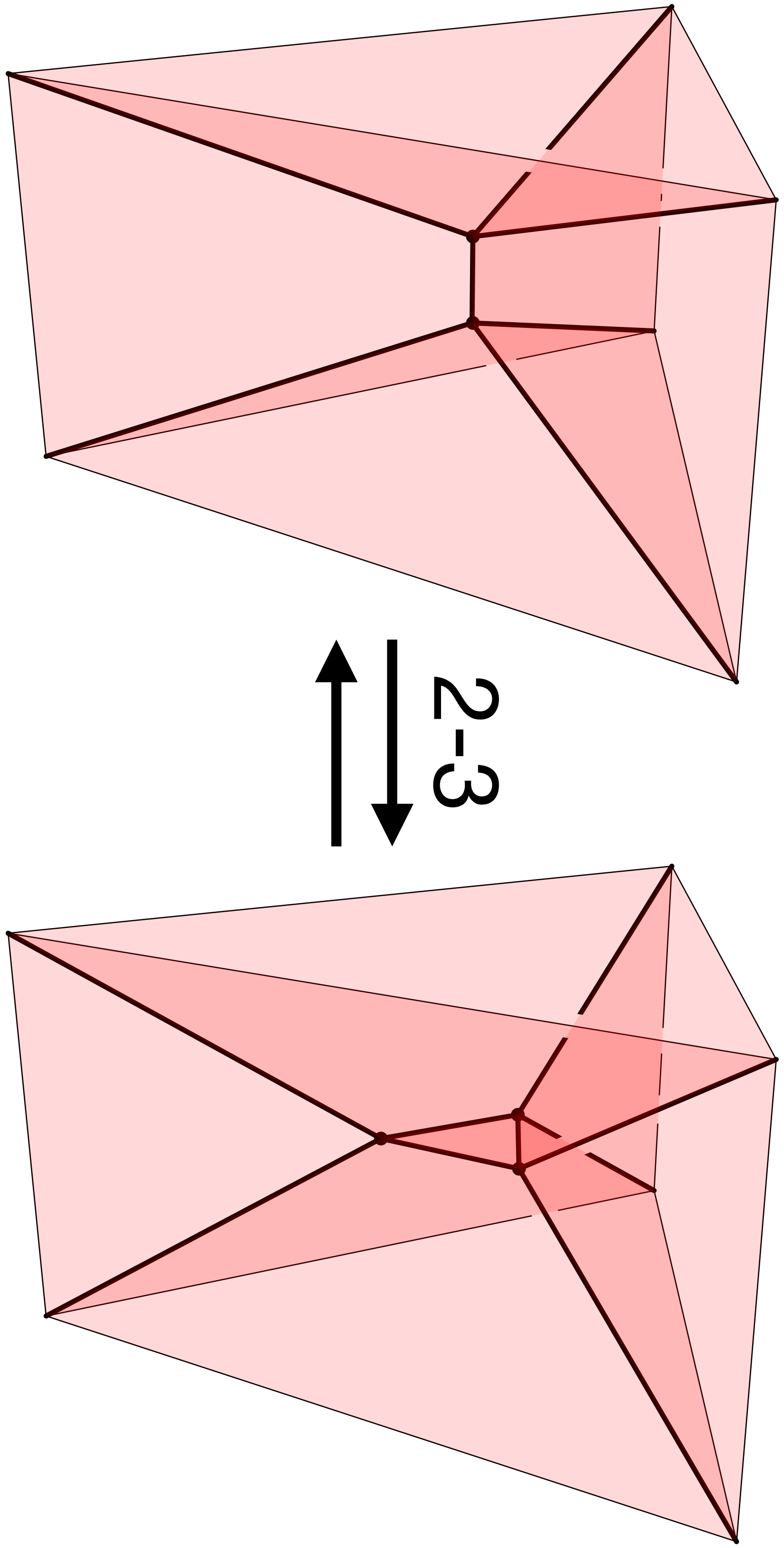

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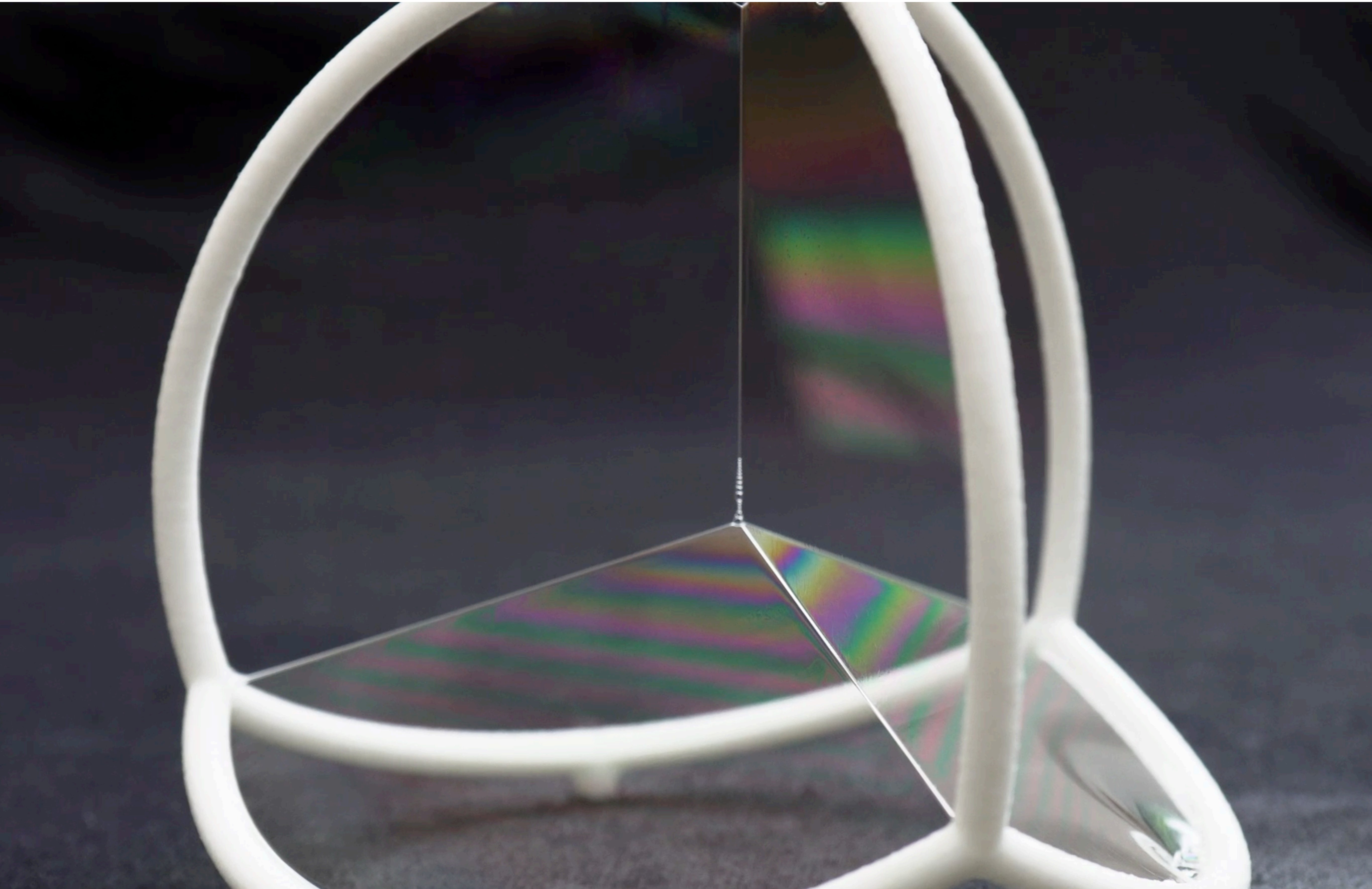
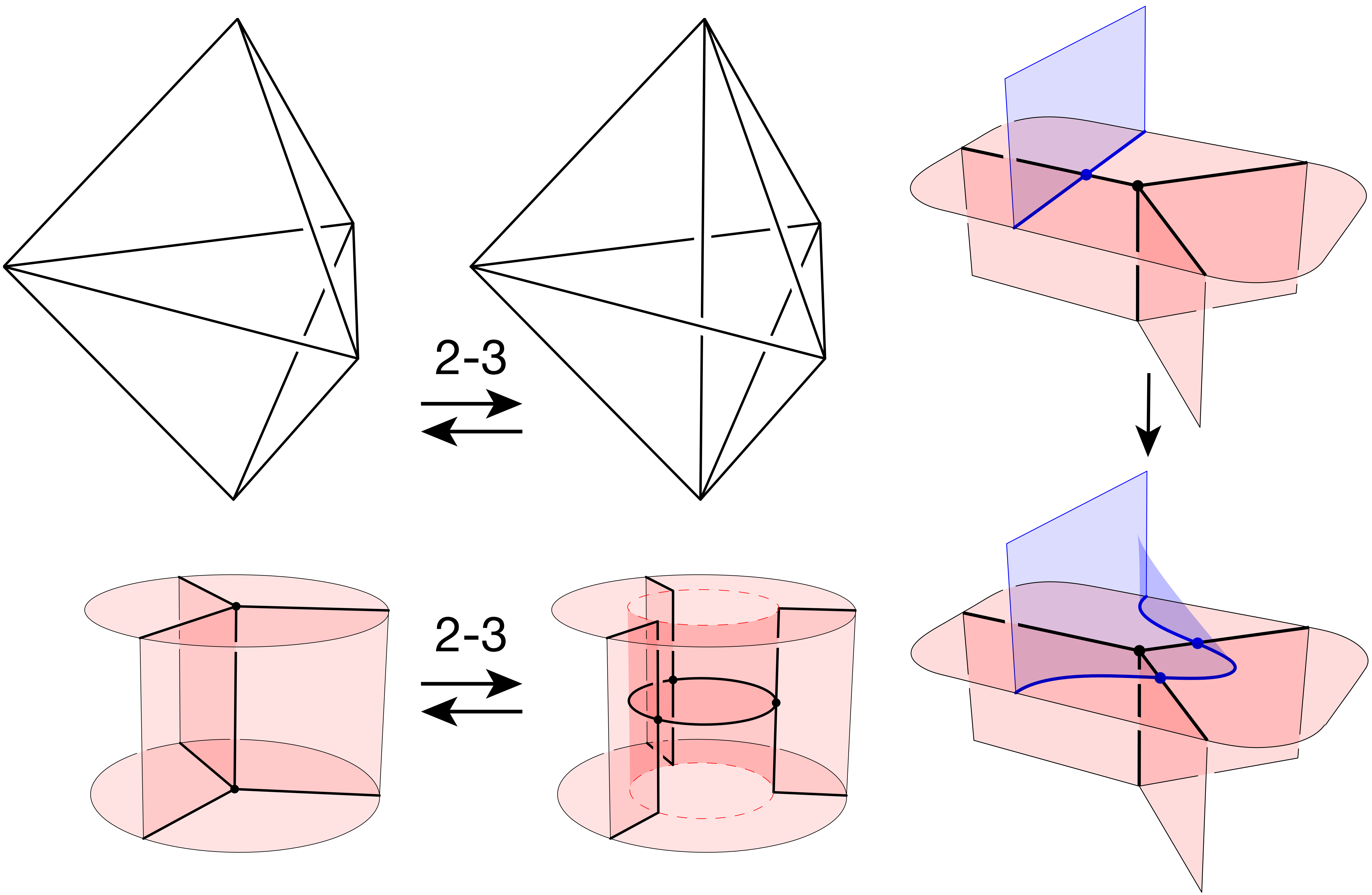


2-3 move

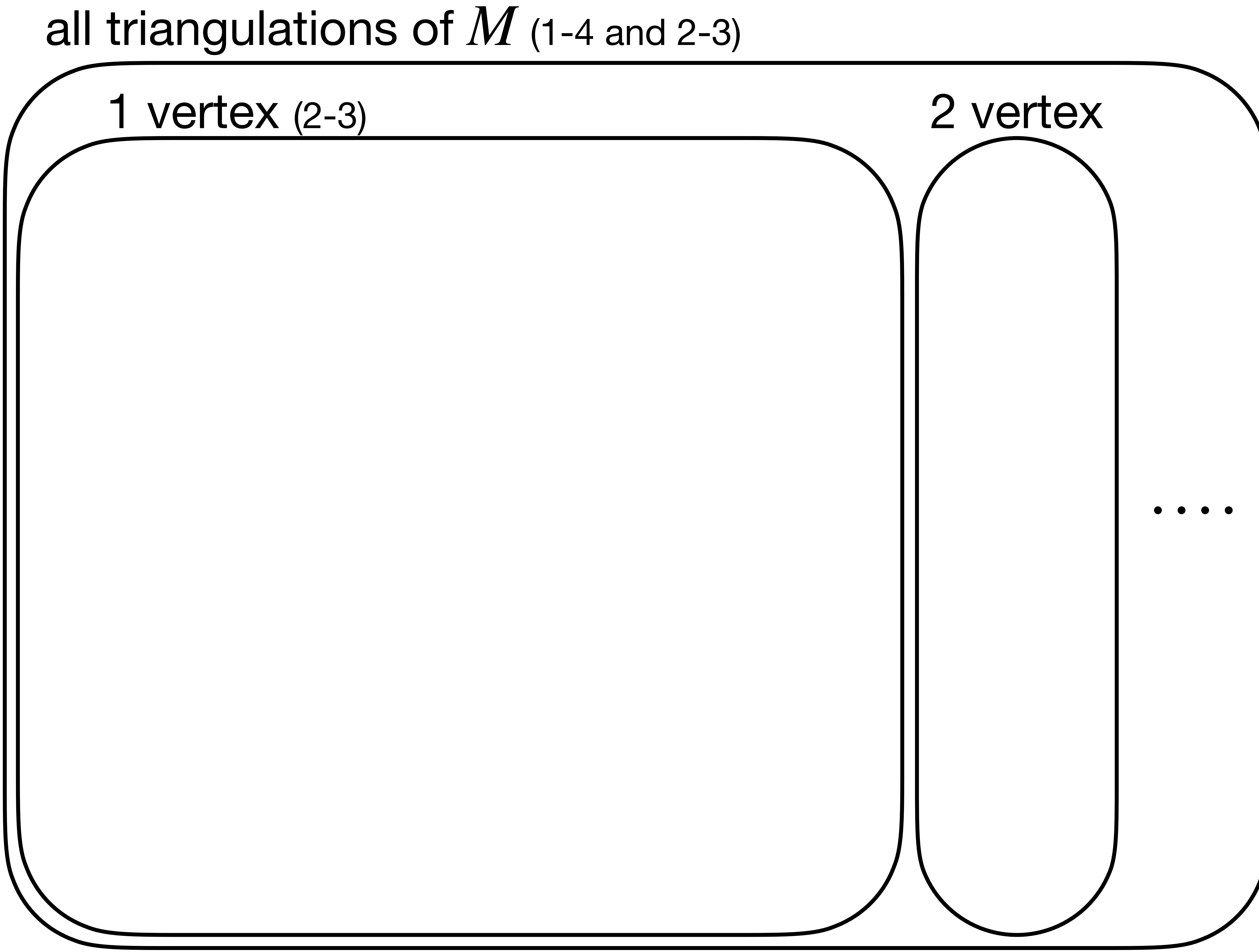


Theorem (Newman '26, Alexander '30, Moise '52, Pachner '78, Banagl-Friedman '04):
The set of all triangulations of a three-dimensional manifold M is connected via 1-4, 2-3, 3-2, and 4-1 moves.

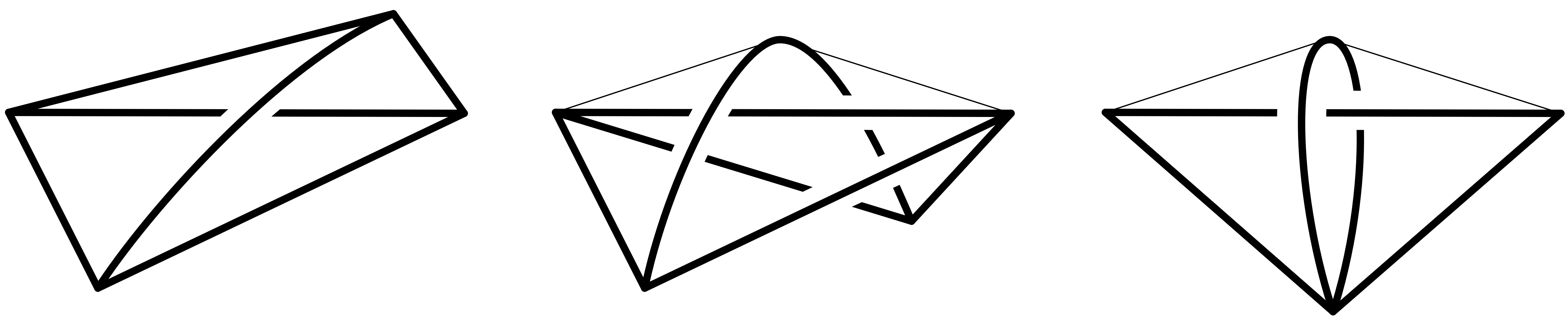
Theorem (Matveev '87, Piergallini '88, Amendola '05):
The set of all triangulations of a three-dimensional manifold M with a fixed number of material vertices is connected via 2-3 and 3-2 moves. (Excepting triangulations with a single tetrahedron.)

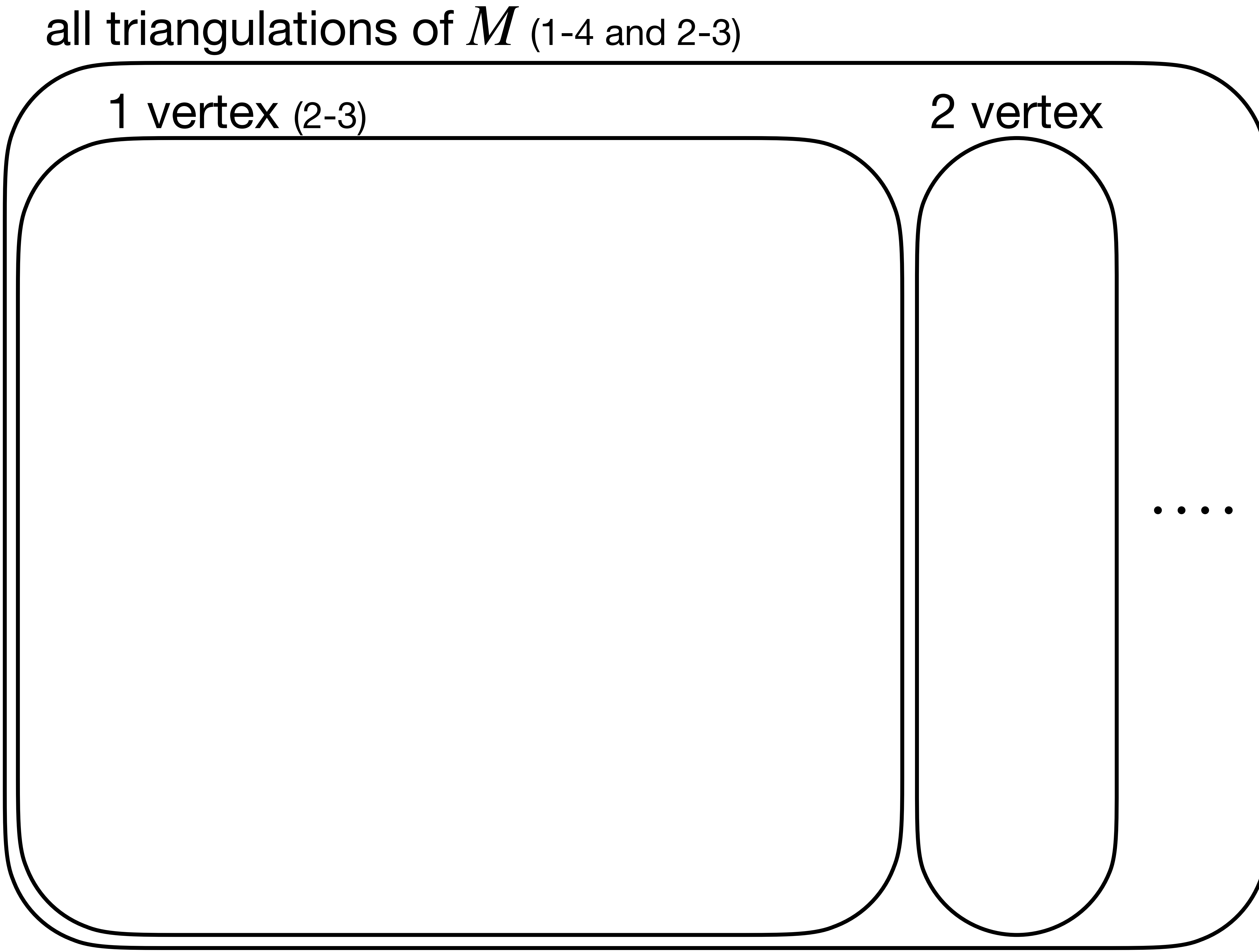
Results like these are useful for building censuses of triangulations, or for defining invariants in terms of triangulations. For example Turaev-Viro's state sums of quantum 6j-symbols.

all triangulations of M (1-4 and 2-3)


all triangulations of M (1-4 and 2-3)
1 vertex (2-3)
2 vertex
....

Theorem (S, ’17):
The set of all triangulations of a three-dimensional manifold with a single vertex and with no degree-one edges is connected under 2-3 and 3-2 moves. (Excepting triangulations with a single tetrahedron and if M is the lens space L(4,1).)

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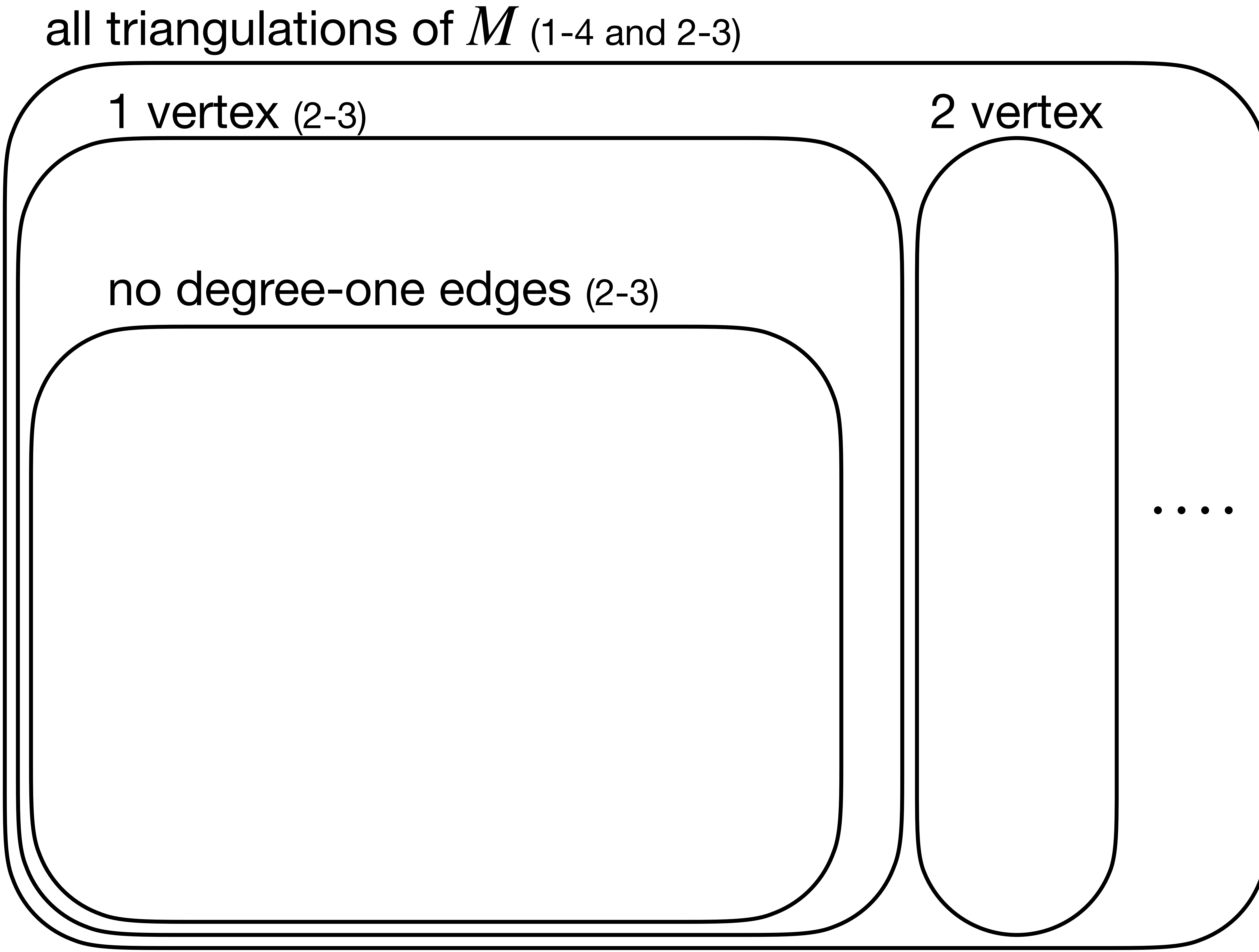
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1 vertex (2-3)
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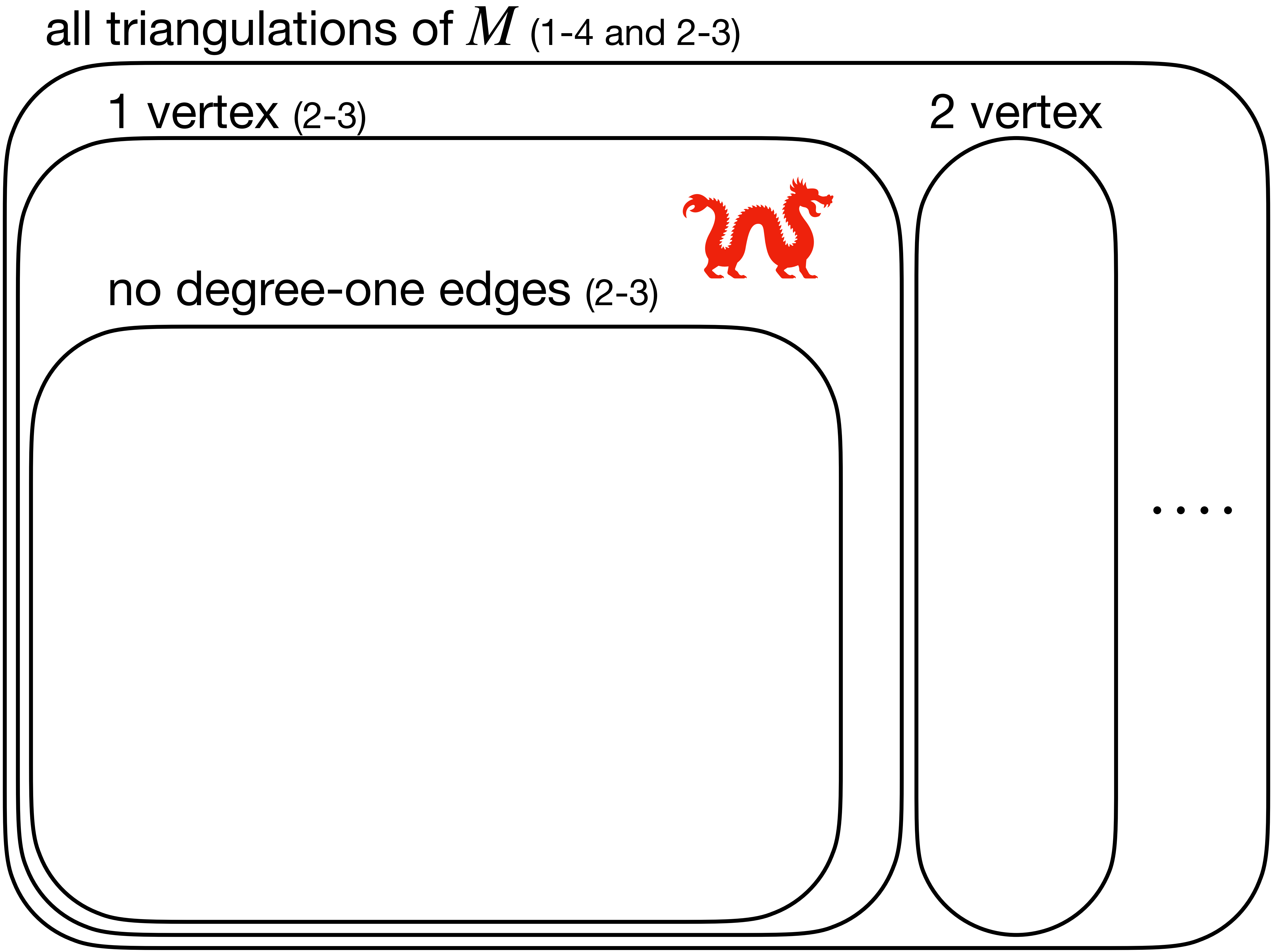
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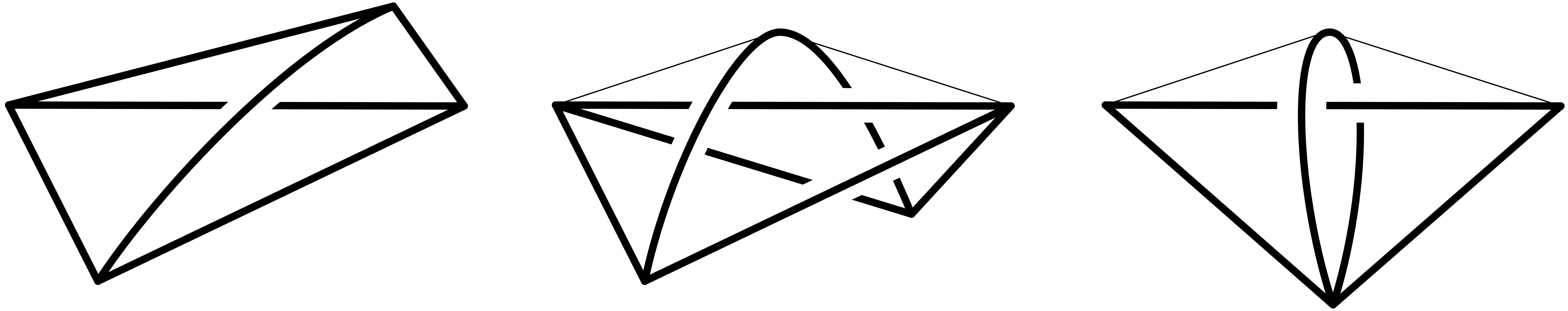
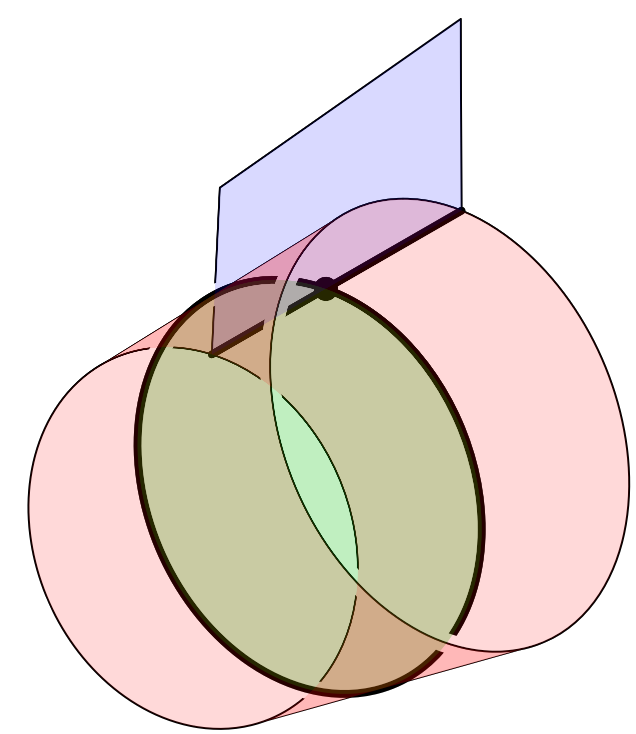


Definition: An edge of a triangulation is *inessential* if it can be homotoped (rel boundary) into a neighbourhood of a vertex.

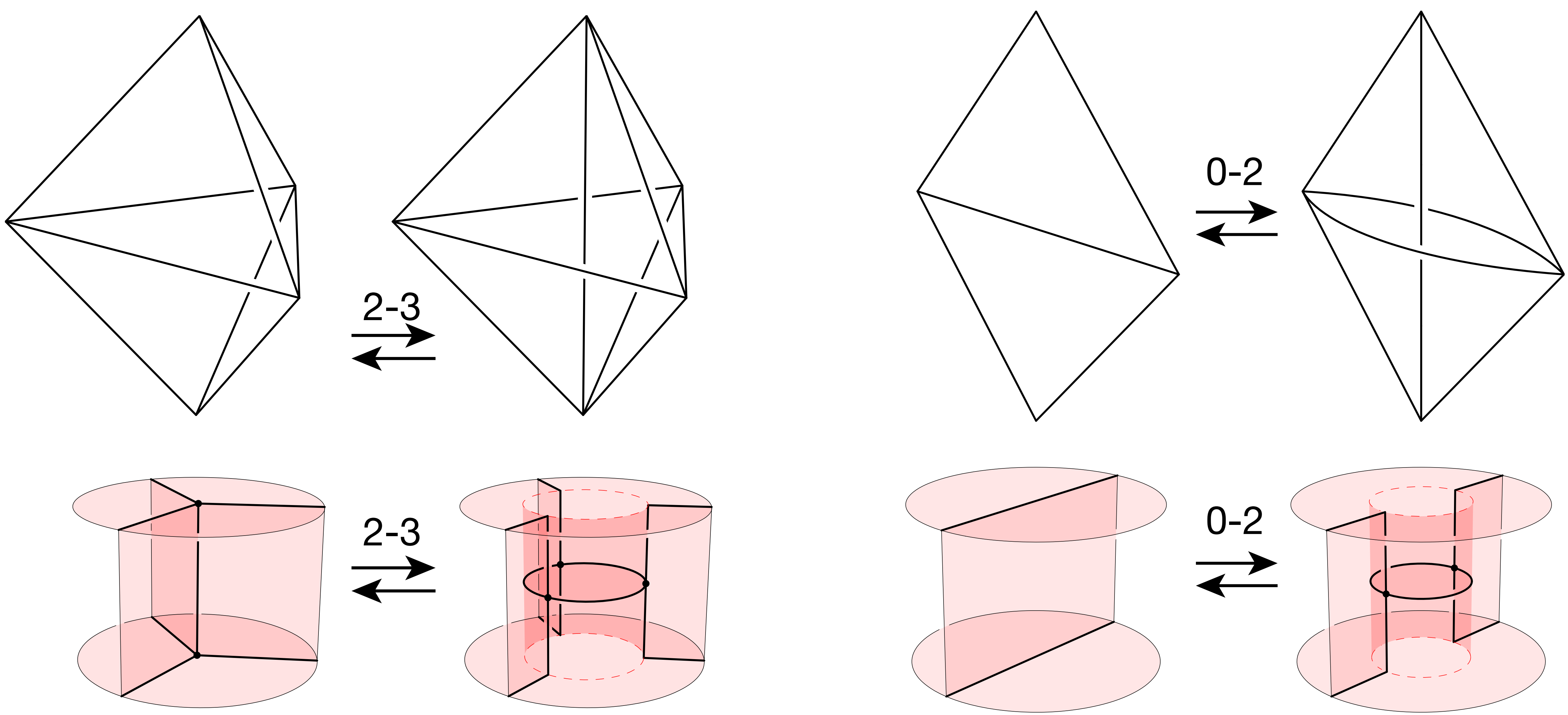
A triangulation is *essential* if it has no inessential edges.

Dually, a face f of a foam is *inessential* if a lift of f to the universal cover meets the same complementary region on both of its sides.

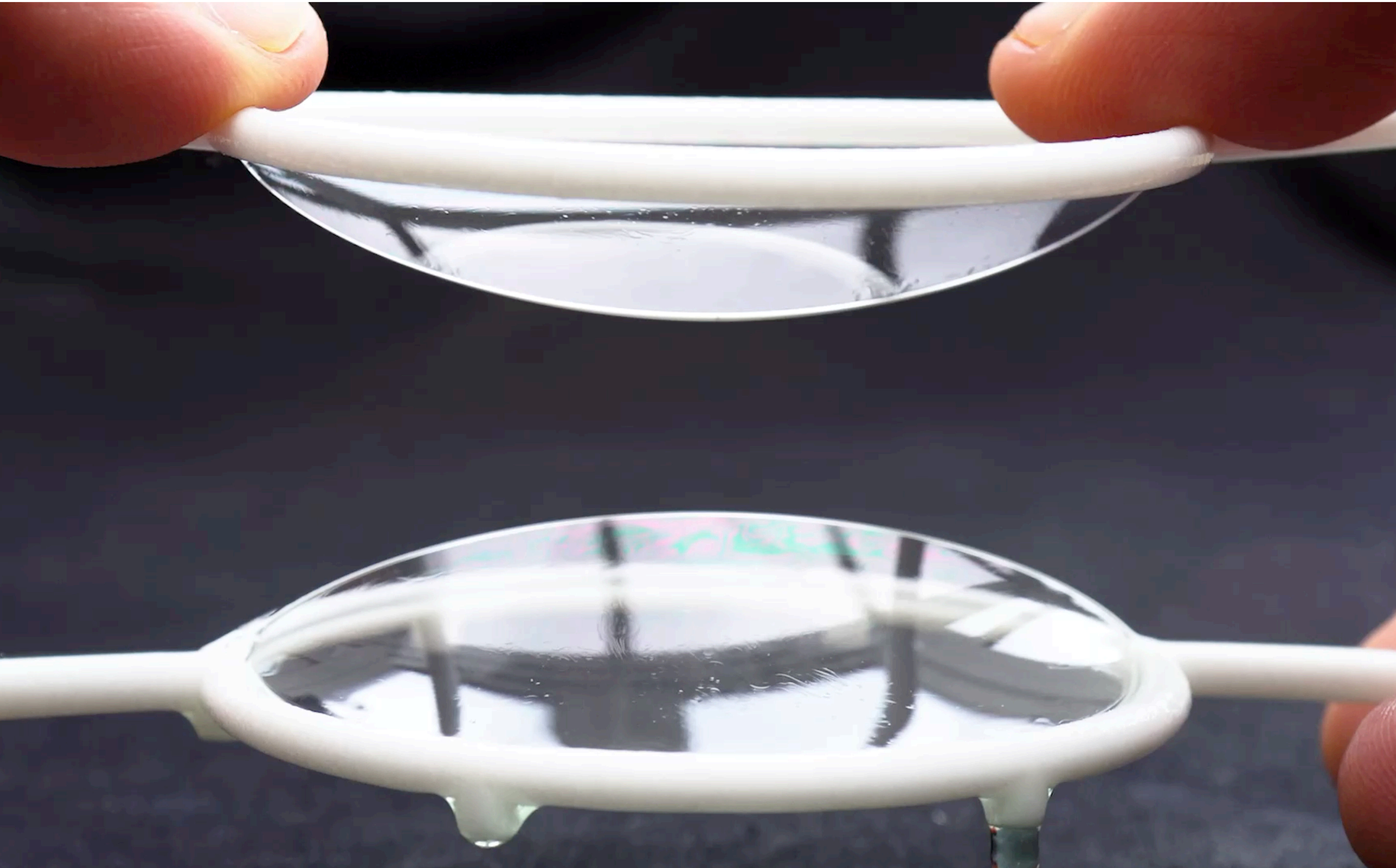


Theorem (Kalelkar, Schleimer, S, ’24):

Suppose that M is a compact, connected three-manifold with boundary. Suppose that the universal cover $\widetilde{M}$ has infinitely many boundary components. Then the set of essential ideal triangulations of M is connected via 2-3, 3-2, 0-2, and 2-0 moves.



0-2 move



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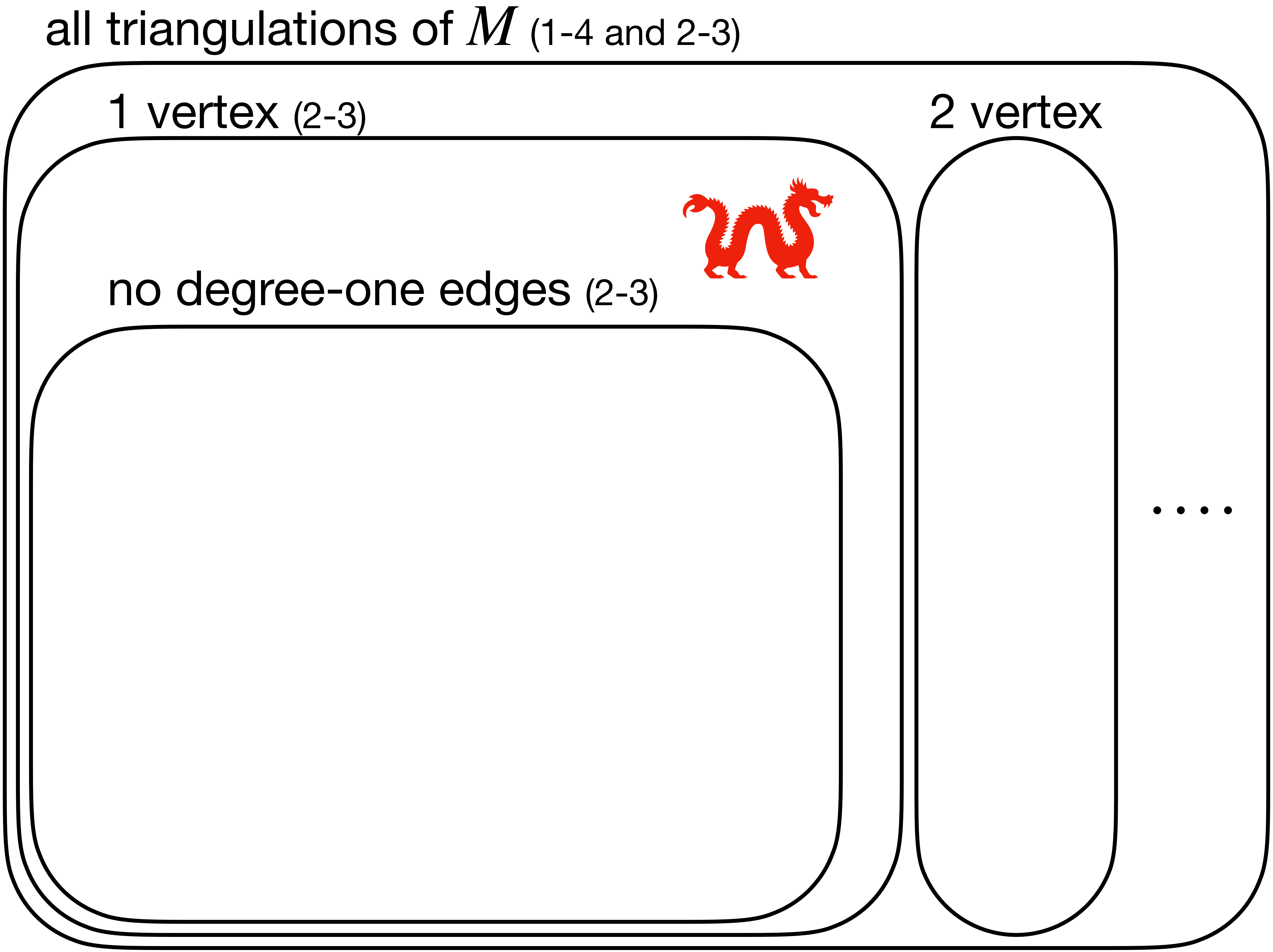
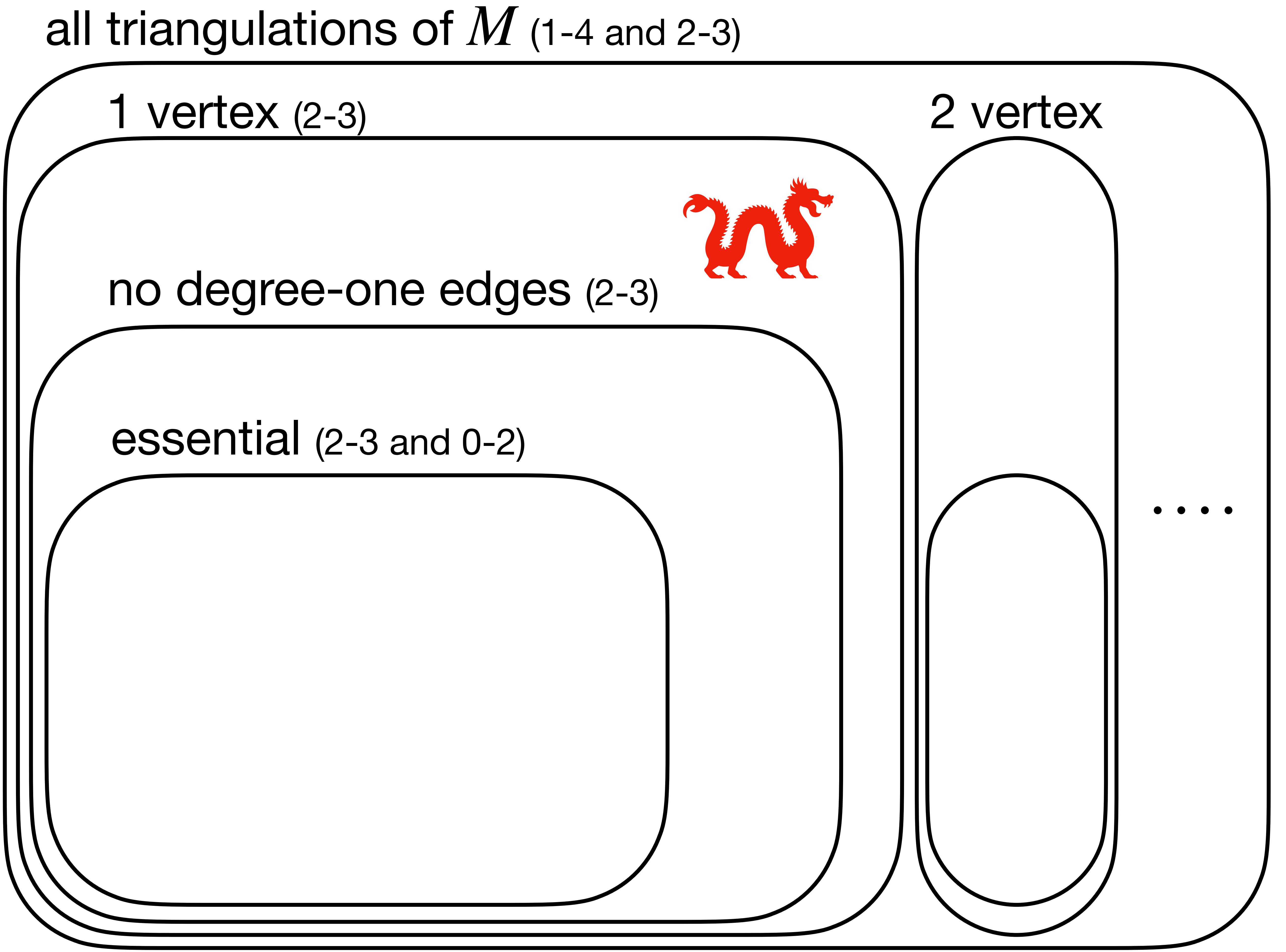
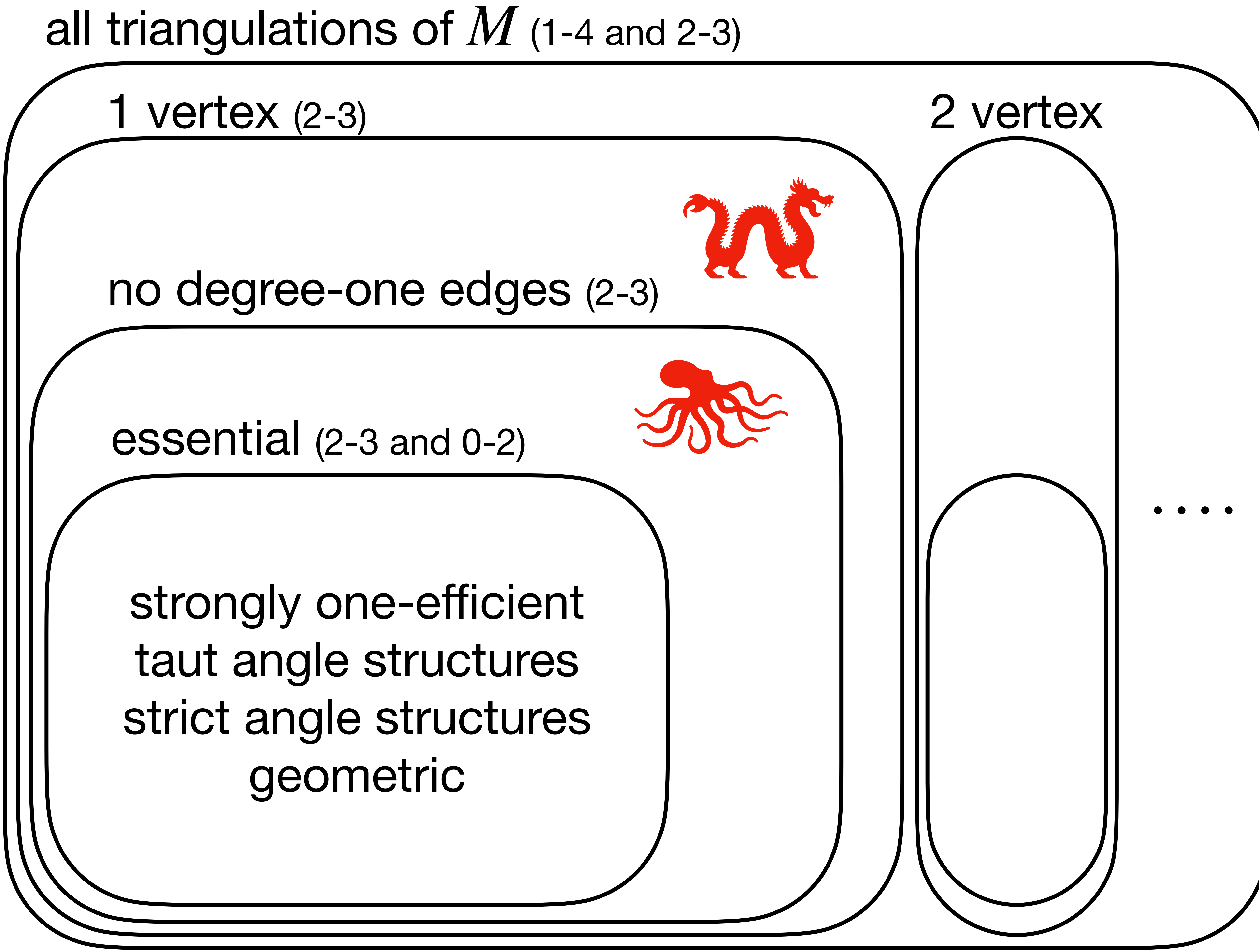
The theorem says that any two foams in M with no self-contacting bubbles in $\widetilde{M}$ can be connected by 2-3, 3-2, 0-2, and 2-0 moves so that no intermediate foam has a self-contacting bubble.



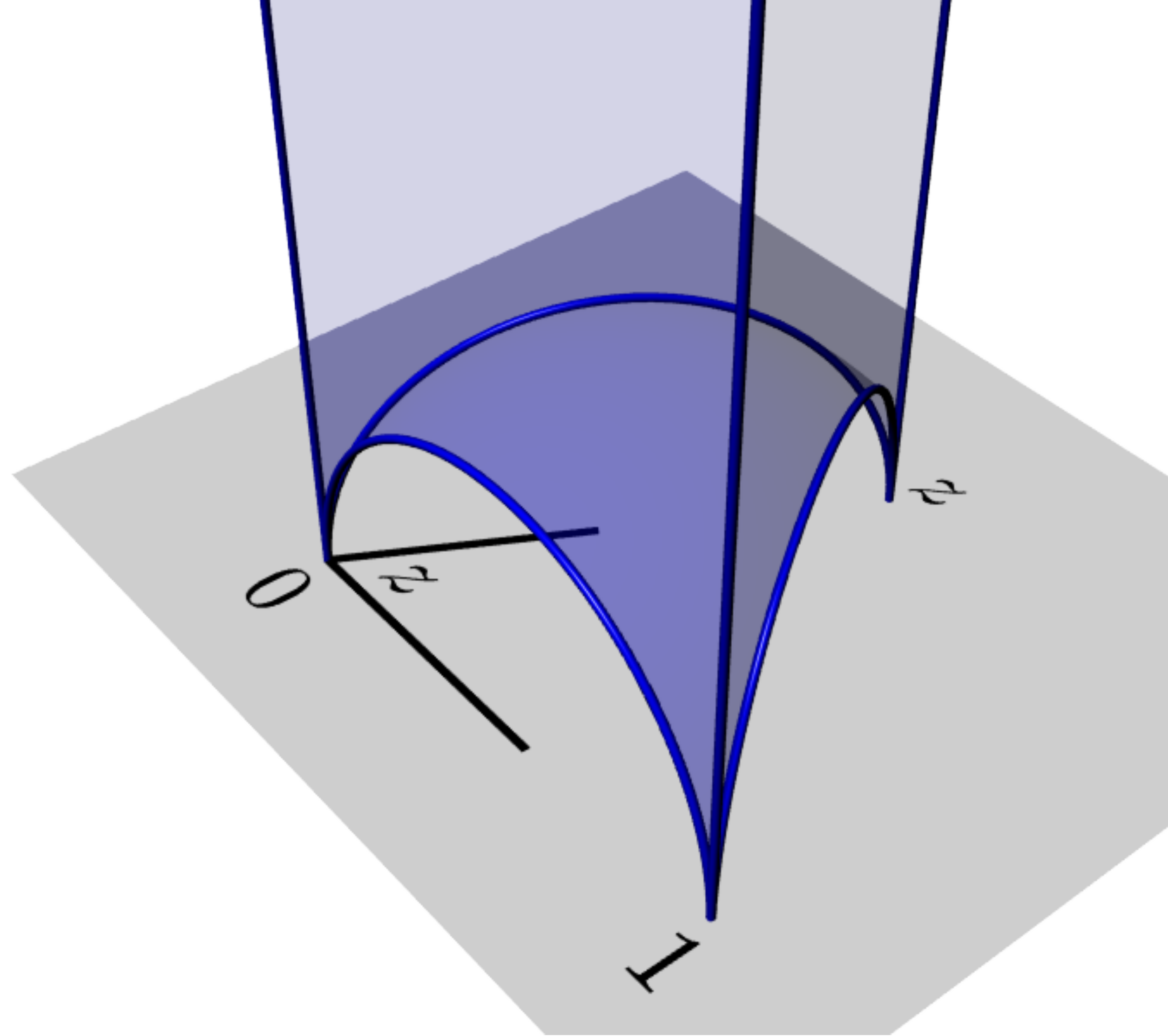
all triangulations of M (1-4 and 2-3)
1 vertex (2-3)
2 vertex
no degree-one edges (2-3)
. . . .



all triangulations of M (1-4 and 2-3)
1 vertex (2-3)
2 vertex
no degree-one edges (2-3)
essential (2-3 and 0-2)

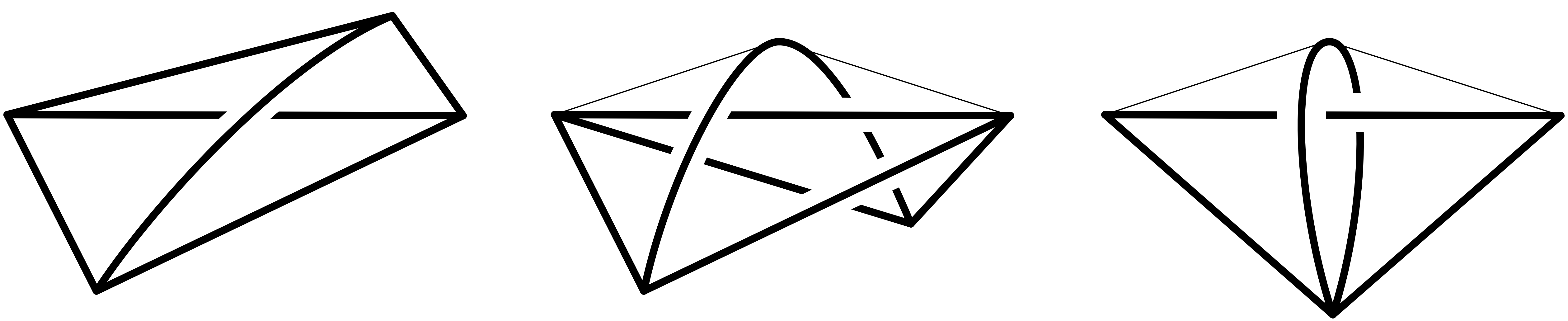

all triangulations of M (1-4 and 2-3)
1 vertex (2-3)
2 vertex
no degree-one edges (2-3)
essential (2-3 and 0-2)
strongly one-efficient
taut angle structures
strict angle structures
geometric
....

Application in quantum topology



Dimofte and Garoufalidis define the *1-loop invariant* τ_T for an ideal triangulation T of a hyperbolic three-manifold.

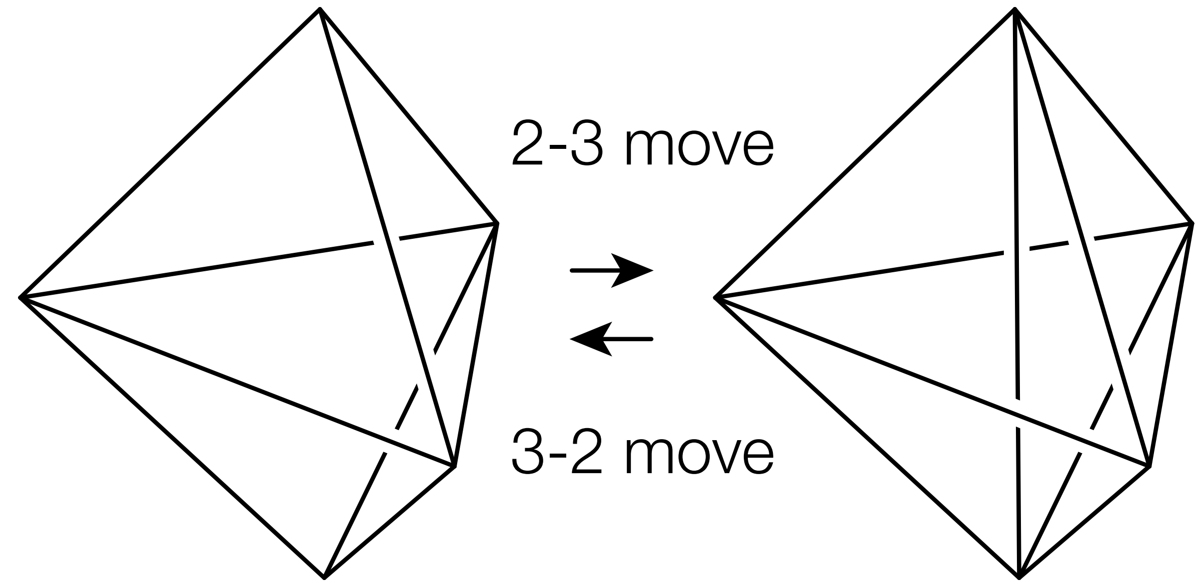
The 1-loop invariant is defined in terms of a solution to Thurston's gluing equations on T.



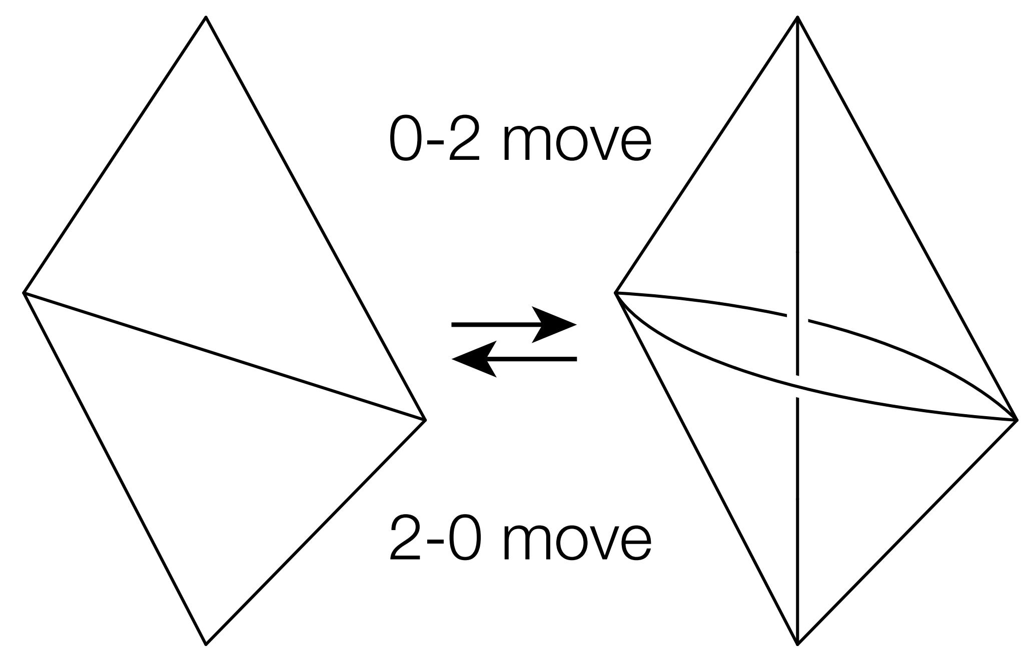
It turns out that a solution exists if and only if T is essential.

Application in quantum topology

Dimofte and Garoufalidis show that τ_T is invariant under 2-3 moves between essential triangulations.

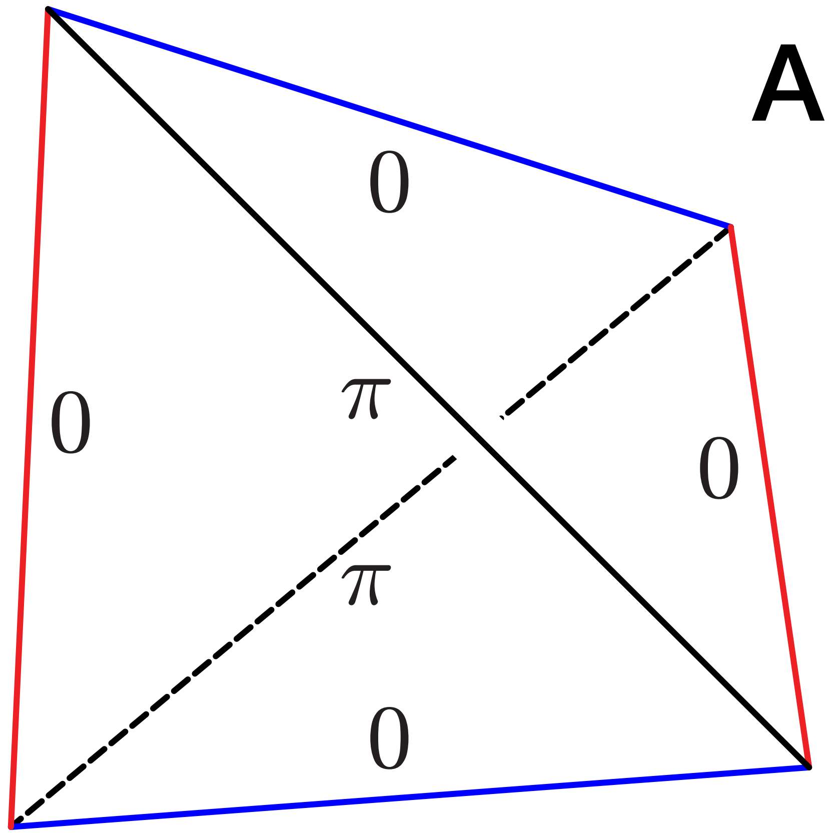


Pandey and Wong show that τ_T is invariant under 0-2 moves between essential triangulations.



Our result then proves that τ_T does not depend on the choice of essential triangulation.

Application to veering triangulations



Conjecture:
Each manifold admits a finite number of veering triangulations.

Schmalian (March 2024) finds many manifolds in the SnapPy census that have *zero* veering triangulations.

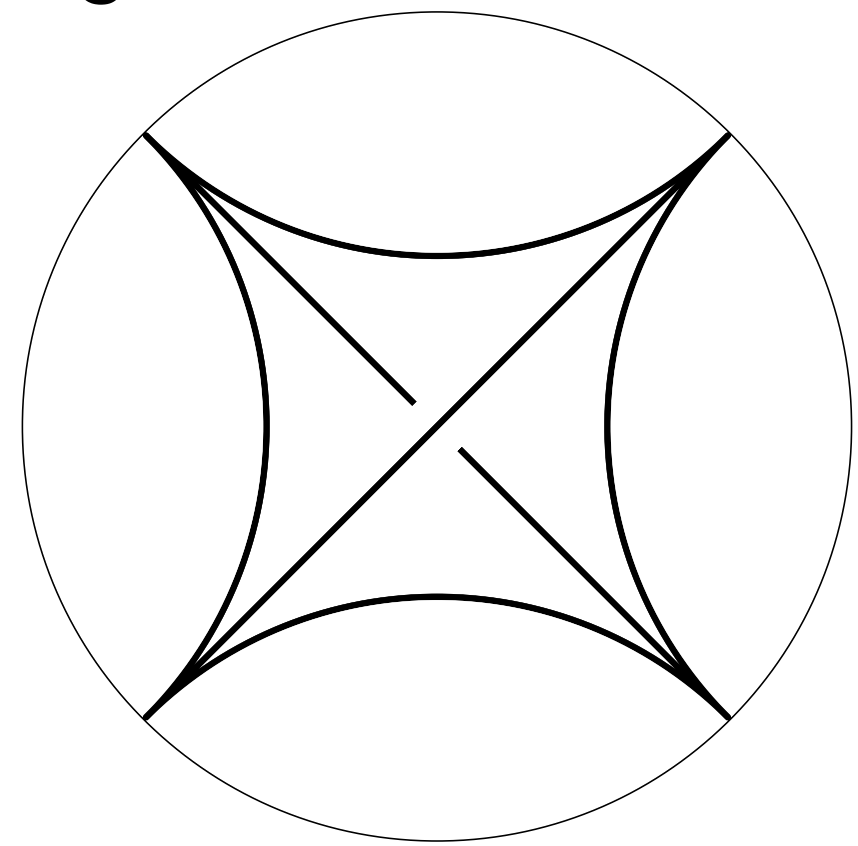
(m006, m007, m011, m029, m030, m037, m047, m049, m060, m064, m081, m082, m095, m116, m117, m129, m130, m142, m143)

(Schleimer-S, work in progress) The figure-eight knot complement has precisely *one* veering triangulation.

Application to veering triangulations

(Schleimer-S, work in progress) The figure-eight knot complement has precisely *one* veering triangulation.

A veering triangulation T canonically determines a circular ordering on the vertices of $\widetilde{T}$ which is *compatible* with a "flattening" of each tetrahedron.

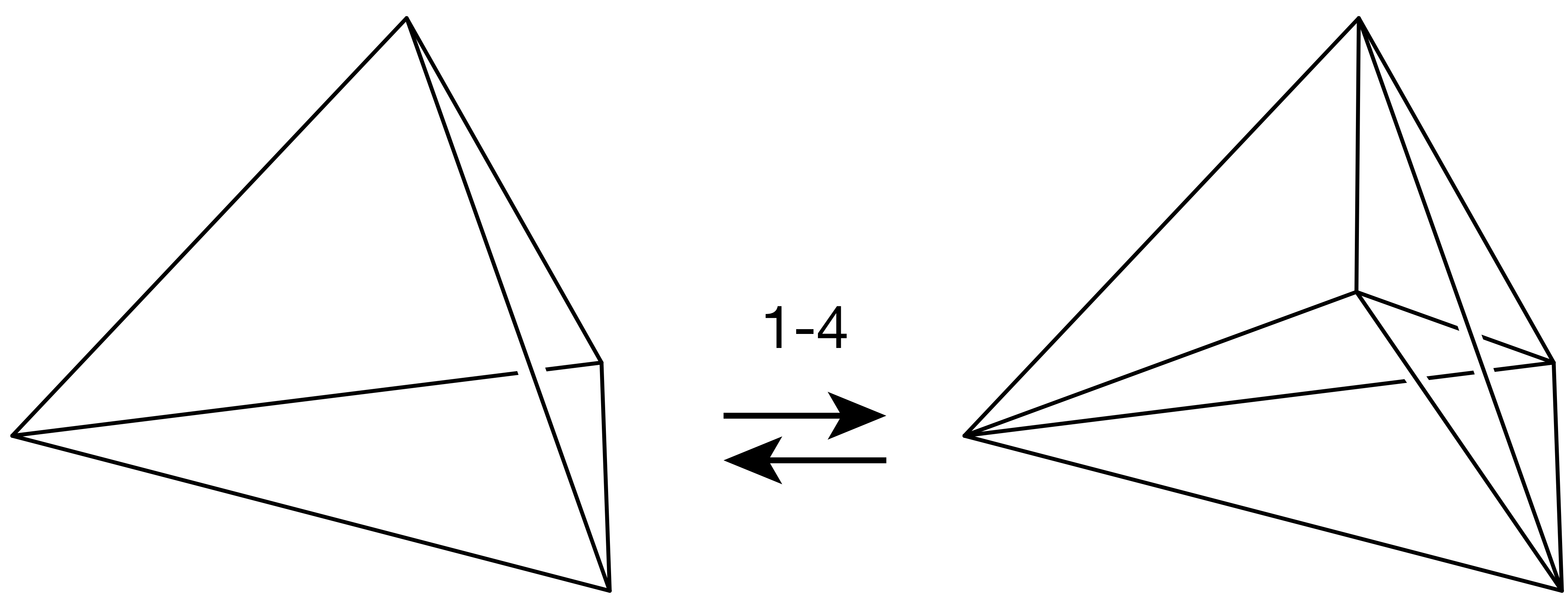


In part of the proof, we have to carry the circular order and flattening through 2-3, 3-2, 0-2, and 2-0 moves, from a potential veering triangulation to a known triangulation.

The intermediate triangulations must have essential edges in order to carry the circular order and flattening.

Ideas in the proof of connectivity

Lemma (Kalelkar, Schleimer, S, following Casali):
Let M be a three-manifold with non-empty boundary.
Suppose that T and T' are essential ideal triangulations of M.
Then there is a sequence of essential partially ideal triangulations connecting T to T', where consecutive triangulations are related by 2-3, 3-2, 1-4, and 4-1 moves.



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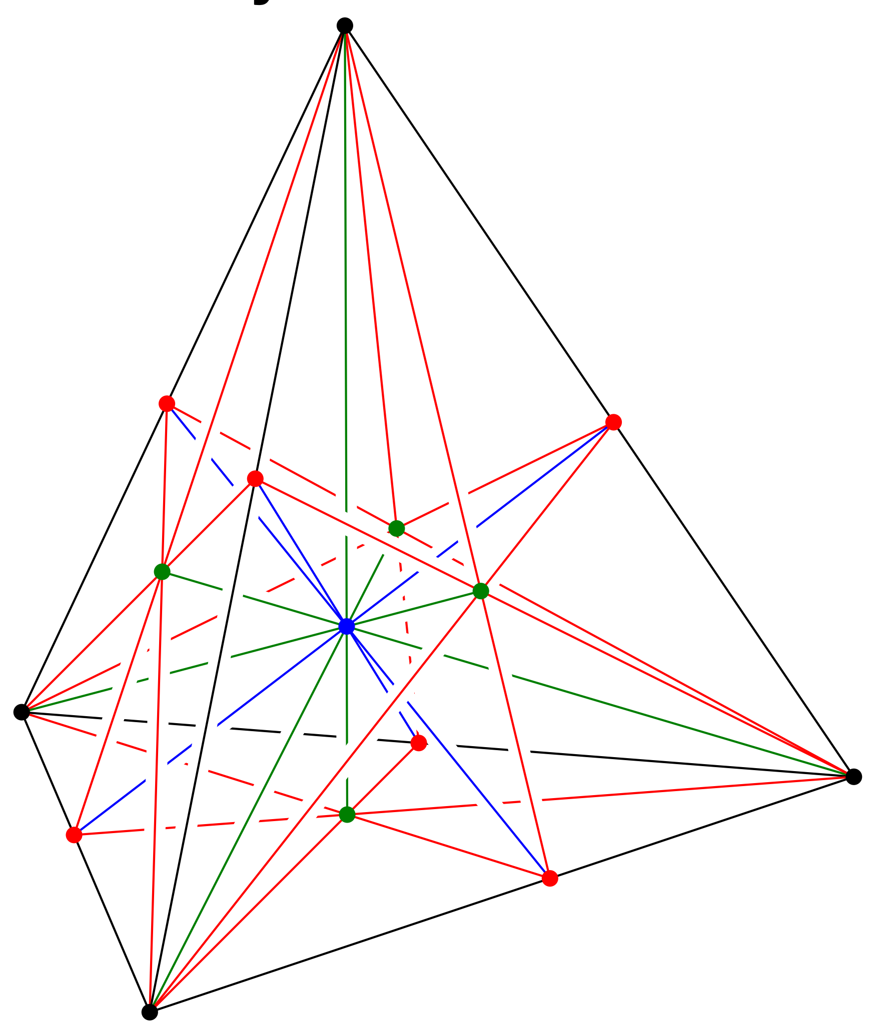

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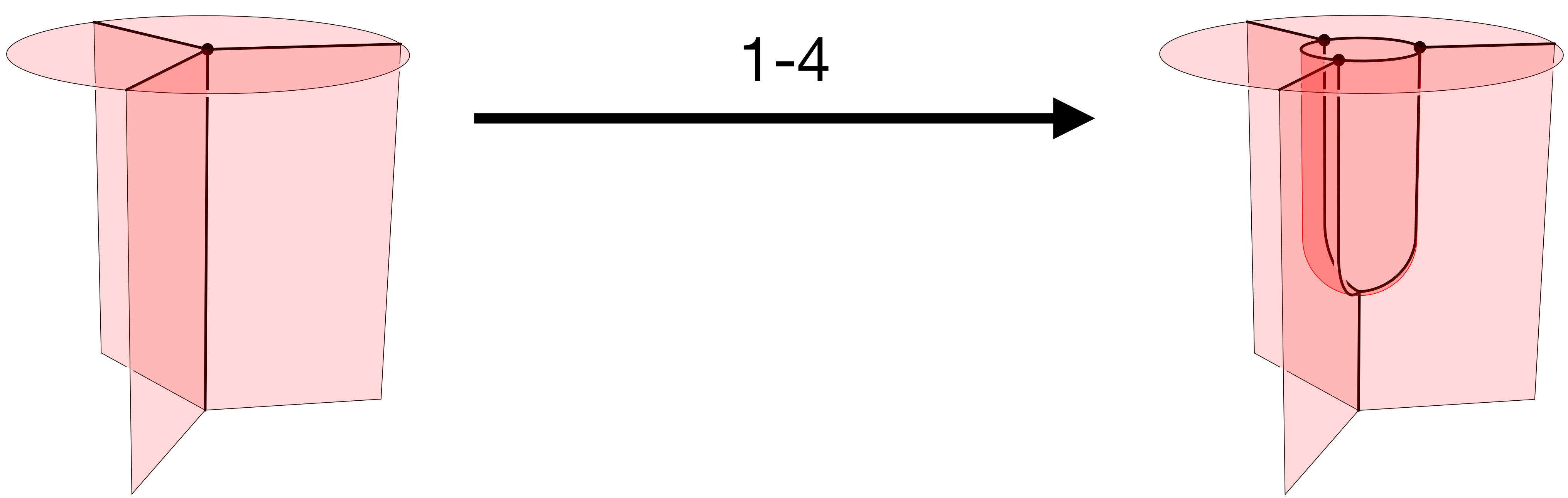

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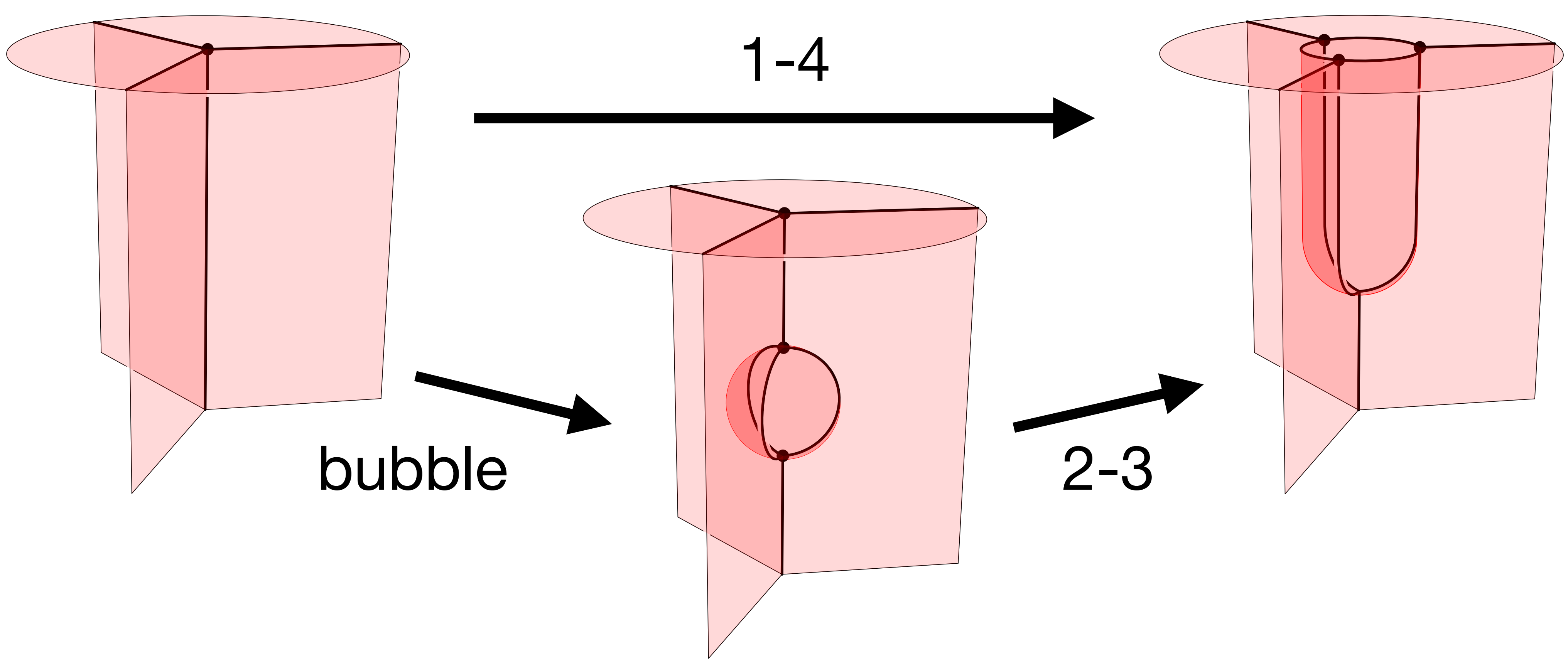
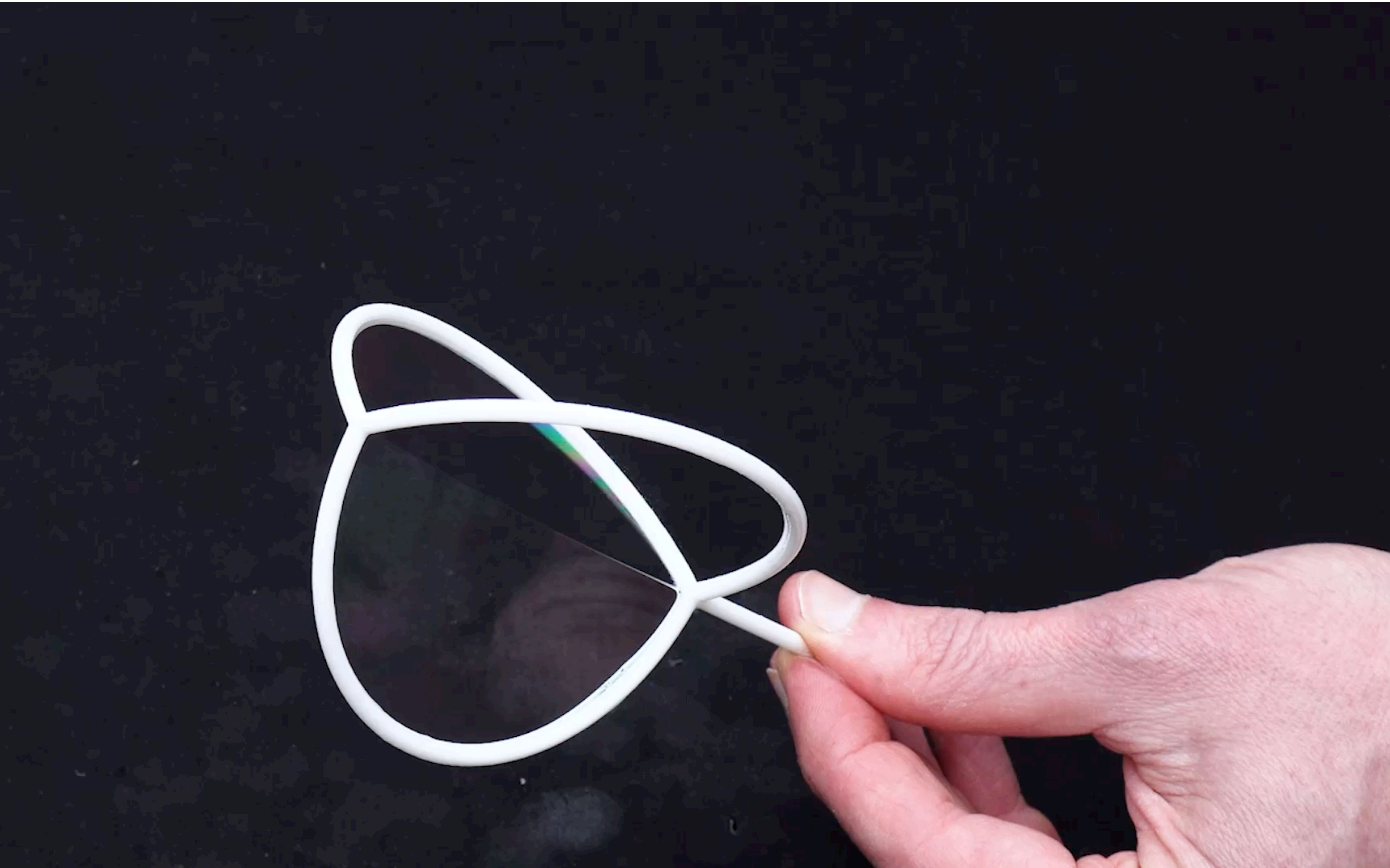


Bubble move



Ideas in the proof

so that $\widetilde{M}$ has infinitely many boundary components

Theorem (Kalelkar, Schleimer, S):

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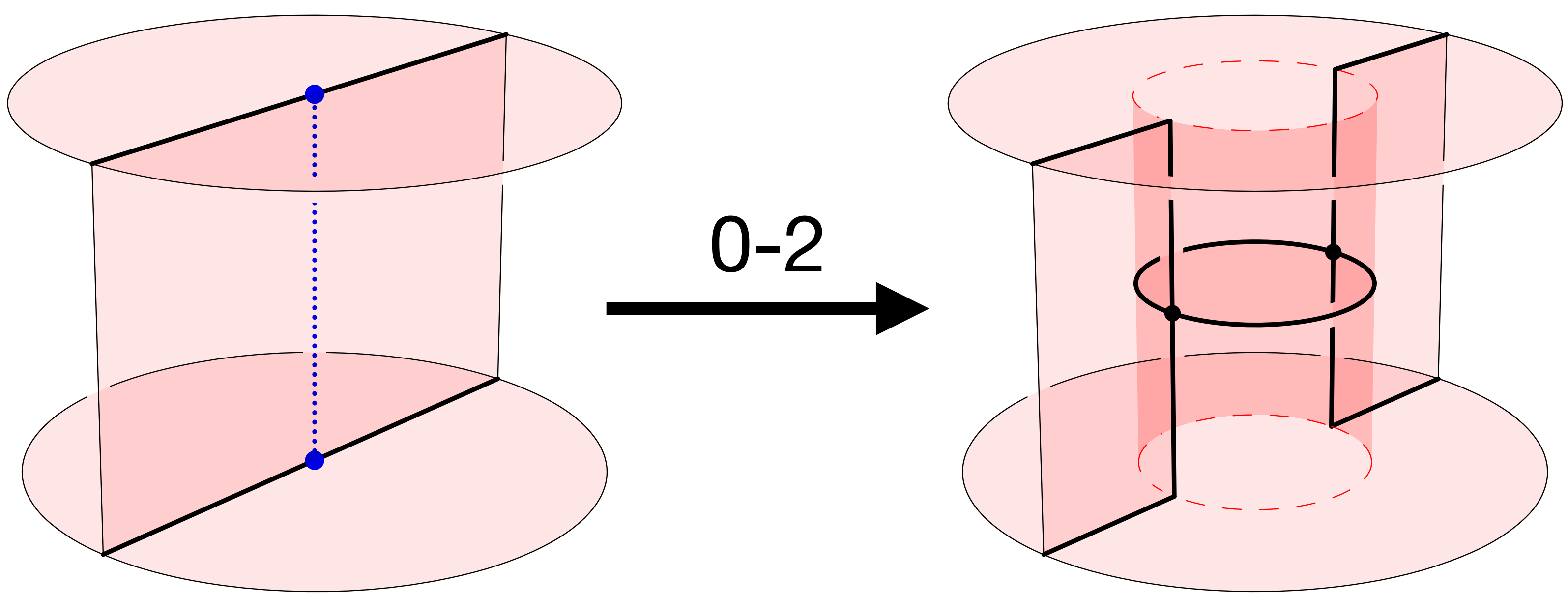
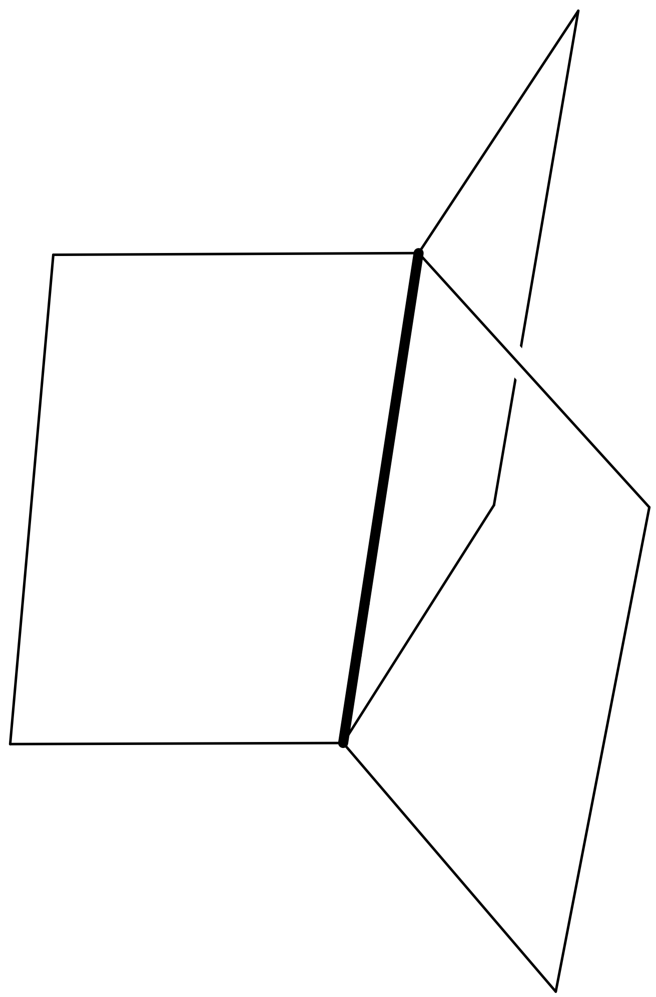
Then there is a sequence of essential ~~partially~~ ideal triangulations connecting T to T', where consecutive triangulations are related by 2-3, 3-2, ~~1-4, and 4-1~~ ~~bubble and reverse bubble~~ 0-2 and 2-0 moves.



We must closely shadow a sequence of triangulations that includes bubble moves, but not actually *do* any bubble moves.

If you need to do a bubble move…



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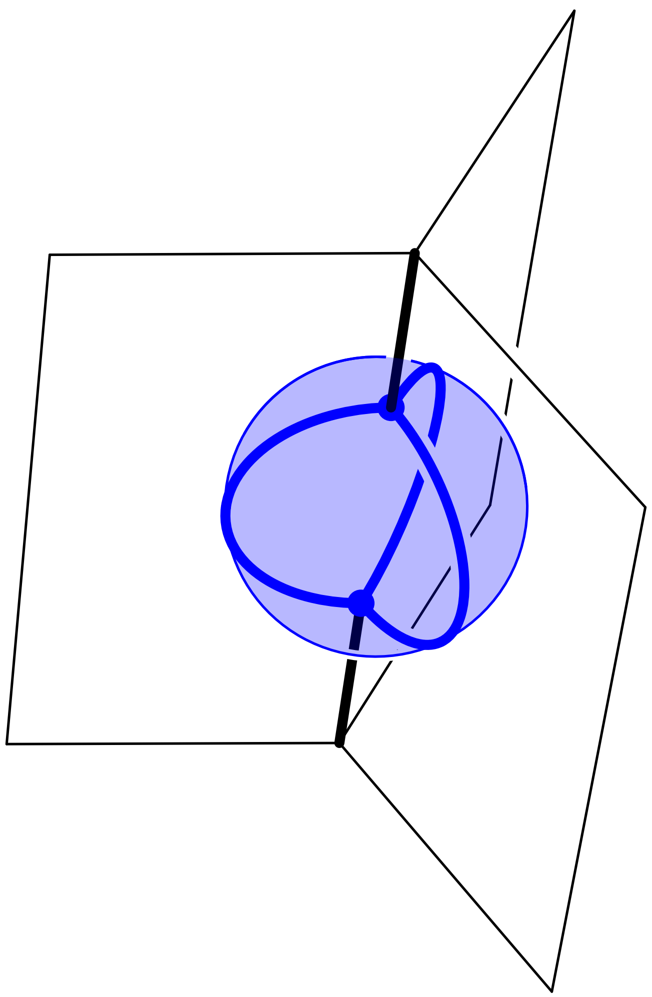
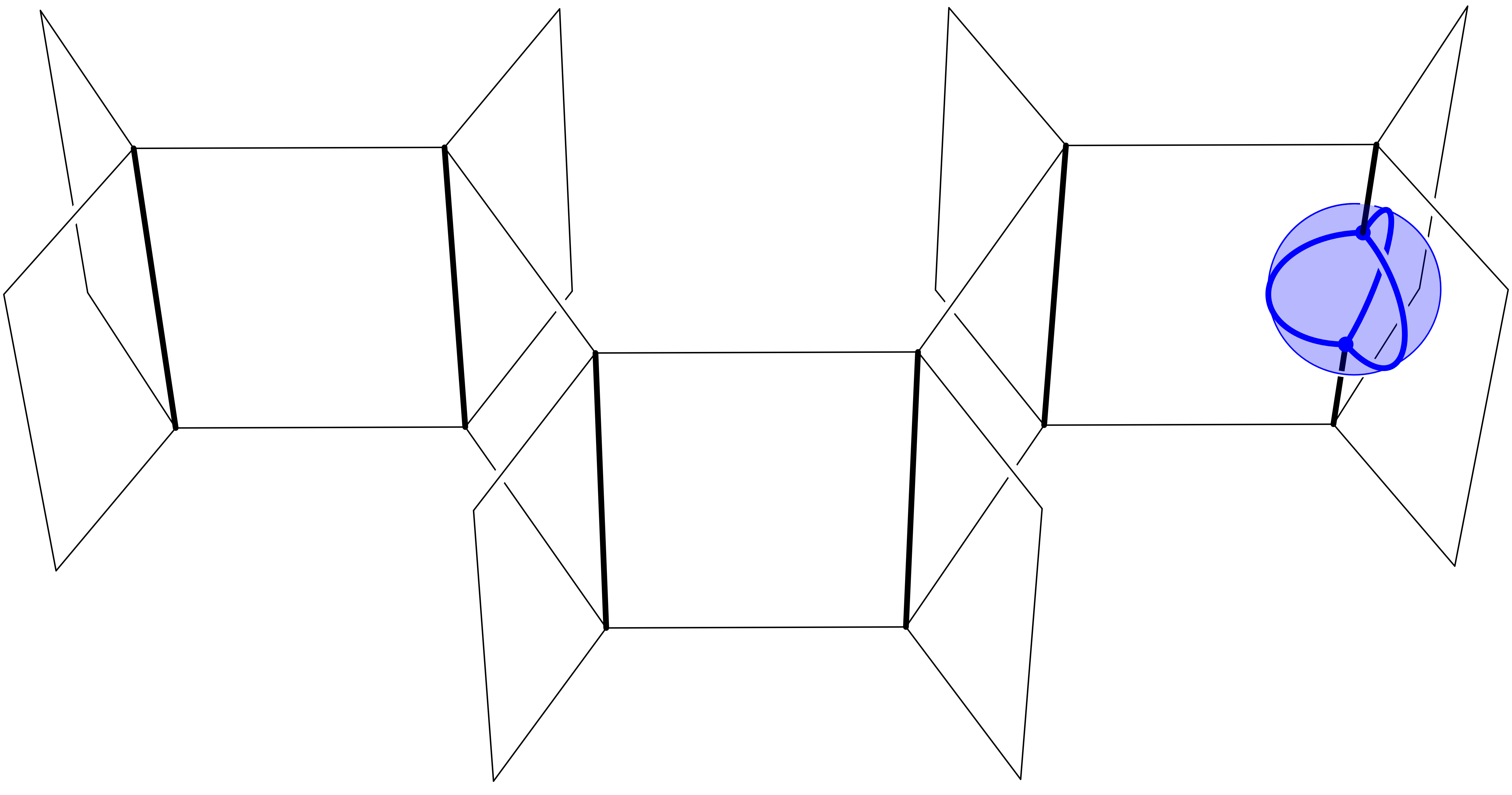


If you need to do a bubble move…

Find a “distant” complementary region ($\widetilde{M}$ has infinitely many).



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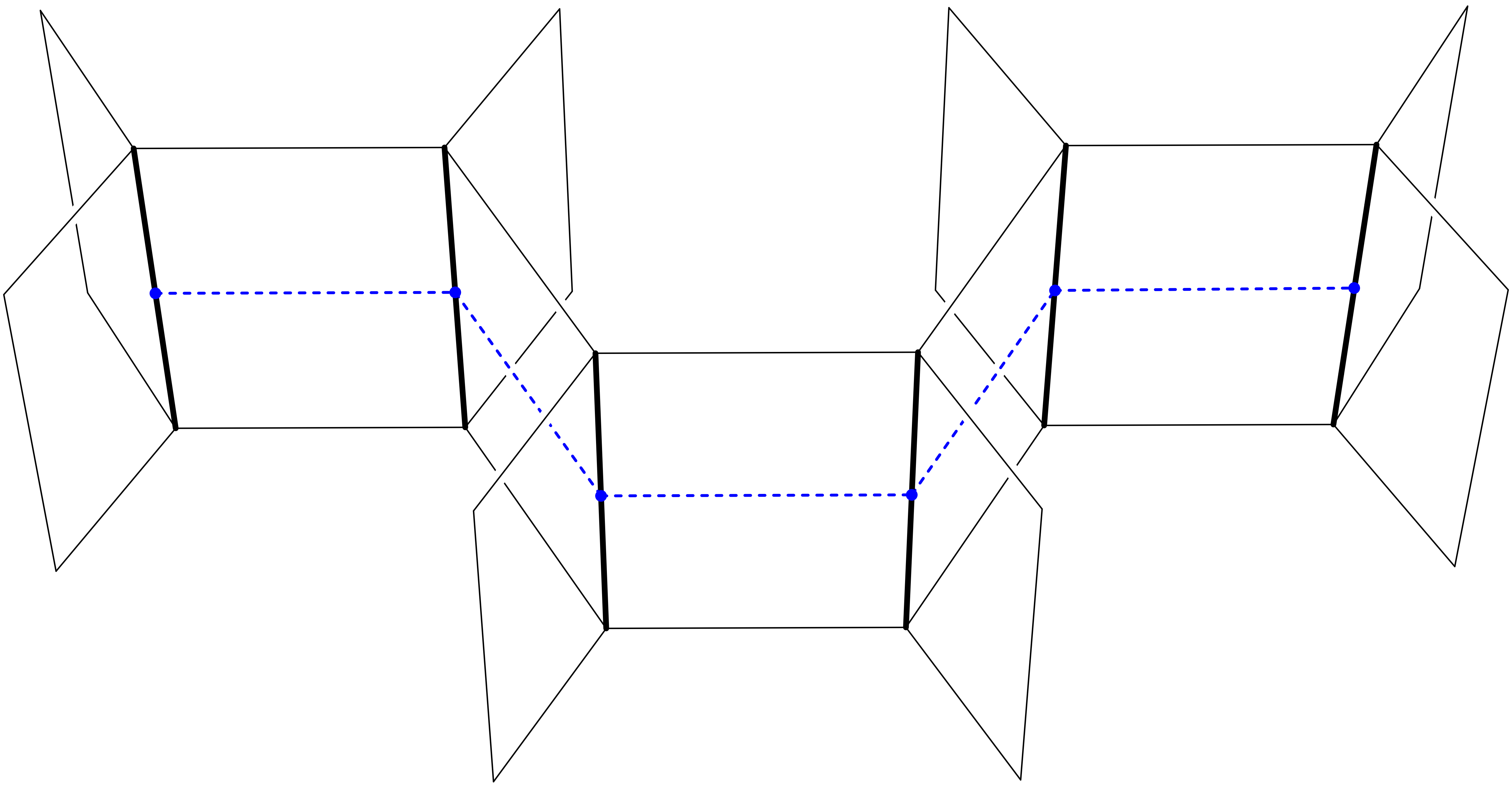

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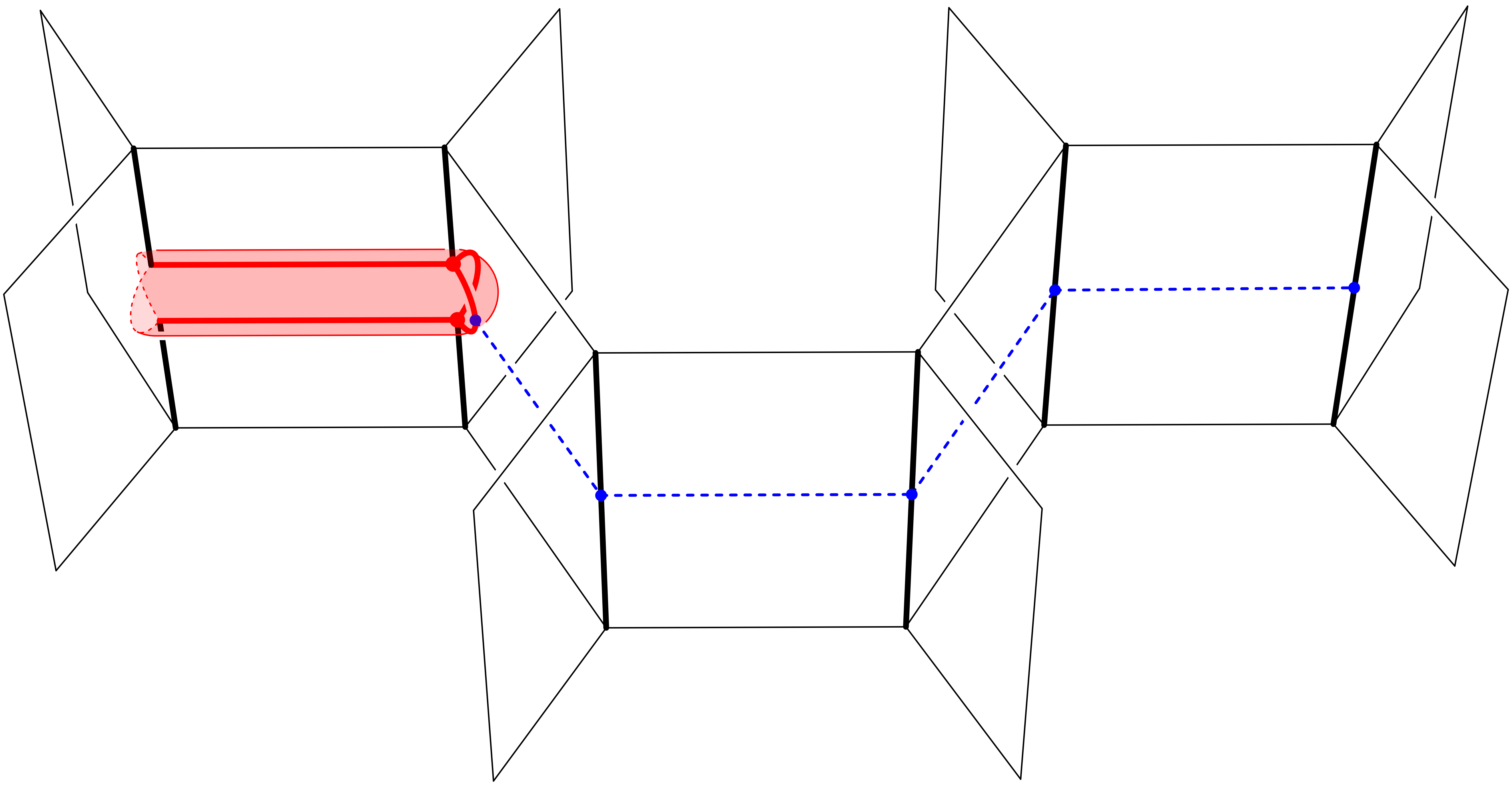
Build a *snake* out of 0-2 moves that connects to that distant region.



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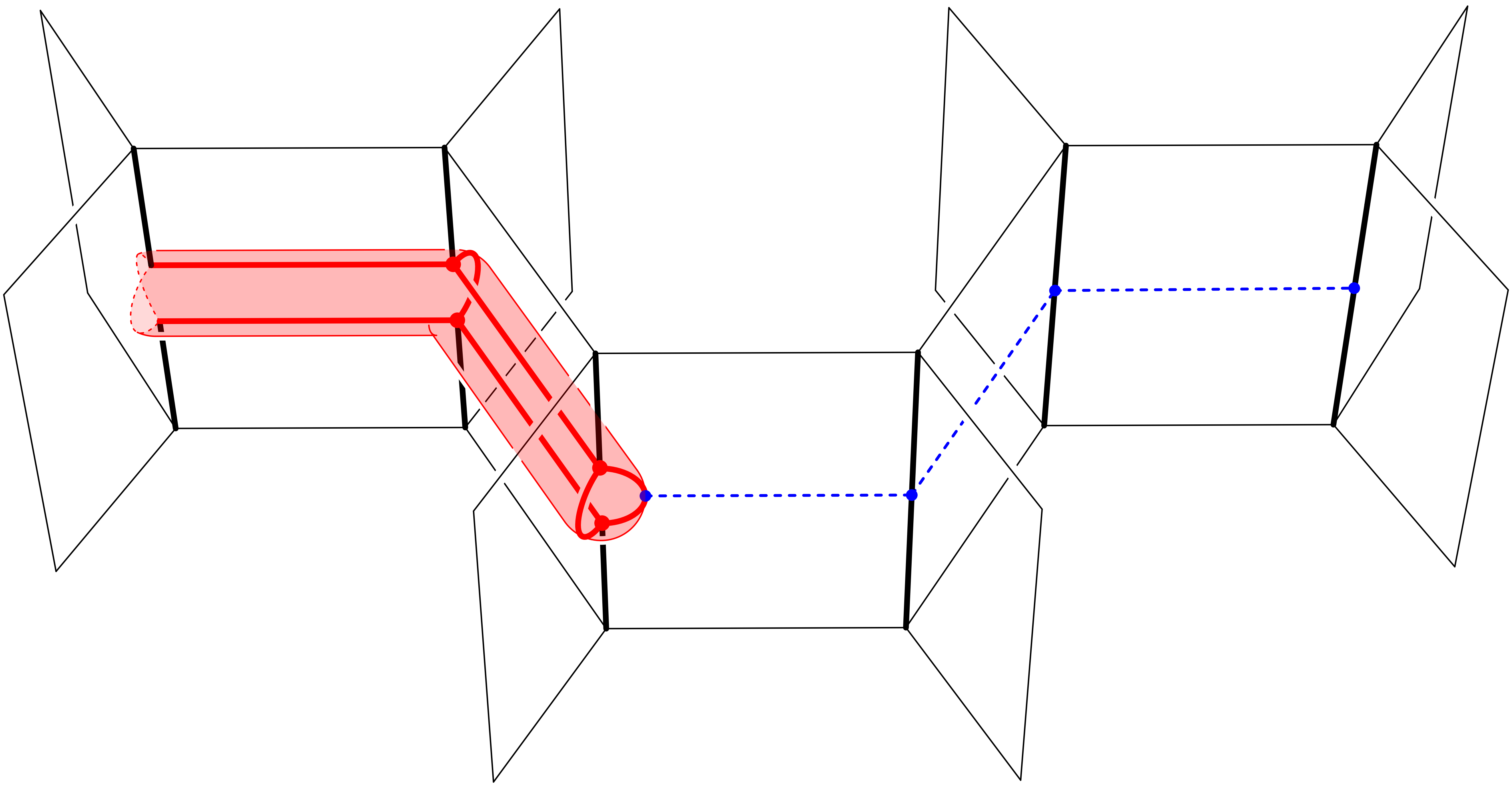

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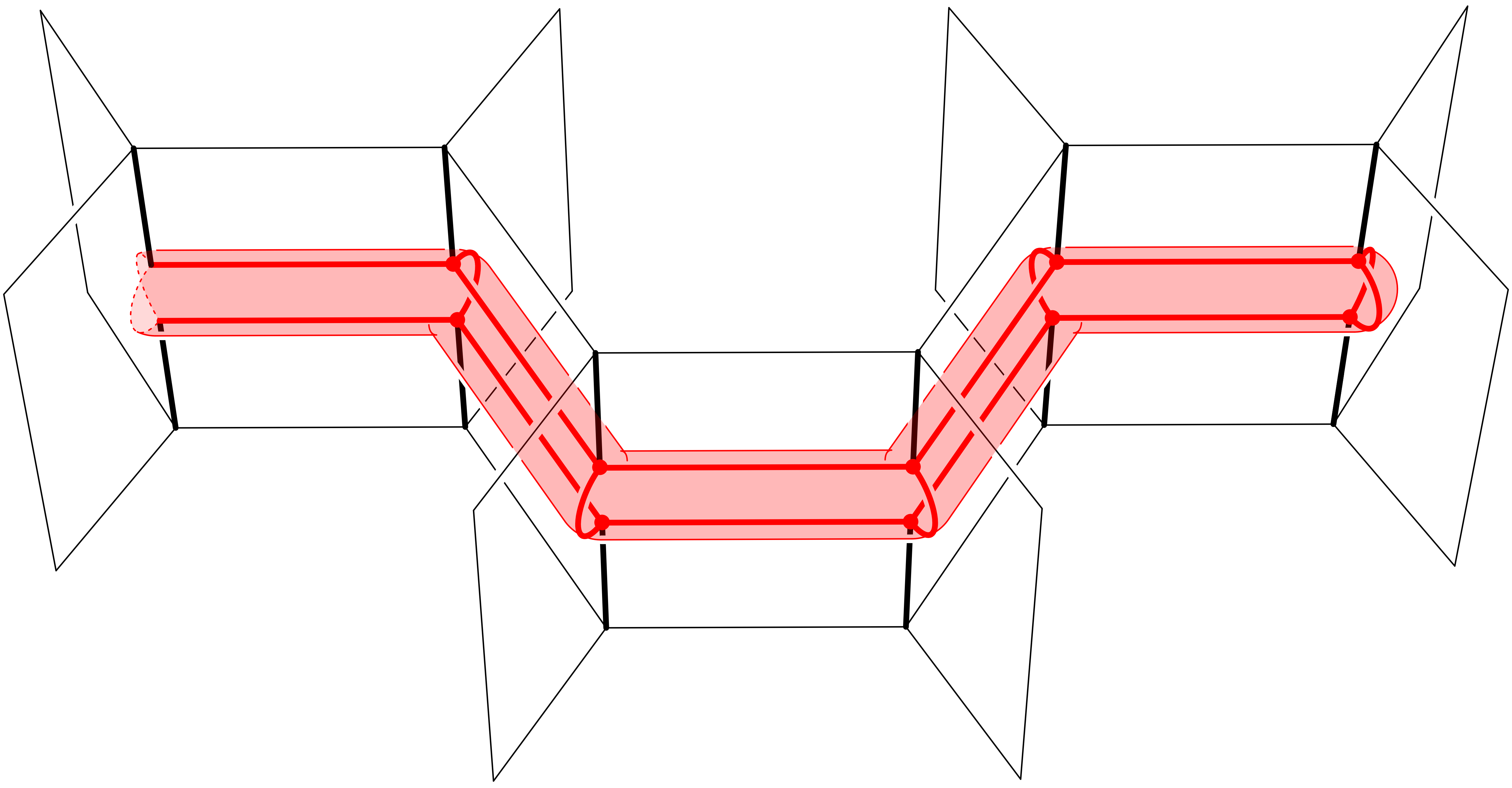
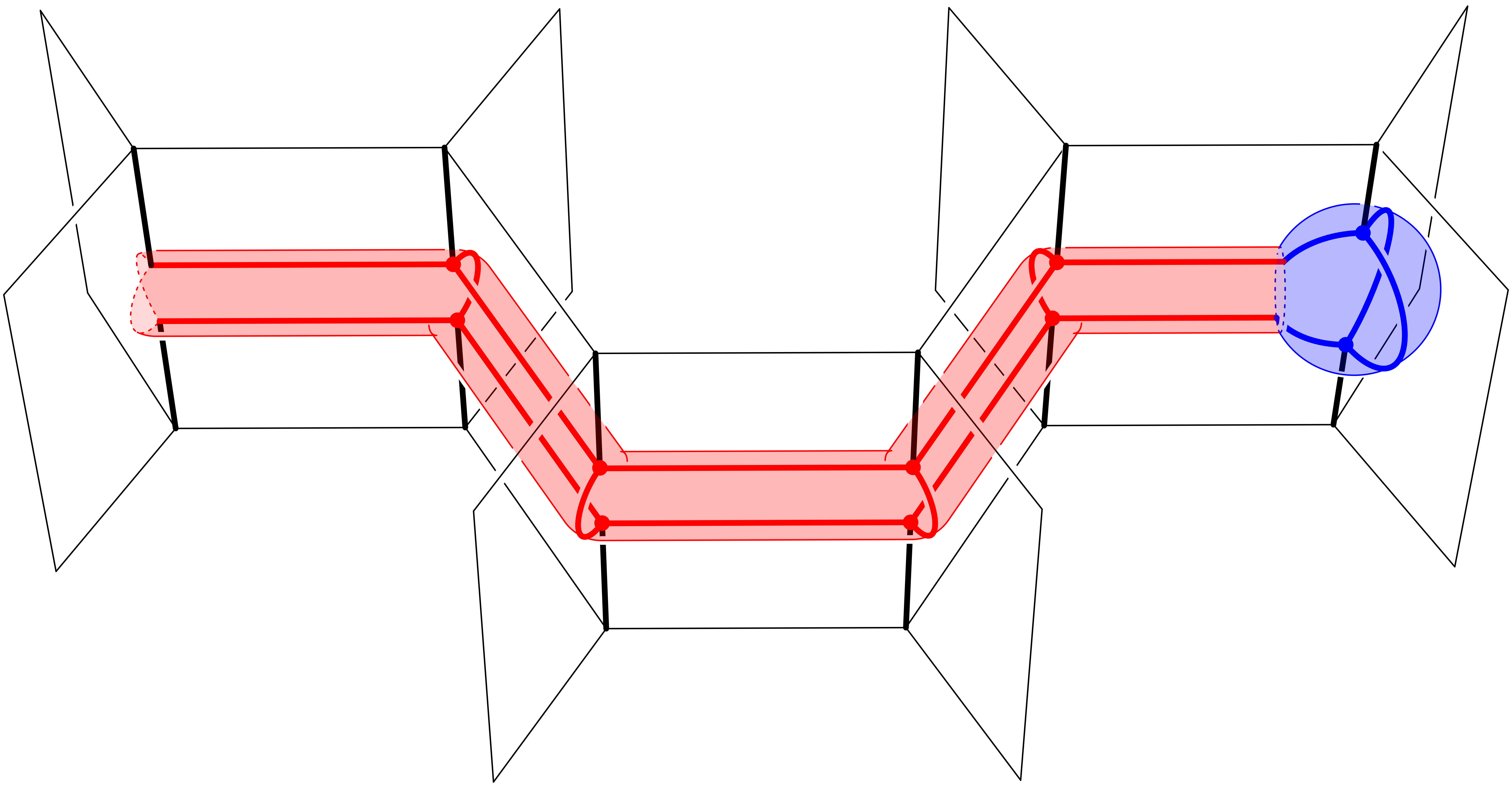


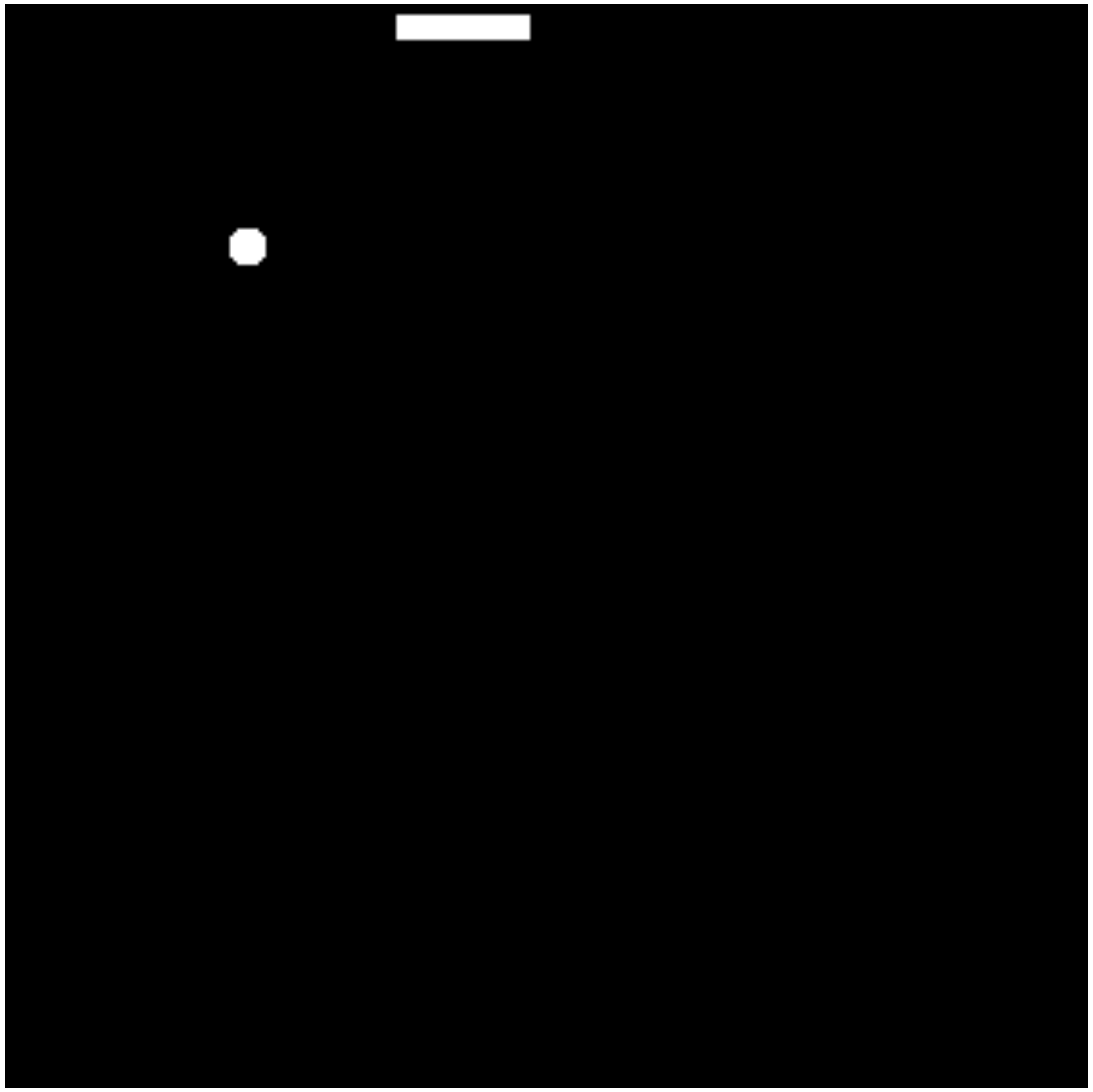
If you need to do a bubble move…

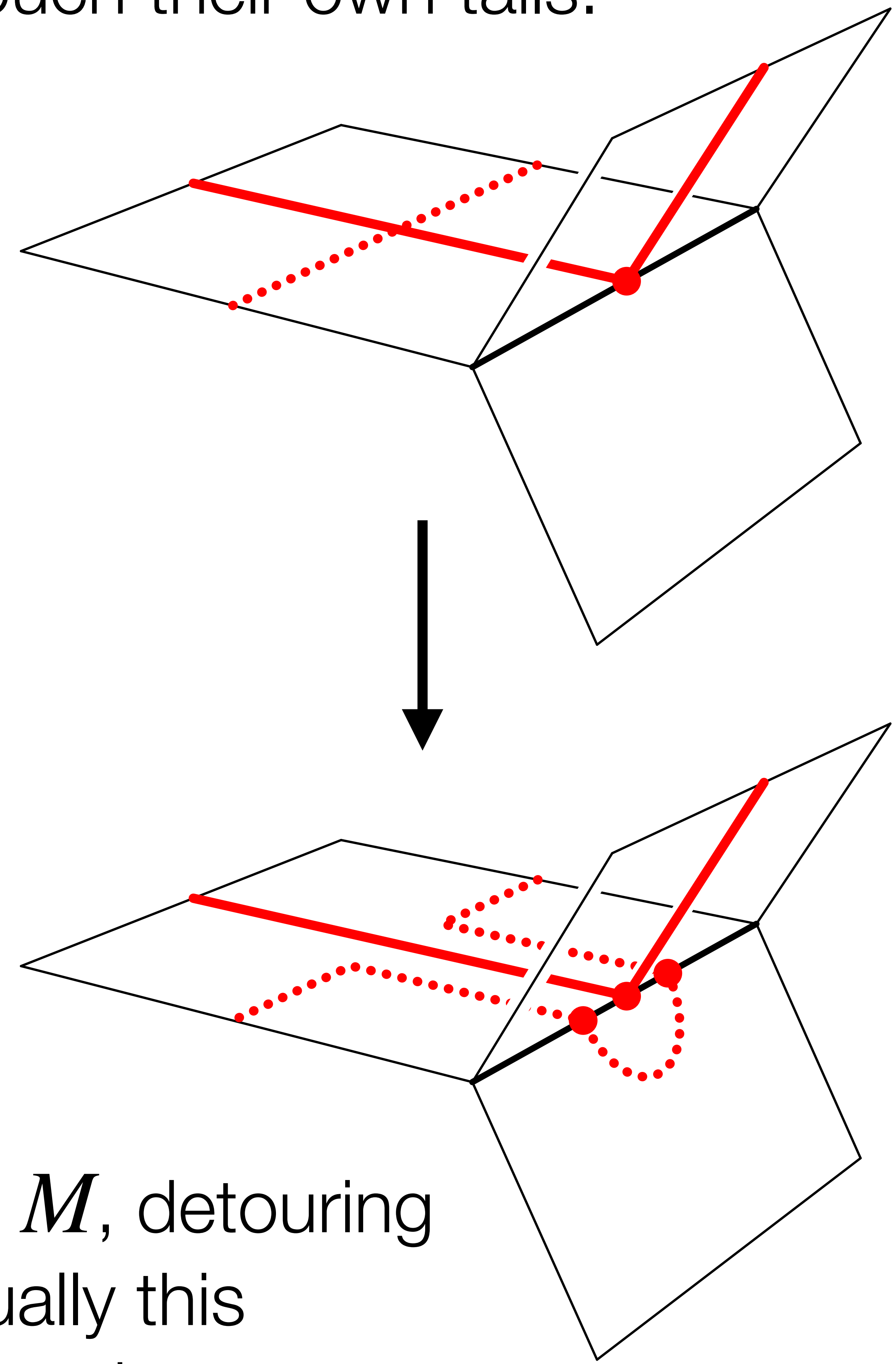
Find a "distant" complementary region ($\widetilde{M}$ has infinitely many).

Build a *snake* out of 0-2 moves that connects to that distant region and makes a "fake bubble".



As is well known, snakes cannot touch their own tails.

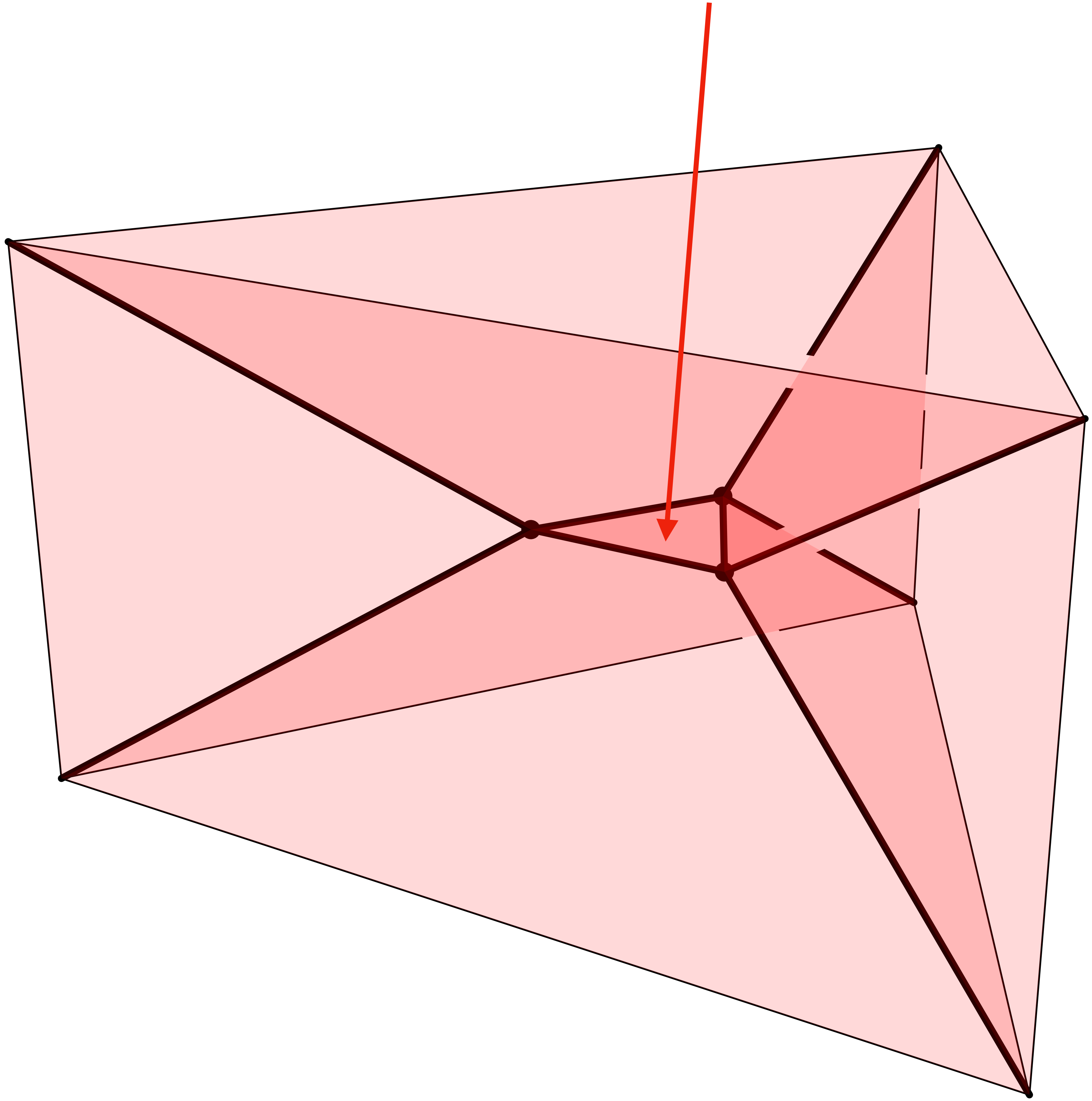




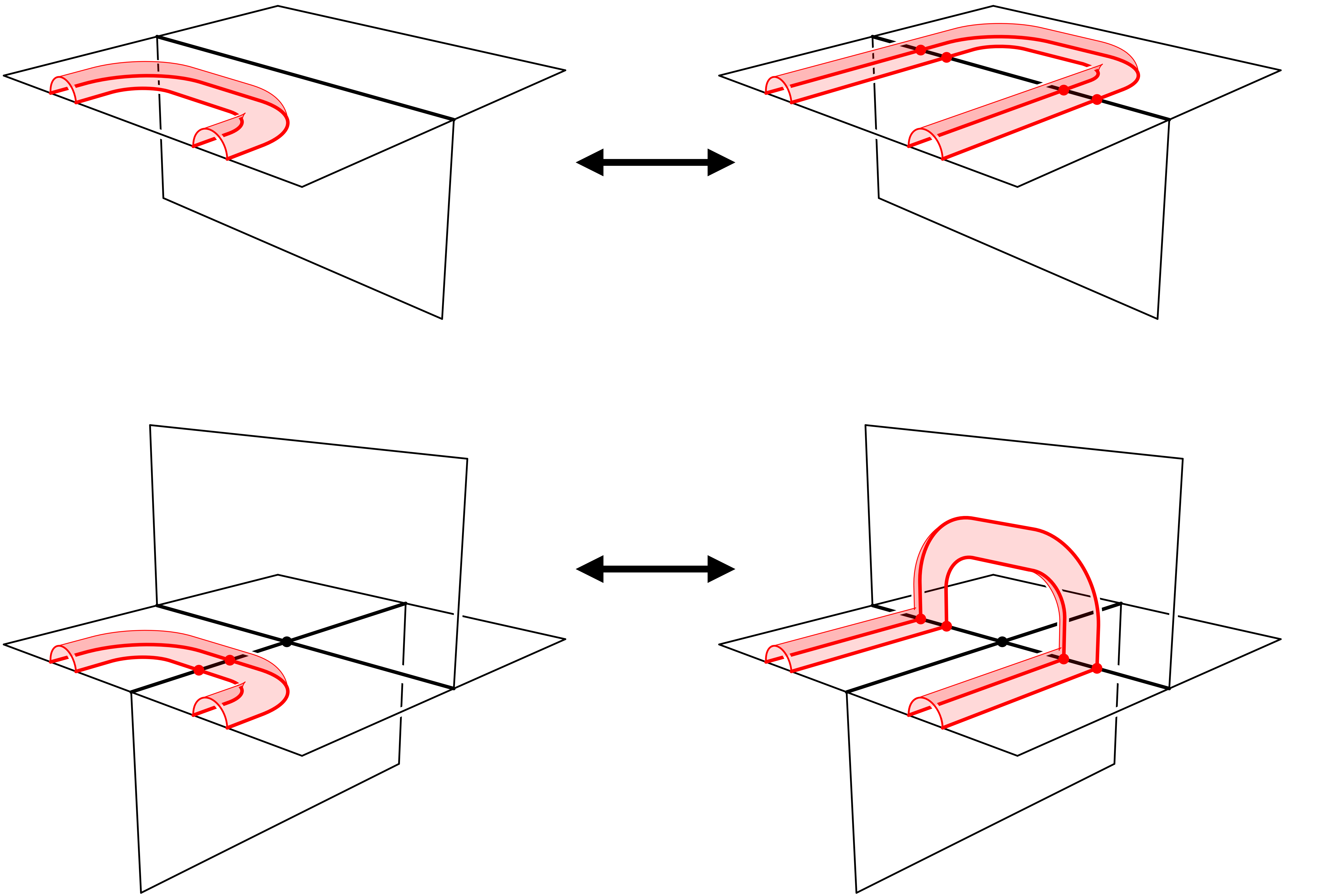
We process along an input path in M, detouring around previous segments. Eventually this produces a path with no self-intersections.

If you need to do a 3-2 move…

I’ve had it with these snakes on this subset of the plane



Slithering snakes



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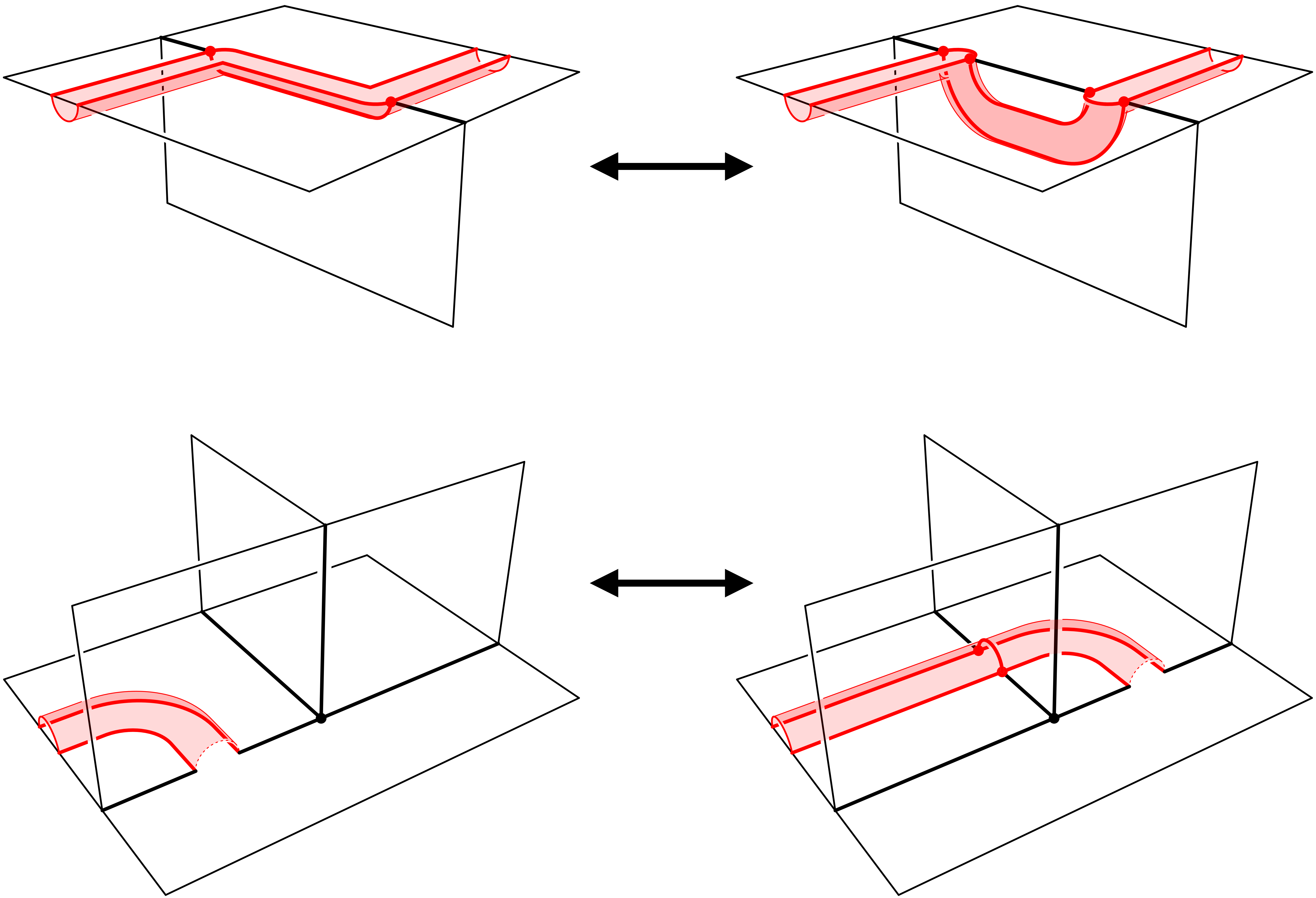

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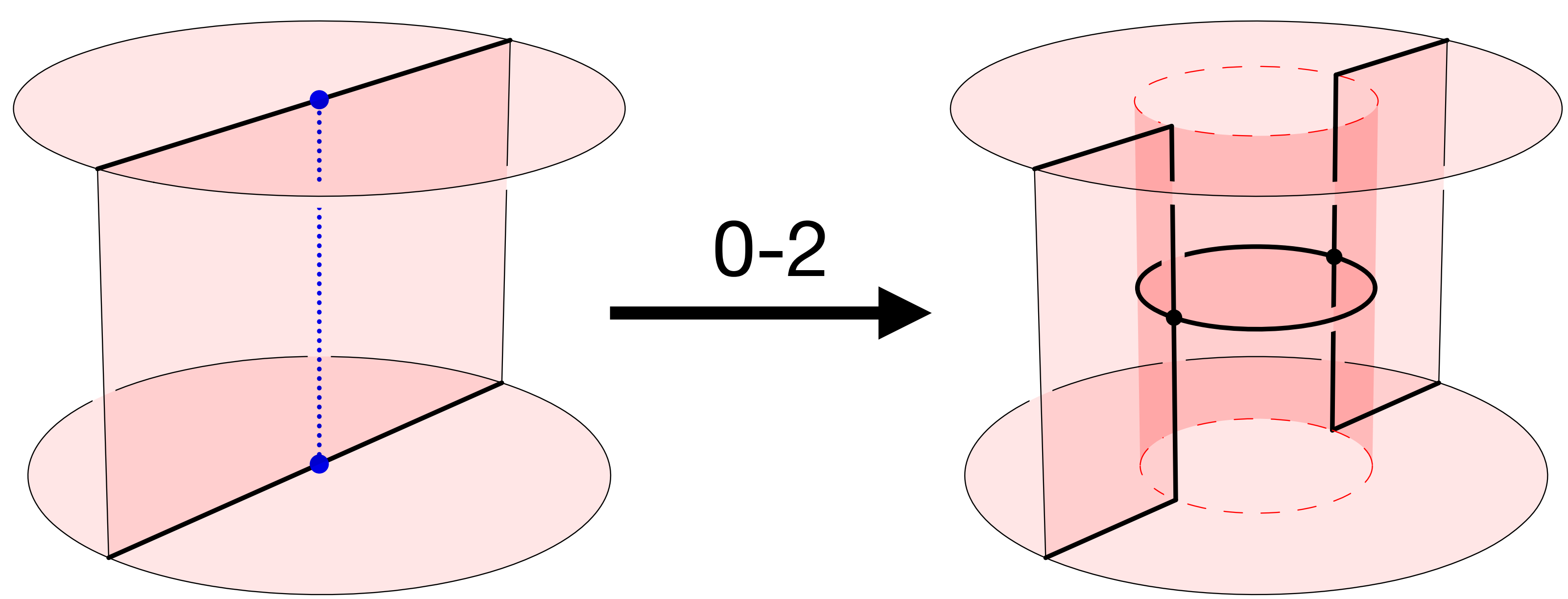
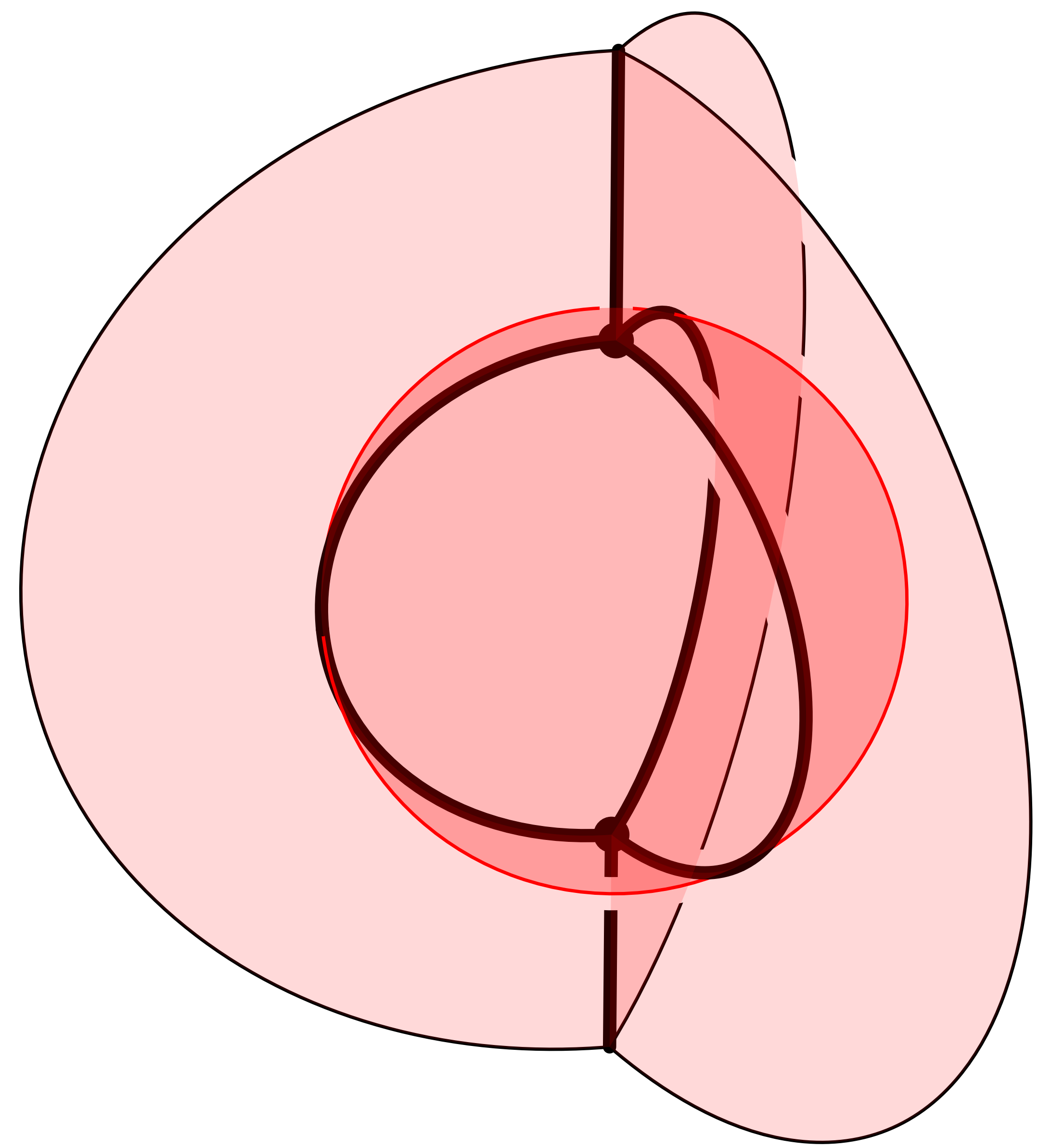


Can we remove the 0-2 and 2-0 moves as well?

This foam of S^3 has two vertices and four complementary regions.

All faces are essential (have different regions either side) but any 2-3 move makes an inessential face.



Can we remove the 0-2 and 2-0 moves as well?

Mirror this across a torus to get a foam of $S^2 \times S^1$ with two vertices.

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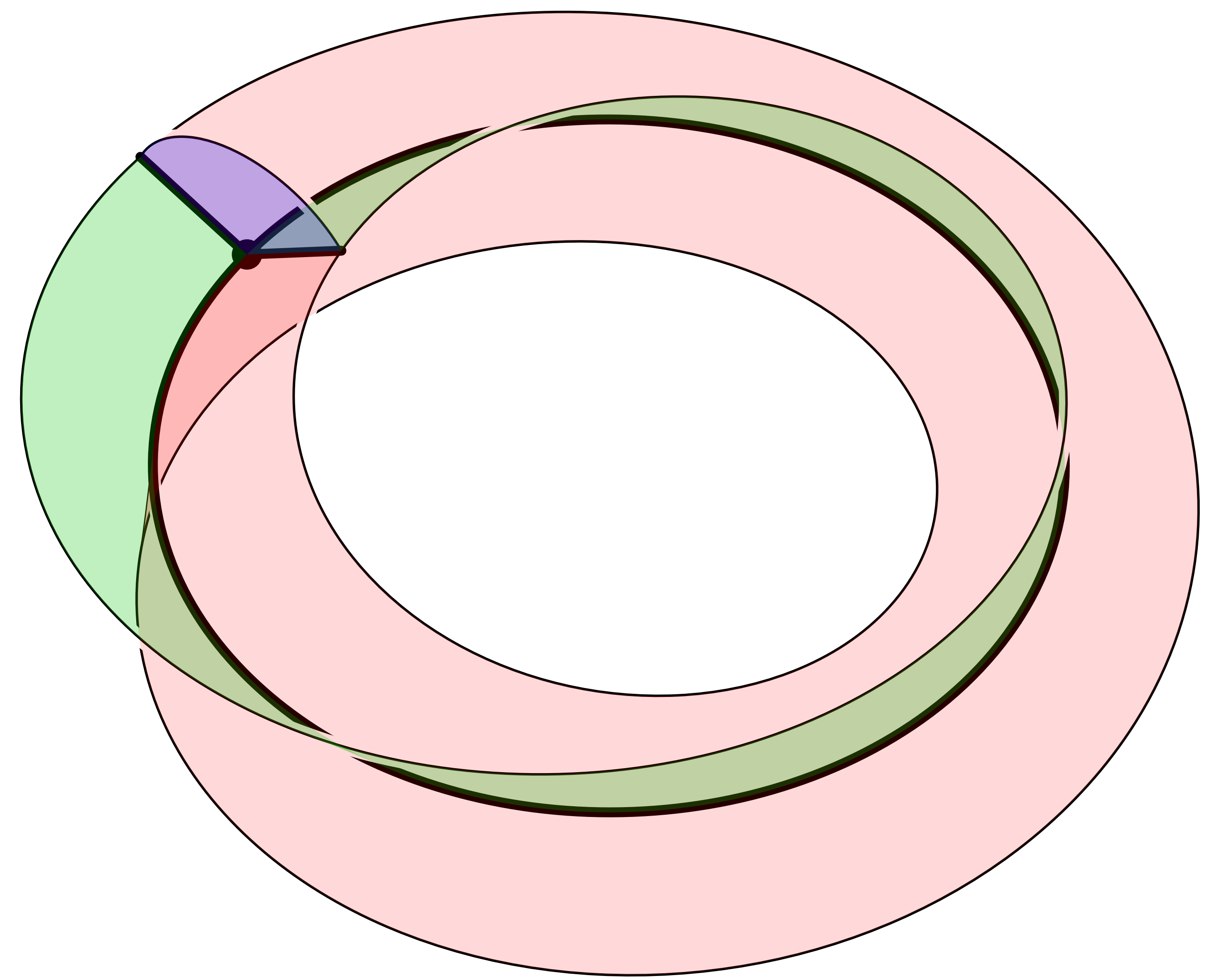
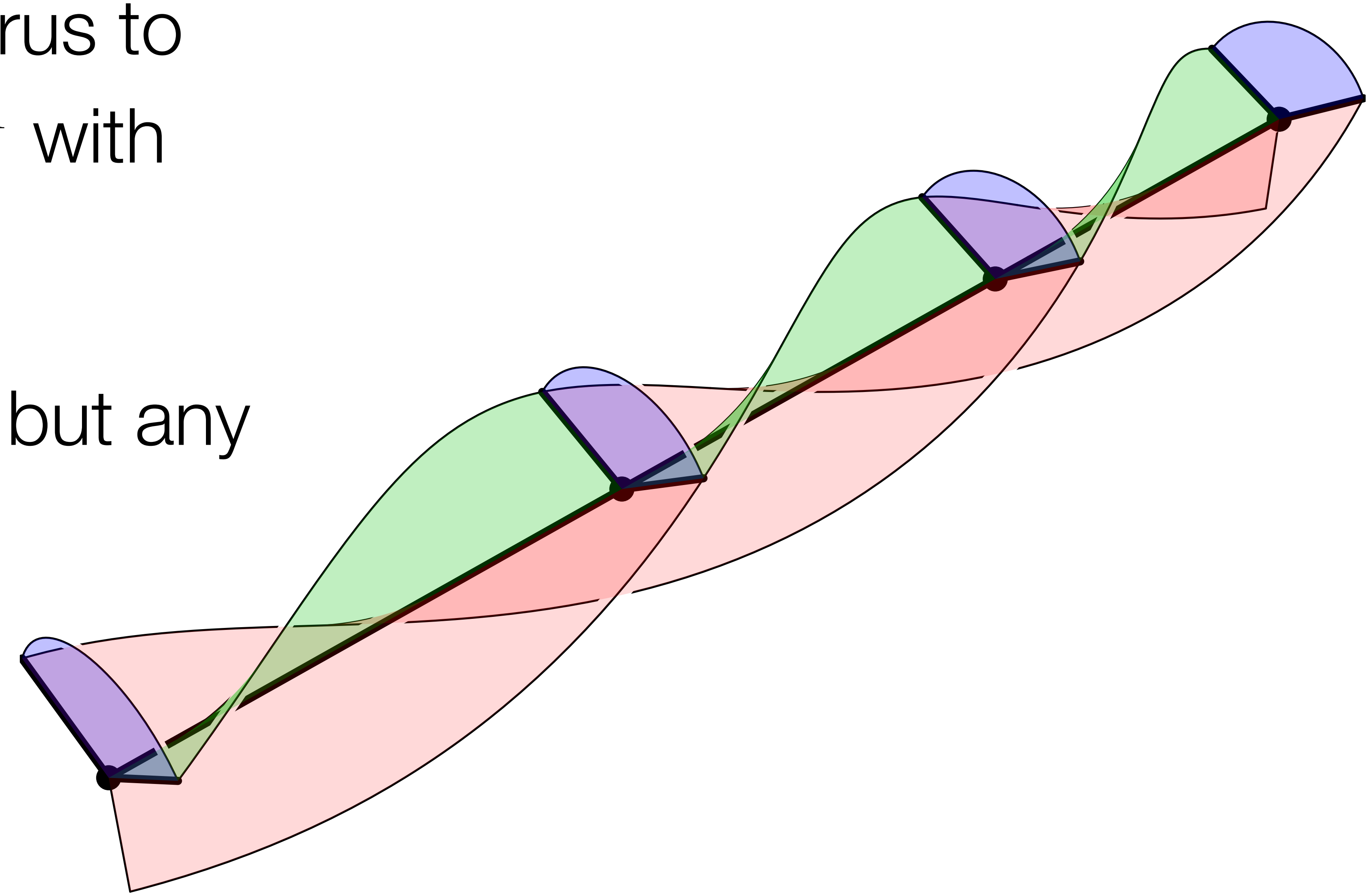


Can we remove the 0-2 and 2-0 moves as well?

Mirror this across a torus to get a foam of $S^2 \times S^1$ with two vertices.

All faces are essential but any 2-3 move makes an inessential face.



Can we remove the 0-2 and 2-0 moves as well?

so that $\widetilde{M}$ has infinitely many boundary components

Theorem* (Kalelkar, Schleimer, S, ’24?):
Let M be a three-manifold with non-empty boundary.
Then the set of essential ideal triangulations of M is connected via 2-3 and 3-2 moves, excepting any isolated ideal triangulations.

The general result

Let Δ_M be the set of boundary components of $\widetilde{M}$. Let $\widetilde{\mathscr{T}}$ be the induced triangulation of $\widetilde{M}$. Given any set of *labels* $\mathscr{L}$, a *labelling* is a $\pi_1(M)$-equivariant function $L : \Delta_M \to \mathscr{L}$.
We say that a triangulation is L-*essential* if no edge of $\widetilde{\mathscr{T}}$ has the same label at either end.

For example, if L is the identity function then a triangulation is L-essential if and only if it is essential.

Theorem (Kalelkar, Schleimer, S):
Suppose that L is a labelling of Δ_M with infinite image. Then:

1. There is an L–essential ideal triangulation of M.
2. The set of L–essential ideal triangulations of M is connected via 2-3, 3-2, 0-2, and 2-0 moves.

The general result

Let Δ_M be the set of boundary components of $\widetilde{M}$. Let $\widetilde{\mathscr{T}}$ be the induced triangulation of $\widetilde{M}$. Given any set of *labels* $\mathscr{L}$, a *labelling* is a $\pi_1(M)$-equivariant function $L : \Delta_M \to \mathscr{L}$. We say that a triangulation is L-*essential* if no edge of $\widetilde{\mathscr{T}}$ has the same label at either end.

Suppose that $\rho : \pi_1(M) \to$ PSL$(2,\mathbb{C})$ is a representation. We can define a labelling L by equivariantly choosing, for each $c \in \Delta_M$, a fixed point at infinity of $\rho(\text{Stab}(c))$.

Then a triangulation $\mathscr{T}$ is L-*essential* if and only if there is a solution to Thurston's gluing equations on $\mathscr{T}$ that recovers ρ.


Thanks!