Research Statement

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My research lies in the area of partial differential equations (PDEs). In particular, I am interested in the mathematical analysis of equations related to fluid dynamics, such as the Navier-Stokes (NSE) and the Euler (EE) equations, as well as equations arising from NSE and EE through various approximations. These equations are used to model a broad range of phenomena in the natural sciences including but not limited to meteorology, oceanography and engineering (aeronautical construction, plasma physics, and lasers). The solutions of these equations exhibit very complex (chaotic or turbulent) behavior, thus rigorous mathematical treatment is crucial for a better understanding of those models and to provide proper guidelines for numerical computations and other applications.

The Navier-Stokes equations read

$$\partial_t u - \nu \Delta u + \sum_{j=1}^{3} \partial_j (u_j u) + \nabla p = f$$

$$\text{div u} = 0$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p(x,t)$ denote the unknown velocity and the pressure, $f(x,t)$ is the given external force. We consider the Cauchy problem with initial data $u(x,0) = u_0(x)$. There are many open problems regarding the NSE when the velocity and the pressure are considered on the whole $\mathbb{R}^3$ or are periodic functions (which amounts to say that $u(x,t)$ and $p(x,t)$ are given on a torus $\mathbb{T}^3$). If the problem is considered on a bounded domain, we equip it with suitable boundary conditions. Due to the effect of viscosity, the fluid is believed to stick to the boundary. Mathematically, this translates into the Diriichlet boundary conditions, namely $u(x,t) = 0$ on the boundary of the domain. The Euler equations are believed to describe the motion of inviscid fluids, that is when $\nu = 0$.

The long term goal of my research program is to broaden the analytical understanding of problems related to the above mentioned equations and to develop general mathematical (analytical) tools useful in the analysis of PDEs. Although the well-posedness theory of the Navier-Stokes and Euler equations gathered great interest of mathematicians in the past decades, the ubiquity of their applications offers a great source of unanswered analytical questions. Throughout my academic career, individually and with a group of mentors who later became collaborators, we obtained a variety of results concerning the above mentioned class of equations. Below, we present a brief summary of our work in the context of present literature and contemporary research directions. The second part of this statement contains a more detailed exposition of these results aimed at readers who are interested in their full formulation and analytical and technical apparatus used in the proofs.

Although the fundamental equations of the fluid dynamics have been formulated in the 18$^{\text{th}}$ and 19$^{\text{th}}$ century, the question of their well-posedness has remained an open problem. Regarding the Navier-Stokes equations, it is believed that the modern mathematical theory was initiated by J. Leray in 1934 in his seminal paper [56], where the global existence of *weak solutions* (potentially not regular in the classical sense) has been proved. T. Kato and H. Fujita in 1962 ([40]) and 1983 ([29]) offered a different way of approaching the problem via *mild solutions* which are known to be regular on their time of existence. The generality of the method limits however the global existence only to the class of initial data which are small in certain scale-invariant norms. In the 1980's and 1990's a french group of mathematicians led by Y. Meyer (see [62] and references therein) embarked on a quest in order to construct global solutions with large initial data. In particular, M. Cannone and Y. Meyer in [14] were able to construct a class of data which are arbitrarily large in the sense of spaces considered by Fujita and Kato. However those data still have to be small in the Koch-Tataru space (BMO^{-1}) to warrant global regularity (the data oscillate fast in all directions making a certain scale-invariant Besov norm sufficiently small). The breakthrough came in the early 2000's with the works of Y. Chemain, I. Gallagher and their collaborators where first examples of truly large (in the sense of the Koch-Tataru space) initial data leading to global regular solutions were constructed ([18, 19, 20]). In [47, 48], in joint work with I. Kukavica and M. Ziane, we provide very general conditions on the initial data which guarantee that the resulting solution is global in time. In particular, in [48] we exhibit a class of initial data, oscillating only in one direction, whose BMO^{-1} norm has algebraic dependence on $1/h$, where h is the period of oscillation. Since our methods allow for inclusion of external forces, the resulting solutions do not decay in time. This greatly extends other results in this direction present in the literature. For instance, the results of I. Gallagher and J.Y. Chemin allow data of the size $|\log 1/h|$. Conditions presented in [48] admit data of order $(1/h)^{1/2}$. In addition, methods used by Gallagher and Chemin in [18, 19, 20] necessitate the absence of external forces and require a rapid decay of corresponding solutions. Since our proofs are based on new anisotropic estimates and a careful analysis of the effect of incorporating oscillations into the original system of equations, they allow non-zero forcing and thus non-decaying solutions. Finally in [49] we prove that the solutions of NSE are smooth under some structural conditions and some natural conditions on the horizontal derivatives of the horizontal components of the velocity and the derivative in the vertical direction. The obtained conditions admit data whose vertical average is large in BMO^{-1}. In this case, unlike previously, our data do not need to oscillate in any direction. The results also allow non-zero external forces and lead to solutions which do not decay in time. A more detailed exposition of the result is contained in the Section 1.1.

With V. Šverák, under the assumption that the NSE admit solutions which become singular in finite time, we prove in [85] that there exists a non-empty set of initial data which lead to singularities, and whose norm is minimal in the sense of the Kato-Fujita condition. Moreover the set of such data is strongly compact modulo translations and dilations consistent with the scaling of the equations. In the same spirit but using different techniques, in [82] we consider the stability of global solutions, where we analyze sets of minimal perturbations, which cause finite time singularities. There is strong evidence that the described properties do not hold for many equations with the same scaling as NSE, thus analysis of this problem sheds new light on features specific to NSE. Interested readers may find more details in Section 1.2.

A different way of tackling the problem of regularity of weak solutions of the NSE was proposed in 1976 by V. Sheffer [86, 87] and later in 1984 by L. Cafarelli, R. Kohn and L. Nirenberg [12]. The goal is to estimate the size of the set of possible singular points. This approach has been dubbed as the *partial regularity*. The inherent feature of this theory is that it is local, that is the considered scale-invariant quantities are localized to a space-time cylinder $Q_r(x_0, t_0)$ in some sense centered at the point of interest (x_0, t_0). Apart of estimating the measure of the possible singular set, one of the main results of this theory are regularity criteria which state that if a certain scale-invariant quantity is sufficiently small on $Q_r(x_0, t_0)$ then (x_0, t_0) is a regular point. The partial regularity of the Navier-Stokes equations has been addressed in various contexts and many regularity criteria have been proposed. Motivated by the work on anisotropic estimates, with I. Kukavica and M. Ziane in [50] we propose a local regularity criterion which takes into account only one component of the velocity field, say $u_3(x, t)$. In addition, we prove a partial regularity criterion based on the pressure $p(x, t)$. The presented approach and obtained criteria allow also for improvement of the possible blow-up rates described by Leray in [56]. For the precise statement of the results and methodology of the proof please refer to Section 1.3.

In order to exhibit specific features of certain physical situations and capture only the relevant dynamics of the model not obscured by phenomena unrelated to the considered setting, one often makes an ansatz transforming the full system of Navier-Stokes equations into a related system suitable for further analysis. A particular example of this procedure are the primitive equations (PE) used in meteorology and climate modeling. The main mathematical difference between the PE and NSE is that the pressure is two-dimensional, however the third component of the velocity field is more singular. The mathematical theory of the primitive equations dates back to the works of P. Lions, R. Temam and S. Wang. In terms of global well-posedness of solutions, the state of the art result proved by C. Cao and E.Titi concerns solutions with initial data in H^1, that is with square-integrable gradient. In [46] we address the question whether there exists a class of initial data which does not require differentiability (even in the weak sense), yet still yields global unique solutions. The answer is affirmative and we are able to prove that the primitive equations are well-posed for initial data in the class of continuous functions. We note that the choice of this class is mostly motivated by the uniqueness questions. Section 1.4 contains a more extensive description of the result and its place in literature.

In order to describe physical phenomena, the fluid dynamics equations (in particular NSE) are often coupled with equations describing the evolution of other physical quantities relevant to the considered model. In the case of the *geodynamo*, which is the essential process by which the rotating, convecting, electrically conducting molten iron in the Earth's fluid core maintains the geomagnetic field counteracting the ohmic decay, NSE are coupled with the magnetic Maxwell equations to obtain the magneto-hydrodynamic equations (MHD). The full MHD system is very hard to tract mathematically and numerically, thus Moffat and Loper [63, 64, 65] proposed an appropriate model known as the magneto-geostrophic equation. Physically the model postulates that slow cooling of the Earth leads to slow solidification of the liquid metal core onto the solid inner core and releases latent heat of solidification that drives compositional convection in the fluid core. In [27], S. Friedlander, V. Vicol, and I address the problem of the well-posedness of an active scalar equation (MG) arising in the study of the magneto-geostrophic turbulence. The equation shares a number of properties with the extensively studied surface quasigeostrophic equation (SQG) and the incompressible porous media equation (IPM). However it has also a number of novel features due to the strong singularity and anisotropy of the operator via which the drift velocity is obtained from the underlying scalar. In comparison to SQG and IPM, the singular character of the symbol of the nonlinearity presents additional significant analytical difficulties. The diffusion in this equation is introduced via the fractional Laplacian $(-\Delta)^\gamma$. In [27], we consider the mathematically challenging super-critical regime $\gamma < 1$. We show that there exists another "critical" threshold value ($\gamma = 1/2$), with respect to which the properties of the corresponding solutions may change drastically. In particular, for $\gamma > 1/2$, we obtain a local well-posedness result for arbitrarily large data in Sobolev spaces and global well-posedness if we impose a smallness condition on the size of the data. For $\gamma < 1/2$ the equations are ill-posed in the sense of Hadamard. For the transitional case $\gamma = 1/2$, we show that, while the small data result is still true, there exist data, whose size is large, and for which the associated linear operator has arbitrarily large unstable eigenvalues. This information is used to prove that, in this case, the equation is Hadamard ill-posed. Such dichotomy is not known for other active scalar equations. In turn, recent works of S. Friedlander and A. Suen concern the case where the multiplier symbol incorporates a *viscous* effect making it a smoothing operator of degree -2. They show in [28] that the non-diffusive case exhibits features better than the three-dimensional Euler equations, in particular they attempt to address the problem of the single versus double exponential growth of the norms and persistence of regularity of solutions. The growth estimate is based on the L^3-norm of the gradient of the initial data. In a recent paper (see [84]), using frequency localization techniques, we show that the regularity is controlled only by the L^3-norm of the initial data (not the gradient) and obtain a much sharper growth estimate of all norms. A technical exposition of the above mentioned results is contained in Section 1.5.

For nonlinear PDEs, the well-posedness questions may depend not only on the actual structure of the equation but also on the choice of the functional setting of the analysis. In order to gain insight into the dynamics described by the equations it is often beneficial to know the functional analytic properties of the nonlinearity as a mapping between Banach spaces. In the context of fluid dynamics such problems date back to the works of Y. Meyer, P. Germain, J. Bourgain and N. Pavlović. In [26], S. Friedlander and I consider an iterative resolution scheme for critically diffusive active scalar equations, where the drift velocity is obtained from the scalar via a Fourier multiplier of order zero. The particular examples of physical and mathematical problems we have in mind are the surface quasigeostrophic equation (SQG) and the porous media equation (IPM). The critical regime for such active scalar equations has been extensively studied and a number of fundamental properties has been addressed (see for instance [13, 23, 41]). In particular, the well-posedness in the critical setting has been established. However, most of the results in literature involve quantities which are subcritical with respect to the inherent scaling of the equation.The goal of our analysis is to gain insight into the behavior of the nonlinearity in order to understand if the well-posedness may be pushed further into the supercritical realm. If the answer is negative, what is the mechanism preventing it. We focus on the second iterate and analyze the properties of the associated bilinear form. The results presented in [26] involve only critical (scale-invariant) or super-critical norms. A more detailed description can be found in Section 1.6.

Another fundamental problem regarding the Navier-Stokes and Euler equations is the question of the inviscid limit. Note that formally, the Euler equations can be obtained from the Navier-Stokes equations via a singular limit (viscosity $\nu \to 0$). In the presence of the boundary the dynamics of inviscid fluids (described by the Euler equations) versus the viscous fluids (described by the Navier-Stokes equations) can be diametrally different. In the case of Dirichlet boundary conditions (the fluid is believed to stick to the boundary due to viscosity), we observe the formation of the boundary layer satisfying the Prandtl equations, which seem to be generally ill-posed. There exists an extensive body of literature on this topic. We address the inviscid limit problem for the Navier-Stokes equations in [80], and with P.B. Mucha in [69]. In [69, 80], we consider the Navier slip-type boundary conditions and show that under suitable assumptions the convergence holds. In particular, in [80] we admit the situation where the vorticity is not bounded. In [69], we obtain an explicit algebraic rate of convergence, previously unknown for general settings. The inviscid limit for active scalar equations is the subject of [83]. We show that convergence in the energy norm holds under much lower regularity assumptions than those considered in literature (c.f. [99]), allowing for mild singularities of gradients of solutions. As in the case of the NSE, we obtain an explicit algebraic rate of convergence.

The equations of fluid dynamics find their applications also in combustion theory which we address in [3], jointly with S. Benachour, I. Kukavica, and M. Ziane. We obtain criteria which guarantee global existence of solutions to the two-dimensional Kuramoto-Sivashinsky (KS) equation. Careful analysis of the anisotropy introduced to the problem by the geometry of the non-homogeneous rectangular domain allows us to obtain results which admit much larger initial data than previously known.

In [81] we analyze the partial regularity of a parabolic approximation of NSE. The considered system falls into the category of artificial compressibility methods and shares a number of important features with the NSE. In particular, the energy estimate or the skew-symmetry of the nonlinearity are preserved. The NSE can be recovered as a singular limit, very similar to low-mach limits for compressible Navier-Stokes equations (the speed of sound becomes infinite). We show that this limiting process is rigorous in the regime of both mild and weak solutions. This result is particularly interesting from the point of view of numerical simulations since such penalization models are used in the context of finite element methods.

1. Technical Exposition of Research

1.1. **Global well-posedness of NSE with large initial data.**

In [47, 48], we consider the three-dimensional incompressible Navier-Stokes system

$$\begin{aligned} \partial_t u - \Delta u + (u \cdot \nabla) u + \nabla p &= f \\ \operatorname{div} u &= 0 \\ u(x,0) &= u_0(x), \end{aligned} \tag{NSE}$$

on the domain $Q = [0,1]^3$ with periodic boundary conditions. These equations arise from applying Newton's second law and the law of conservation of mass to the fluid motion. The unknown functions are the velocity vector field $u(x,t)$ and the scalar pressure function $p(x,t)$, while the external force $f(x,t)$ is given.

Beginning with the seminal paper of J. Leray [56] the mathematical properties of solutions of NSE, such as regularity and uniqueness, have been extensively studied and there exists a vast body of literature on that topic (see [55] and references therein). In particular, H. Fujita and T. Kato in [29] showed that the global existence holds if $\|u_0\|_{\dot{H}^{1/2}}$ is sufficiently small. This result has been subsequently extended to other functional spaces. In particular, M. Cannone, Y. Meyer, and F. Planchon [15] obtained global solutions in Besov spaces $\dot{B}^{-1+3/p}_{p,\infty}$ (allowing for big oscillatory L^3 data). In [42], H. Koch and D. Tataru constructed solutions in the space BMO^{-1}. In the sense of inclusion, this is the biggest space in which one can hope for well-posedness of the Navier-Stokes equations. On the other hand, a negative result of J. Bourgain and N. Pavlović [4] establishes ill-posedness in the space $\dot{B}^{-1}_{\infty,\infty}$.

It has been discovered by G. Raugel and G. Sell that high oscillations in one or more directions regularize the solutions. In the papers [74, 75], they proved that global existence holds for a large class of data $R(\epsilon)$, oscillating with the frequency $1/\epsilon$ in the vertical direction (c.f. also [34, 35]). Subsequent works [1, 18, 19, 20, 31, 37, 38, 52, 53, 67, 68, 92, 93] complemented and extended these results.

Except in the two-dimensional case, it is extremely difficult to construct global solutions with large BMO^{-1} data and non-zero force. Namely, even if the H^1 norm is large, the BMO^{-1} norm can be small, especially if a solution oscillates in one direction. J.Y. Chemin and I. Gallagher in [18, 19] explored the oscillatory character of the initial data to produce a specific class of data with large $\dot{B}^{-1}_{\infty,\infty}$ norm yet admit the existence of a global solution. We would like to point out that due to the oscillations, the solutions in [18, 19] decay vary rapidly to zero.

In [47], in collaboration with I. Kukavica and M. Ziane, we obtain global regularity of the Navier-Stokes equations for a wide class of initial data oscillating with a period $1/h$ in the third direction. The only subcritical condition we require is $\|\nabla_T u_0\|_{L^2(Q)} \leq C h^{-1} |\log h|^{-1/2}$ where ∇_T represents a tangential gradient. Compared to earlier results, there is no restriction on the Fourier support of solutions and the data are genuinely three-dimensional, i.e., the third component does not need to vanish. Also, in [18], the condition is nonlinear, whereas the one presented in [47] is based solely on the size of the data. The proofs are based on delicate anisotropic estimates which explore the effect of the oscillatory ansatz on the considered problem. We introduce averaging operators in the horizontal and vertical directions ($M_T[\cdot]$ and $M[\cdot]$, respectively) and we use certain cancellations between such operators in the estimates of $\|\nabla_T u\|_{L^2}$, $\|M_T[u_T]\|_{L^2}$, and $\|M[u_3]\|_{L^p}$ for certain $p \geq 3$.

In [48], we choose a set of quantities more consistent with the effects of oscillations. Using techniques similar to those in [47], we obtain estimates on $\|\nabla_T u_T\|_{L^2}$, $\|\partial_3 u\|_{L^2}$, and $\|M[u_3]\|_{L^p}$, for certain $p \geq 3$. The main result of [48] states that the solution is regular if $\|\nabla_T u_T(0)\|_{L^2} \leq C^{-1}h^{-1}$, $\|\partial_3 u(0)\|_{L^2} \leq C^{-1}h^{-1}$, $\|M[u_3](0)\|_{L^p} \leq C^{-1}h^{2/p-1}$, where $p \in [3, \infty)$. The proof relies on special cancellations which occur when considering the set of the chosen quantities. The choice of these functionals is particularly suitable for the study of 2.5-dimensional Navier-Stokes equations and their (possibly large) perturbations, which is the main heuristics behind our results. Estimates obtained in the proof dictate a balance between the size of tangential derivatives of the horizontal components of velocity and the vertical derivative of the velocity. In [48] we also take advantage of new estimates which furthermore allow for the difference in order of the size of the above mentioned quantities. We prove that the solution is regular if $\|\nabla_T u_T(0)\|_{L^2} \leq C^{-1}h^{2/3p-1}$, $\|M[u_3](0)\|_{L^p} \leq C^{-1}h^{3/p-1}$, and $\|\partial_3 u(0)\|_{L^2} \leq C^{-1}h^{-1-1/3p}$, where $p \in [4, \infty)$. We exhibit a class of initial data which generate global regular solutions and whose size is $\|u_0\|_{BMO^{-1}} \sim h^{-1+2/3p}$ or $\|u_0\|_{BMO^{-1}} \sim h^{-1+\epsilon}$ (which can be generalized to $\|u_0\|_{BMO^{-1}} \sim h^{-1}$ up to a logarithm). Currently, we are working on the extension of our results to the domain $\mathbb{R}^3$.

The results presented in [49] surprisingly do not require any fast oscillations. We consider a class of functions $u = (u_1, u_2, u_3)$ on $Q = [0, 1]^3$ such $u_1(x_1, x_2, x_3)$ is an odd function of x_1 and an even function of x_2, $u_2(x_1, x_2, x_3)$ is an even function of x_1 and an odd function of x_2 and $u_3(x_1, x_2, x_3)$ is an even function of x_1 and x_2. The considered quantities are the same as in [48]. The main result states that the solution is regular if $\|\nabla_T u_{0k}\|_{L^2} \leq C\eta^2$, $\|\partial_3 u_0\|_{L^2} \leq C\eta^3$, and $\|M[u_{03}]\|_{L^p} \leq C\eta^{-1}$, where $p \in [3, \infty)$, $\eta \in (0, 1)$, and the operator $M[\cdot]$ is the average in the x_3-direction. The proof relies on special cancellations which occur when considering the set of the chosen quantities and their even/odd character as functions of the horizontal variables.

1.2. **Minimal initial data for potential Navier-Stokes singularities.** In [85], with V. Šverák, we consider the Cauchy problem for the Navier-Stokes equations with the initial condition in the homogenous Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$. The $\dot{H}^{1/2}$ norm is invariant under the natural scaling of the initial data $u_0(x) \to \lambda u_0(\lambda x)$, and the Cauchy problem is known to be globally well-posed for sufficiently small $u_0 \in \dot{H}^{1/2}$, and locally well-posed for any $u_0 \in \dot{H}^{1/2}$, as proved by H. Fujita and T. Kato in [29]. More precisely, there exists $\rho > 0$ such that for all $u_0 \in \dot{H}^{1/2}$ with $\|u_0\|_{\dot{H}^{1/2}} < \rho$ there exists a global solution. For Sobolev spaces with higher regularity ($\dot{H}^s(\mathbb{R}^3)$ with $s > 1/2$), scaling properties of the equation allow to match coarse-scale behavior of solutions with the one at fine-scales thus providing valuable information for the latter. However, this principle breaks down for $s = 1/2$ or more generally for any other choice of a critical space, for instance $L^3(\mathbb{R}^3)$ or the Morrey space $\dot{\mathcal{M}}_2^3(\mathbb{R}^3)$. These norms are invariant under the natural scaling of the equations and the scaling procedure seems to loose its value, thus making work in those scale invariant spaces more challenging. Therefore, we define ρ_{max} as the *supremum* of all $\rho > 0$ such that for data $\|u_0\|_{\dot{H}^{1/2}(\mathbb{R}^3)} < \rho$ the corresponding $u(t,x)$ is a global mild solution. It is not known if ρ_{max} is finite or infinite. In [77] and [82] we are interested in a hypothetical situation when ρ_{max} is finite. In fact ρ_{max} could be finite for various reasons, which depend on the exact notion of the solution. However, one can show that with the natural definition of the mild solution, the only reason ρ_{max} could be finite is the appearance of finite-time singularities in the solution $u(t,x)$ for some initial data $u_0(x)$. Such initial data $u_0(x)$ would then necessarily satisfy the condition $\|u_0\|_{\dot{H}^{1/2}} \geq \rho_{max}$. This motivates the following question:

If ρ_{max} is finite, does there exist an initial datum $u_0 \in \dot{H}^{1/2}$ with $\|u_0\|_{\dot{H}^{1/2}} = \rho_{max}$, such that the solution $u(x,t)$ of the Cauchy problem for NSE develops a singularity in finite time?

For many equations with the same scaling as NSE, there is strong evidence that this is not the case (for instance the cubic nonlinear heat equation analyzed by Poláčik and Quittner in [73] or the complex Ginzburg-Landau equations). Hence, analysis of questions similar to the one above, takes into account properties specific for NSE and poses an interesting problem.

In [77] we were able to give an affirmative answer using suitable weak solutions of L. Caffarelli, R. Kohn and L. Nirenberg (c.f. [12]) and local Leray solutions as constructed by P.G. Lemarié-Rieusset (c.f. [55]). In the subsequent paper [82], using the techniques of profile decomposition, we extend the result to a general situation, where ρ_{max} is the distance from given non-zero initial data generating a global solution.

1.3. **Anisotropic partial regularity criteria for the NSE.** The theory of partial regularity for the NSE , whose aim is to estimate the Hausdorff dimension of the singular set and development of interior regularity criteria, was initiated by Scheffer in [86, 87]. In a classical paper [12], Caffarelli, Kohn, and Nirenberg proved that for a suitable weak solution the one-dimensional parabolic Hausdorff measure (parabolic Hausdorff length) of the singular set equals zero. Recall that a point is regular if there exists a neighborhood in which u is bounded (and thus Hölder continuous); otherwise, the point is called singular. Their interior regularity criterion reads as follows: There exist two constants $\epsilon_{\text{CKN}} \in (0,1]$ and $\alpha \in (0,1)$ such that if

$$\int_{Q_1} (|u|^3 + |p|^{3/2})\, dxdt \leq \epsilon_{\text{CKN}}$$

then

$$\|u(x,t)\|_{C^\alpha(Q_{1/2})} < \infty$$

where $Q_r = \{(x,t) : |x| < r, -r^2 \leq t \leq 0\}$. Alternative proofs were given by Lin [57], Ladyzhenskaya and Serëgin [54], Kukavica [43, 44], Vasseur [94], and Wolf [97, 98]. The problem of partial regularity of the solutions of the Navier–Stokes equations has since then been addressed in various contexts

[45, 77, 78, 79, 89, 90] and a variety of interior regularity criteria has been proposed. In particular Wolf proved in [98] the following: There exists $\epsilon_{\mathrm{W}} > 0$ such that if

$$\int_{Q_1} |u|^3 \, dxdt \leq \epsilon_{\mathrm{W}}$$

then the solution $u(x,t)$ is regular at the point $(0,0)$.

In a recent paper [96], Wang and Zhang proved an anisotropic interior regularity criterion, which states: For every $M > 0$ there exists $\epsilon_{\mathrm{WZ}}(M) > 0$ such that if

$$\int_{Q_1} (|u|^3 + |p|^{3/2}) \, dxdt \leq M$$

and

$$\int_{Q_1} |u_h|^3 \, dxdt \leq \epsilon_{\mathrm{WZ}}(M)$$

where $u_h = (u_1, u_2)$, then the solution $u(x,t)$ is regular at the point $(0,0)$. Their result can be viewed as a local version of the component-reduction regularity. Regularity is obtained by imposing conditions only on some components of the velocity, rather that of three.

In [50], with I. Kukavica and M. Ziane, we prove an interior regularity criterion involving only one component of the velocity. Using arguments different than in [96], we prove the following stronger statement: For every $M > 0$ there exists a constant $\epsilon(M) > 0$ such that if

$$\int_{Q_1} (|u|^3 + |p|^{3/2}) \, dxdt \leq M \tag{1}$$

and

$$\int_{Q_1} |u_3|^3 \, dxdt \leq \epsilon(M)$$

then $u(x,t)$ is regular at the point $(0,0)$.

Note that every suitable weak solution satisfies (1) for M sufficiently large. The applied contradiction argument may be also used to prove a new interior regularity criterion based on the pressure. Namely, we prove that if (1) holds and if

$$\int_{Q_1} |p|^{3/2} \, dxdt \leq \epsilon(M) \tag{2}$$

then the solution is regular at $(0,0)$.

As a corollary we obtain a stronger version of the Leray's regularity criterion concerning weak solutions. Namely, by [30, 56], if T is an epoch of irregularity, then for any $q > 3$ there is a sufficiently small $\epsilon > 0$ such that $\|u(\cdot,t)\|_{L^q} \geq \frac{\epsilon}{(T-t)^{(1-3/q)/2}}$ for $t < T$ sufficiently close to T. Recall that T is an epoch of irregularity if T is a singular time for u, while the times $t < T$ sufficiently close to T are regular. We prove that if $T > 0$ is the first singular time, then for all $q \geq 3$

$$\|(u_1, u_2)(\cdot,t)\|_{L^q} \geq \frac{M}{(T-t)^{(1-3/q)/2}}$$

or

$$\|u_3(\cdot,t)\|_{L^q} \geq \frac{\epsilon(M)}{(T-t)^{(1-3/q)/2}},$$

for $t < T$ sufficiently close to T. (A similar statement holds when T is an epoch of irregularity.) Similarly,we show that if T is the first singular time, then

$$\|u(\cdot,t)\|_{L^q} \geq \frac{M}{(T-t)^{(1-3/q)/2}}$$

or

$$\|p(\cdot,t)\|_{L^{q/2}} \geq \frac{\epsilon(M)}{(T-t)^{1-3/q}}.$$

for $t < T$ sufficiently close to T. The proofs are based on a contradiction argument and the regularity of certain limit systems.

1.4. **Well-posedness of primitive equations with continuous initial data.** The primitive equations of the atmosphere and the ocean are widely considered to be the fundamental model for meteorology and climate prediction. Indeed, the full compressible Navier-Stokes equations, which govern the dynamics of the atmosphere and the ocean, are very complicated and contain phenomena which are not interesting from the geophysical point of view, such as shocks and sound waves. The Boussinesq approximation along with the hydrostatic balance lead to the primitive equations

$$\partial_t v_k - \nu \Delta v_k + \sum_{j=1}^{2} \partial_j (v_j v_k) + \partial_3 (w v_k) + \partial_k p = 0, \quad k = 1, 2$$

$$\sum_{k=1}^{2} \partial_k v_k + \partial_3 w = 0. \tag{3}$$

The main part of the system consists of the momentum equations and the conservation of mass, a simplified version of which is given above. The full primitive equations contain also the thermodynamic equations (diffusion of temperature), as well as the diffusion of humidity (for the atmosphere) and diffusion of salinity for the ocean, c.f. [70, 91].

The mathematical theory started with the work of P. Lions, R. Temam, and S. Wang [58, 59, 60] who set the analytical foundation for the equations and established the global existence of weak solutions for square integrable initial data in the spirit of Leray. The H^2 regularity of the associated stationary linear problem was obtained in [102, 103]. This result implied the local existence of strong solutions with initial data in H^1, which was established by Bresch et al [5, 6, 7, 8] and independently by Hu et al [36]. The global existence of strong solutions with initial data in H^1 was first proven by Cao and Titi in [11] in the case of Neumann boundary conditions on the top and bottom and for cylindrical domains. The case of the physical boundary conditions, including the Dirichlet boundary condition on the bottom of the ocean with the general bottom topography, was settled in [51, 52] with uniform gradient bounds obtained in [53]. In conclusion, the global existence of weak solution without uniqueness is known for both two and three space dimensions. Imposing the H^1 regularity for initial data leads leads to global existence and uniqueness of solutions (2D and 3D).

The state of analysis for the primitive equations seems to be much better than the one for the Navier-Stokes equations. However, when considering the uniqueness of weak solutions of the primitive equations in 2D, a classical and elementary fact for the 2D Navier-Stokes equations, we face the obstacle of a derivative loss in the nonlinearity, leading to an outstanding open problem. Furthermore, for the same reason, the well-posedness in L^p for the primitive equations remains open in both the 2D and 3D cases for any $p \geq 1$. We note the derivative loss in the nonlinearity constitutes a primary reason for the ill-posedness and finite time blow-up for the inviscid primitive equations in Sobolev spaces [76, 10].

In [46] we consider the well-posedness problem of the primitive equations for data in a class larger than H^1 for uniqueness of weak solutions, and thus well-posedness. In this spirit Bresch et al proved in [9] the uniqueness in 2D for weak solutions with $\partial_{x_3} v_0 \in H^{1/2}$. In [46], we establish the well-posedness (existence and uniqueness) of solutions with only continuous initial data that require no differentiability. Our approach relies on the splitting of the initial data into a smooth finite energy part and a small bounded part. In our reasoning, we exploit the fact that this splitting is preserved by the equation, the main difficulty being caused by the pressure and the derivative loss terms. We note that the main reason for the choice of the space in which we seek solutions is the possibility of establishing uniqueness.

1.5. **Well-posedness versus instability of the geodynamo.** In [27], we address the well-posedness of a nonlinear active scalar equation (MG) arising in the study of magnetogeostrophic turbulence. The equation in three dimensions reads

$$\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\gamma \theta = S, \tag{MG}$$

where $\kappa \geq 0$ and $S(x,t)$ is a given external force. An explicit operator $M[\theta]$ encodes the physics of the underlying physical process and produces the divergence-free velocity $u(x,t)$ from the scalar "buoyancy" field $\theta(x,t)$. More precisely, the velocity field is obtained from the scalar via the relationship

$$u_j = M_j[\theta],$$

where $M_j[\cdot]$ are Fourier multiplier operators with explicit symbols.

In the past decade, active scalar equations have received considerable attention due to their prevalent presence in many physical models. The MG equation has some features in common with the much studied surface quasigeostrophic equation (SQG) and the incompressible porous media equation (IPM) (c.f. [13, 16, 17, 22, 23, 39, 41, 100]). However MG has also a number of novel and distinctive features due to the strong singularity and anisotropy of the operator $M[\cdot]$ as well as the even nature of its Fourier symbol. Well-posedness of the critical case $\gamma = 1$ has been proven by S. Friedlander and V. Vicol in [24]. In [25], Friedlander and Vicol showed that the corresponding non-dissipative model ($\kappa = 0$) is ill-posed in the sense of Hadamard. In [27], we consider the fractionally dissipative MG equation in the super-critical regime $0 < \gamma < 1$. We show that for

$\gamma > 1/2$ the equation is locally well-posed, while for $\gamma < 1/2$ it is ill-posed, in the sense that there is no Lipschitz solution map. We also explore the anisotropy of the constitutive law for the velocity in order to obtain an improvement in the regularity of the solutions when the initial data and the force have thin Fourier support. For certain data we prove local existence and uniqueness for all values $0 < \gamma < 1$.

In [28] S. Friedlander and A. Suen initiated the analysis of the case where a certain viscous effect is incorporated in the constitutive law defining the drift velocity from the active scalar. In particular, they prove the existence of weak solutions with initial data in L^3 as well as the persistence of regularity for solutions with initial data in $W^{s,3}$, where $s > 0$. In particular, in case $s = 1$, they obtain an explicit growth estimate of the Sobolev norm

$$\|\nabla\theta(t,\cdot)\|_{L^3} \leq C\|\nabla\theta_0\|_{L^3} \exp(Ct\|\theta_0\|_{W^{1,3}}).$$

In [84], we prove that in fact for any initial data in $L^3 \cap B^s_{p,\infty}$, where $s > 0$ and $p \in [1,\infty]$, the regularity is preserved and moreover we have

$$\|\theta(t,\cdot)\|_{L^3 \cap B^s_{p,\infty}} \leq C\|\theta_0\|_{L^3 \cap B^s_{p,\infty}} \exp(Ct\|\theta_0\|_{L^3}).$$

The proof is based on a commutator estimate which takes into account that the drift velocity is two orders more regular than the active scalar.

1.6. **Active scalars and the second iterate.** The singular behavior of solutions of active scalar equations is an outstanding open problem. Such results are particularly interesting because of close ties of the active scalar equations and the three-dimensional incompressible Euler and Navier-Stokes equations. For the latter, well-posedness questions in a variety of scale-invariant spaces have been considered in order to separate features resulting from the structure built into the equation, from the properties of solutions resulting from the particular choice of functional spaces. In [32], P. Germain considered the second iterate for (NSE) and showed that while the corresponding bilinear form is continuous from $BMO^{-1}(\mathbb{R}^3) \times BMO^{-1}(\mathbb{R}^3)$ to $BMO^{-1}(\mathbb{R}^3)$, it is not continuous from $\dot{B}^{-1}_{\infty,q}(\mathbb{R}^3) \times \dot{B}^{-1}_{\infty,q}(\mathbb{R}^3)$ to $\mathcal{S}'(\mathbb{R}^3)$, $q > 2$. The same feature of the NSE nonlinearity used by Germain allowed Bourgain and Pavlović [4] to prove the norm inflation phenomenon for NSE in $\dot{B}^{-1}_{\infty,\infty}$. They constructed initial data, whose size in the space $\dot{B}^{-1}_{\infty,\infty}$ is arbitrarily small, however the resulting solution becomes arbitrarily large in an arbitrarily short time. In [26], we consider the case of critically dissipative active scalar equations where the Fourier symbol, with which the drift velocity is obtained from the scalar, is of order zero. Through analysis of the second iterate, we show that if the Fourier symbol is even, the corresponding bilinear form is not continuous from $\dot{B}^{-1/2}_{\infty,q}(\mathbb{R}^2) \times \dot{B}^{-1/2}_{\infty,q}(\mathbb{R}^2)$ to $\mathcal{S}'(\mathbb{R}^2)$. This is particularly interesting from the point of view of weak solutions. We would like to note that the bilinear form is continuous from $\dot{B}^0_{\infty,1}(\mathbb{R}^2) \times \dot{B}^0_{\infty,1}(\mathbb{R}^2)$ to $\dot{B}^0_{\infty,1}(\mathbb{R}^2)$. The proofs are based on the use of Coifman-Meyer-type theorems and a careful analysis of interactions between different frequency regions. The main difficulties stem from the fact that the nonlinearity in the equation and the diffusion provided by the fractional Laplacian are exactly in balance. This is the reason that an elementary perturbation result for small L^∞ initial data fails.

This balance is especially visible in the interaction of lower frequencies where the lack of sufficient diffusion dictates the choice of the space $\dot{B}^0_{\infty,1}$ rather than L^∞ or BMO. The results obtained in [26] are a first step to proving the norm inflation for active scalar equations which, due to the scalar nature of the problem, may be used to construct singular solutions. The results apply to a broader class of active scalar equations which preserve the L^∞ scaling.

1.7. **Vanishing viscosity limits.** In [80], we consider the inviscid limit of the two-dimensional incompressible Navier-Stokes equations when the Navier slip-with-friction conditions are prescribed on the impermeable boundaries of a bounded domain Ω.

The question of convergence of the non-stationary incompressible Navier-Stokes flow to the Euler flow as the kinematic viscosity $\nu \to 0$ has been an outstanding open problem for bounded domains. The main difficulty lies in the fact that strong solutions are known to exist only for a short time interval that tends to zero as $\nu \to 0$. Weak solutions exist for all time but the proof of uniqueness of such solutions presents major challenges. The situation is furthermore strongly influenced by the type of assumed boundary conditions. In the case of the Dirichlet condition $u(x,t) = 0$, where $x \in \partial\Omega$, T. Kato showed in [40] that the above convergence holds in $L^2(\Omega)$ uniformly in time on the interval of existence $[0,T]$ of the Euler flow if and only if $\nu \int_0^T \|\nabla u^\nu\|^2_{L^2(\Omega_{c\nu})} \to 0$ as $\nu \to 0$, where $\Omega_{c\nu}$ is the boundary strip of width $c\nu$.

In 1823, C. L. M. H. Navier proposed slip-type boundary conditions, which claim that the component of the fluid velocity tangent to the surface should be proportional to the rate of strain at the surface. The velocity component normal to the surface is naturally zero as mass cannot penetrate an impermeable solid surface. Recent experiments, generally with typical dimensions microns or smaller, have demonstrated that the phenomenon of slip actually occurs. For example, the Dirichlet condition is no longer true, when moderate pressure is involved, such as in high altitude aerodynamics. We also stress that the Navier slip-type condition was derived in the kinematic theory of gases by Maxwell. Under such boundary conditions, T. Clopeau, A. Mikelić and R. Robert in [21] proved the convergence to the Euler equations with wall conditions for an L^∞ initial vorticity and for an L^∞ forcing vorticity. This result was extended to L^p initial vorticity but without forcing term, for $p > 2$, by M. C. Lopes Filho, H. J. Nussenzveig Lopes and G. Planas [61].

In [80] we present the case of a C^1 initial velocity with a forcing vorticity in $L^1((0,T), L^q(\Omega))$, for $q > 4$. The idea is to perform some estimates in $L^\infty((0,T), L^q(\Omega))$ of the vorticity, uniformly with respect to the viscosity coefficient. Because of the boundary term an estimate in $L^\infty_{t,x}$ of the tangential velocity is needed. Thanks to some elliptic basic results in Sobolev spaces and an embedding theorem of some (parabolic) Hölder spaces in some (parabolic) Sobolev spaces, we estimate in return the L^∞ norm of the tangential velocity as a function of the $L^\infty((0,T), L^q(\Omega))$ norm of the vorticity. These estimates allow us to apply the Banach fixed point theorem and prove the existence and uniqueness of a solution with suitable estimates on vorticity, enabling – via a compactness method – passage to the inviscid limit. This method – more precisely the use of the

(Besov-Nikolskii-Ladyzhenskaya) embedding theorem – requires $q > 4$. Let us stress that this need for an $L^\infty_{t,x}$ estimate of the tangential velocity is also the reason for the restriction to $p > 2$ in [61].

In [69], jointly with P.B. Mucha, we consider the Cauchy problem for Euler equations and prove the uniqueness of weak solutions of the two and three dimensional incompressible Euler equations when the velocity $u(x,t)$ belongs to $L^\infty((0,T), L^{2+\epsilon}(\Omega))$ (for some $\epsilon > 0$) and ∇u lies in $L^1((0,T); BMO(\Omega))$. This improves a classical result due to V.I. Yudovich [101], as well as a result due to M.M. Vishik [95]. The key point in the analysis is an estimate on the difference between L^1 and the Hardy space $\mathcal{H}^1$ based on properties of the Zygmund space $L \ln L$. We also analyze the inviscid limit, assuming the Navier slip-type boundary conditions, and obtain an explicit algebraic rate of convergence.

In addition, in [83], we address the question of inviscid limits for a class of active scalar equations. The classical approach to the inviscid limit problem relies on energy methods to the equation governing the difference $\theta(x,t) = \theta_\kappa(x,t) - \theta_0(x,t)$, where $\theta_\kappa(x,t)$ and $\theta_0(x,t)$ are the solutions of the viscous and inviscid problems, respectively. Regarding the energy estimates, the nonlinearity in the equation creates the term

$$\int_0^T \int_{\mathbb{R}^n} \theta \cdot \nabla \theta_0 \theta \; dxdt, \tag{4}$$

which needs to be controlled. The most direct condition which enables us to estimate (4) is to require that $\nabla \theta_0 \in L^1((0,T), L^\infty(\mathbb{R}^n))$. We then obtain the following immediate inequality

$$\left| \int_0^T \int_{\mathbb{R}^n} \theta \cdot \nabla \theta_0 \theta \; dxdt \right| \leq C \|\nabla \theta_0\|_{L^1((0,T),L^\infty(\mathbb{R}^n))} \|\theta\|^2_{L^\infty((0,T),L^2(\mathbb{R}^n))}, \tag{5}$$

which is the core of the energy approach. Given this estimate, Gronwall's inequality allows us to pass to the limit $\kappa \to 0$. Note that the regularity imposed on the solution is quite high (and it is not clear that it follows from energy methods). Moreover, for equations in 2D, the classical existence result yields local solutions θ for data in Sobolev space H^s with $s > 2$. For solutions in the borderline space H^2, we have $\nabla \theta \in H^1$. This, however, is not enough to conclude that $\nabla \theta \in L^\infty$ and motivates the choice of BMO (being the limiting space of the Sobolev embedding theorem) rather than L^∞.

Inviscid limits for active scalar equations have been considered by J. Wu in [99], where solutions in Sobolev spaces H^s with $s \geq 3$ are considered. This assumption provides enough smoothness for the use of estimate (5). In [83] we consider a situation where $\nabla \theta = g + b$ with $g \in L^1((0,T), L^\infty(\mathbb{R}^n))$ and $b \in L^1((0,T), BMO(\mathbb{R}^n))$ such that the support of $b(\cdot, t)$ has finite measure for a.e. $t \in (0,T)$. More precisely, we prove that under such conditions, we are able to pass to the limit $\kappa \to 0$. Our proofs are different from the usual ones encountered in the active scalar equations literature and rely on the inequality

$$\left| \int_{\mathbb{R}^n} fg \; dx \right| \leq C \|f\|_{BMO} \|g\|_{L^1} \left[1 + |\ln \|g\|_{L^1}| + \ln(1 + \|g\|_{L^\infty})\right]. \tag{6}$$

rather than on commutator estimates based on the Littlewood-Paley decomposition. Techniques used in [83] apply to a general class of active scalar equations with symbols of order zero. As a byproduct of our method, we obtain an explicit rate of convergence in terms of the viscosity coefficient κ.

1.8. **Anisotropic estimates and global well-posedness of the two-dimensional KS.** In [3], we consider the two-dimensional Kuramoto-Sivashinsky equation

$$\partial_t\varphi + \Delta^2\varphi - \Delta\varphi + \frac{1}{2}|\nabla\varphi|^2 = 0, \tag{KS}$$

on the domain $\Omega = [0, L_1] \times [0, L_2]$. The role of KS is well known in the contemporary nonlinear mechanics and physics. This equation arises for instance as a model in hydrodynamics (a thin film flow down an inclined plane in the presence of an electric field), in combustion theory (propagation of flame fronts), phase turbulence and plasma physics, as well as a model for spatio-temporal chaos (c.f. [2, 71] for a short review of applications with key references).

Mathematically, the KS has been extensively studied in dimension one, but the main physical interest is in the two-dimensional case, since the equation models the flame propagation fronts. However, the global well-posedness for KS in the general 2D setting is an open problem. This is related to the fact that, although the equation is locally well-posed in $L^2(\Omega)$, it does not preserve the L^2 norm. The first global well-posedness result has been given by G. Sell and M. Taboada in [88], who showed the existence of a bounded local absorbing set in $H^1_{\text{per}}([0, 2\pi] \times [0, 2\pi\epsilon])$ for ϵ small enough. In [66] L. Molinet obtained a more transparent result on the local dissipativity of the Kuramoto-Sivashinsky equation in a thin rectangular domain and gave sufficient conditions on L_2, depending on L_1, so that the equation KS admits a global solution. The size of admissible initial data depends on L_1 and L_2. Our result takes advantage of anisotropic estimates and allows much larger initial data than previously known.

1.9. **Singular limit and partial regularity for a model of NSE.** In [81], we consider the following system of equations

$$\begin{aligned} &\partial_t u - \Delta u + (u\cdot\nabla)u + \frac{1}{2}u \operatorname{div} u - \frac{1}{\epsilon}\nabla \operatorname{div} u = 0 \\ &u(x,0) = u_0(x). \end{aligned} \tag{7}$$

This system shares a number of features with the three-dimensional incompressible Navier-Stokes equations. For instance, it has the same scaling, the non-linearity is skew-symmetric, and it satisfies the energy estimate

$$\|u(t)\|^2_{L^2} + \int_0^t \|\nabla u(s)\|^2_{L^2}\, ds + \frac{1}{\epsilon}\int_0^t \|\operatorname{div} u(s)\|^2_{L^2}\, ds \le \|u_0\|^2_{L^2}. \tag{8}$$

As $\epsilon \to 0$, we expect from (8) that solutions of (7) converge to solutions of NSE. The main interest of such result lies in applications to numerical computations where similar penalization schemes have been used in context of finite element methods (c.f. [33]). The system (7) has been also addressed by Šverák and Plecháč in [72] in the context of finite-time blow-up of radially symmetric solutions. In [81], we establish convergence of weak solutions and mild solutions thus validating the above expectations. We also address the problem of regularity of solutions of (7) and obtain partial regularity criteria.

References

[1] J.D. Avrin, *Large-eigenvalue global existence and regularity results for the Navier-Stokes equation*, J. Differential Equations **127** (1996), 365–390.

[2] H. Bellout, S. Benachour, and E. Titi, *Finite time regularity versus global regularity for hyperviscous Hamilton-Jacobi-like equations*. Nonlinearity **16** (2003), 1967–1989.

[3] S. Benachour, I. Kukavica, W. Rusin, and M. Ziane, *Anisotropic estimates for the two-dimensional Kuramoto-Sivashinsky equation*. Submitted.

[4] J. Bourgain and N. Pavlović, *Ill-posedness of the Navier-Stokes equations in a critical space in 3D*. J. Funct. Anal. **255**, No. 9 (2008), 2233–2247.

[5] D. Bresch, F. Guillén-González, N. Masmoudi, and M.A. Rodríguez-Bellido, *Uniqueness of solution for the 2D primitive equations with friction condition on the bottom*, Seventh Zaragoza-Pau Conference on Applied Mathematics and Statistics (Spanish) (Jaca, 2001), Monogr. Semin. Mat. García Galdeano, vol. 27, Univ. Zaragoza, Zaragoza, 2003, pp. 135–143.

[6] D. Bresch, F. Guillén-González, N. Masmoudi, and M.A. Rodríguez-Bellido, *Asymptotic derivation of a Navier condition for the primitive equations*, Asymptot. Anal. **33** (2003), no. 3-4, 237–259.

[7] D. Bresch, F. Guillén-González, N. Masmoudi, and M.A. Rodríguez-Bellido, *On the uniqueness of weak solutions of the two-dimensional primitive equations*, Differential Integral Equations **16** (2003), no. 1, 77–94.

[8] D. Bresch, F. Guillén-González, N. Masmoudi, and M.A. Rodríguiz-Bellido, *In the uniqueness of weak solutions of the two-dimensional primitive equations*, Differential Integral Equations, **16**, (2003), 77–94.

[9] D. Bresch, A. Kazhikhov, and J. Lemoine, *On the two-dimensional hydrostatic Navier-Stokes equations*, SIAM J. Math. Anal. **36** (2004/05), no. 3, 796–814 (electronic).

[10] C. Cao, S. Ibrahim, K. Nakanishi, and E.S. Titi, *Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics*, http://arxiv.org/pdf/1210.7337.

[11] C. Cao and E.S. Titi, *Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics*, Ann. of Math. (2) **166** (2007), no. 1, 245–267.

[12] L. Caffarelli, R. Kohn, and L. Nirenberg, *Partial regularity of suitable weak solutions of the Navier-Stokes equations*, Comm. Pure Appl. Math. **35** (1982), 771–831.

[13] L.A. Caffarelli and A. Vasseur. *Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation*. Ann. of Math. (2), 171(3) (2010), 1903–1930.

[14] M. Cannone, Y. Meyer, *Littlewood-Paley decomposition and Navier-Stokes equations*. Methods Appl. Anal. 2 (1995), no. 3, 307–319.

[15] M. Cannone, Y. Meyer, and F. Planchon, *Solutions autosimilaires des équations de Navier-Stokes*, Séminaire "Équations aux Dérivées Partielles" de l'École polytechnique, Exposé VIII, 1993–1994.

[16] D. Chae, P. Constantin, D. Córdoba, F. Gancedo, and J. Wu. *Generalized surface quasi-geostrophic equations with singular velocities*. Comm. Pure Appl. Math. 65 (8) (2012), 1037–1066.

[17] D. Chae, P. Constantin, and J. Wu, *Inviscid models generalizing the 2D Euler and the surface quasi-geostrophic equations*. Arch. Ration. Mech. Anal. 202(1) (2012), 35–62.

[18] J. Y. Chemin and I. Gallagher, *On the global wellposedness of the 3-D Navier-Stokes equations with large initial data*. Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 4, 679–698.

[19] J. Y. Chemin and I. Gallagher, *Large, global solutions to the Navier-Stokes equations, slowly varying in one direction*. Trans. Amer. Math. Soc. **362** (2010), no. 6, 2859–2873.

[20] J. Y. Chemin and I. Gallagher, M. Paicu, *Global regularity for some classes of large solutions to the Navier-Stokes equations*. Ann. of Math. (2) **173** (2011), no. 2, 983–1012.

[21] T. Clopeau, A. Mikelić, and R. Robert, *On the vanishing viscosity limit for the 2D incompressible Navier-Stokes equations with the friction type boundary conditions*. Nonlinearity **11** (1998), no. 6, 1625–1636.

[22] P. Constantin and J. Wu, *Behavior of solutions of 2D quasi-geostrophic equations*. SIAM J. Math. Anal. 30(5) (1999), 937–948.

[23] A. Córdoba and D. Córdoba, *A maximum principle applied to quasi-geostrophic equations*. Comm. Math. Phys. 249(3) (2004), 511–528.

[24] S. Friedlander and V. Vicol, *Global well-posedness for an advection-diffusion equation arising in magneto-geostrophic dynamics*. Ann. Inst. H. Poincaré Anal. Non Linéaire, 28(2) (2011), 283–301.

[25] S. Friedlander and V. Vicol, *On the ill/well-posedness and nonlinear instability of the magneto-geostrophic equations*. Nonlinearity, 24 (11) (2011), 3019–3042.

[26] S. Friedlander and W. Rusin, *On the second iterate for critically diffusive active scalar equations*. Accepted for publication in the Journal of Mathematical Fluid Mechanics.

[27] S. Friedlander, W. Rusin, and V. Vicol, *On the supercritically diffusive magnetogeostrophic equations*. Nonlinearity 25 (2012), doi:10.1088/0951-7715/25/11/3071.

[28] S. Friedlander, A. Suen, *Existence, uniqueness, regularity and instability results for the viscous magneto-geostrophic equation*. Nonlinearity **28** (2015), 3193–3217.

[29] H. Fujita and T. Kato, *On the nonsationary Navier-Stokes system*, Rend. Sem. Mat. Univ. Padova, **32** (1962), 243–260

[30] G.P. Galdi, *Introduction to the Navier-Stokes initial-boundary value problem. Fundamental directions in mathematical fluid mechanics*. Adv. Math. Fluid Mech., Birkhäuser, Basel, 2000, 1–70.

[31] I. Gallagher, *The tridimensional Navier-Stokes equations with almost bidimensional data: stability, uniqueness, and life span*, Internat. Math. Res. Notices **18** (1997), 919–935.

[32] P. Germain, *The second iterate for the Navier-Stokes equation*. J. Funct. Anal. 255 (2008), no. 9, 2248–2264.

[33] P. Gresho and R.L. Sani,*Incompressible Flow and the Finite Element Method*. New York:Wiley & Sons, 2000.

[34] J. K. Hale and G. Raugel, *A damped hyperbolic equation on thin domains*, Trans. Amer. Math. Soc. **329** (1992), 185–219.

[35] J. K. Hale and G. Raugel, *Partial differential equations on thin domains*, in "Differential Equations and Mathematical Physics, Birmingham, AL, 1990," pp. 63–97, Academic Press, Boston, 1992.

[36] C. Hu, R. Temam, and M. Ziane, *Regularity results for linear elliptic problems related to the primitive equations [mr1924143]*, Frontiers in mathematical analysis and numerical methods, World Sci. Publ., River Edge, NJ, 2004, pp. 149–170.

[37] D. Iftimie, *The* 3D *Navier-Stokes equations seen as a perturbation of the* 2D *Navier-Stokes equations*, Bull. Soc. Math. France **127** (1999), 473–517.

[38] D. Iftimie and G. Raugel, *Some results on the Navier-Stokes equations in thin 3D domains*, J. Differential Equations **169** (2001), 281–331.

[39] N. Ju, *Existence and uniqueness of the solution to the dissipative 2D quasi-geostrophic equations in the Sobolev space*. Comm. Math. Phys. 251(2) (2004), 365–376.

[40] T. Kato, *Remarks on zero viscosity limit for nonstationary Navier-Stokes flows with boundary*. Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983), 85–98, Math. Sci. Res. Inst. Publ., 2, Springer, New York, 1984.

[41] A. Kiselev, F. Nazarov, and A. Volberg, *Global well-posedness for the critical 2D dissipative quasi-geostrophic equation.* Invent. Math. 167(3) (2007), 445–453.

[42] H. Koch and D. Tataru, *Well-posedness for the Navier-Stokes equations.* Adv. Math. **157** (2001), no. 1, 22–35.

[43] I. Kukavica, *On partial regularity for the Navier-Stokes equations*, Discrete Contin. Dyn. Syst. **21** (2008), no. 3, 717–728.

[44] I. Kukavica, *The partial regularity results for the Navier-Stokes equations*, Proceedings of the workshop on "Partial differential equations and fluid mechanics," Warwick, U.K., 2008.

[45] I. Kukavica and Y. Pei, *An estimate on the parabolic fractal dimension of the singular set for solutions of the Navier-Stokes system*, Nonlinearity **25** (2012), no. 9, 2775–2783.

[46] I. Kukavica, Y. Pei, W. Rusin, and M. Ziane, *Primitive Equations With Continuous Initial Data.* Nonlinearity 27 (2014), 1135-1155.

[47] I. Kukavica, W. Rusin, and M. Ziane, *Solutions to Navier-Stokes equations for large oscillatory data.* Accepted for publication in Advances in Differential Equations.

[48] I. Kukavica, W Rusin, and M. Ziane, *A class of large* BMO^{-1} *non-oscillatory data for the Navier-Stokes equations.* Journal of Mathematical Fluid Mechanics, Volume 16, Issue 2, 293–305.

[49] I. Kukavica, W Rusin, and M. Ziane, *A class of solutions of the Navier-Stokes equations with large data.* Journal of Differential Equations, Volume 255, Issue 7, p. 1492-1514.

[50] I. Kukavica, W. Rusin, and M. Ziane, *An anisotropic partial regularity criterion for the Navier-Stokes equations.* Submitted.

[51] I. Kukavica and M. Ziane, *The regularity of solutions of the primitive equations of the ocean in space dimension three*, C. R. Math. Acad. Sci. Paris **345** (2007), no. 5, 257–260.

[52] I. Kukavica and M. Ziane, *Regularity of the Navier-Stokes equation in a thin periodic domain with large data*, Discrete Contin. Dynam. System **16** (2006), 67–86.

[53] I. Kukavica and M. Ziane, *On the regularity of the Navier-Stokes equation in a thin periodic domain*, J. Differential Equations **234** (2007), 485–506.

[54] O.A. Ladyzhenskaya and G.A. Seregin, *On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations*, J. Math. Fluid Mech. **1** (1999), no. 4, 356–387.

[55] P.G. Lemarié-Rieusset, *Recent developments in the Navier-Stokes problem*, Chapman & Hall/CRC Research Notes in Mathematics, 431. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[56] J. Leray, *Sur le mouvement d'un liquide visquex emplissant l'espace*, Acta Math. **63** (1934), 193–248.

[57] F. Lin, *A new proof of the Caffarelli-Kohn-Nirenberg theorem*, Comm. Pure Appl. Math. **51** (1998), no. 3, 241–257.

[58] J.-L. Lions, R. Temam, and S.H. Wang, *New formulations of the primitive equations of atmosphere and applications*, Nonlinearity **5** (1992), no. 2, 237–288.

[59] J.-L. Lions, R. Temam, and S.H. Wang, *On the equations of the large-scale ocean*, Nonlinearity **5** (1992), no. 5, 1007–1053.

[60] J.-L. Lions, R. Temam, and S.H. Wang, *Mathematical theory for the coupled atmosphere-ocean models. (CAO III)*, J. Math. Pures Appl. (9) **74** (1995), no. 2, 105–163.

[61] M. C. Lopes Filho, H. J. Nussenzveig Lopes, and G. Planas, *On the inviscid limit for two-dimensional incompressible flow with Navier friction condition.* SIAM J. Math. Anal. **36** (2005), no. 4, 1130–1141 (electronic).

[62] Y. Meyer, *Wavelets, paraproducts, and Navier-Stokes equations.* Current developments in mathematics, 1996 (Cambridge, MA), 105–212, Int. Press, Boston, MA, 1997.

[63] H.K. Moffat, D.E. Loper, *The mangnetostrophic rise of a buoyant parcel in the earth's core.* Geophysical Journal International **117**(2) (1994), 394–402.

[64] H.K. Moffat, *Magnetic Field Generation in Elecrically Conducting Fluids*. Cambridge: Cambridge University Press, 1983.

[65] H.K. Moffatt, *Magnetostrophic turbulence and the geodynamo* in *IUTAM Symposium on Computational Physics and New Perspectives in Turbulence*, volume 4 of *IUTAM Bookser.*, pages 339–346. Springer, Dordrecht, 2008.

[66] L. Molinet, *Local dissipativity in L^2 for the Kuramoto-Sivashinsky equation in spatial dimension 2.* J. Dynam. Diff. Eq. **12** (2000), no. 3, 533–556.

[67] S. Montgomery-Smith, *Global regularity of the Navier-Stokes equation on thin three dimensional domains with periodic boundary conditions*, Electronic J. Differential Equations **11** (1999), 1–19.

[68] I. Moise, R. Temam, and M. Ziane, *Asymptotic analysis of the Navier-Stokes equations in thin domains*, Topol. Methods Nonlinear Anal. **10** (1997), 249–282.

[69] P.B. Mucha and W. Rusin, *Zygmund spaces, inviscid limit and uniqueness of Euler flows*. Communications in Mathematical Physics 280 (2008), no. 3, 831–841.

[70] M. Petcu, R.M. Temam, and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of numerical analysis. Vol. XIV. Special volume: computational methods for the atmosphere and the oceans, Handb. Numer. Anal., vol. 14, Elsevier/North-Holland, Amsterdam, 2009, pp. 577–750.

[71] D.T. Papageorgiou and D. Tseluko, *A global contracting set for nonlinear KuramotoSivashinsky equations arising in interfacial electrohydrodynamics*. European J. Appl. Math. **17** (2006), 677–703.

[72] P. Plecháč and V. Šverák, *Singular and regular solutions of a nonlinear parabolic system.* Nonlinearity 16 (2003),2083–2097.

[73] P. Poláčik and P. Quittner, *Asymptotic behavior of threshold and sub-threshold solutions of a semilinear heat equation* Asymptot. Anal. **57** (2008), 125–141.

[74] G. Raugel and G. R. Sell, *Navier-Stokes equations on thin* 3D *domains. I. Global attractors and global regularity of solutions*, J. Amer. Math. Soc. **6** (1993), 503–568.

[75] G. Raugel and G. R. Sell, *Navier-Stokes equations on thin* 3D *domains. II. Global regularity of spatially periodic solutions*, in "Nonlinear Partial Differential Equations and Their Applications," Colège de France Seminar, Vol. XI, pp. 205–247, Longman, Harlow, 1994.

[76] M. Renardy, *Ill-posedness of the hydrostatic Euler and Navier-Stokes equations*, Arch. Ration. Mech. Anal. **194** (2009), no. 3, 877–886.

[77] J.C. Robinson and W. Sadowski, *Decay of weak solutions and the singular set of the three-dimensional Navier-Stokes equations*, Nonlinearity **20** (2007), no. 5, 1185–1191.

[78] J.C. Robinson and W. Sadowski, *A criterion for uniqueness of Lagrangian trajectories for weak solutions of the 3D Navier-Stokes equations*, Comm. Math. Phys. **290** (2009), no. 1, 15–22.

[79] J.C. Robinson and W. Sadowski, *Almost-everywhere uniqueness of Lagrangian trajectories for suitable weak solutions of the three-dimensional Navier-Stokes equations*, Nonlinearity **22** (2009), no. 9, 2093–2099.

[80] W. Rusin, *On the inviscid limit for the solutions of two-dimensional incompressible Navier-Stokes equations with slip-type boundary conditions.* Nonlinearity 19 (2006), no. 6, 1349–1363.

[81] W. Rusin, *Incompressible Navier-Stokes equations as a limit of a nonlinear parabolic system.* Journal of Mathematical Fluid Mechanics 14 (2012), no. 2, 383–405.

[82] W. Rusin, *Navier-Stokes equations, stability and minimal perturbations of global solutions.* Journal of Mathematical Analysis and Applications 386 (2012), no. 1, 115–124.

[83] W. Rusin, *Inviscid limits for active scalar equations with mildly singular gradients.* Journal of Mathematical Fluid Mechanics 15 (2013), no. 2, 415–423.

[84] W. Rusin, *Persistence of regularity for the non-dissipative viscous magneto-geostrophic equation.* Submitted.

[85] W. Rusin and V. Šverák, *Minimal initial data for potential Navier-Stokes singularities.* Journal of Functional Analysis, 260 (2011), no. 3, 879–891.

[86] V. Scheffer, *Partial regularity of solutions to the Navier-Stokes equations.* Pacific J. Math. 66 (1976), no. 2, 535–552.

[87] V. Scheffer, *Hausdorff measure and the Navier-Stokes equations.* Comm. Math. Phys. 55 (1977), no. 2, 97–112.

[88] G.R. Sell and M. Taboada, *Local dissipativity and attractors for the Kuramoto-Sivashinsky equation in thin 2D domains.* Nonlin. Anal. **18** (1992), 671–687.

[89] G.A. Seregin, *Estimates of suitable weak solutions to the Navier-Stokes equations in critical Morrey spaces.* (English, Russian summary) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 336 (2006), Kraev. Zadachi Mat. Fiz. i Smezh. Vopr. Teor. Funkts. 37, 199–210, 277; translation in J. Math. Sci. (N. Y.) 143 (2007), no. 2, 29612968

[90] G. Seregin, *Local regularity for suitable weak solutions of the Navier-Stokes equations.* Russian Math. Surveys **62** (2007), pp. 595–614.

[91] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of mathematical fluid dynamics. Vol. III, North-Holland, Amsterdam, 2004, pp. 535–657.

[92] R. Temam and M. Ziane, *Navier-Stokes equations in three-dimensional thin domains with various boundary conditions*, Adv. Differential Equations **1** (1996), 499–546.

[93] R. Temam and M. Ziane, *Navier-Stokes equations in thin spherical domains*, Contemp. Math. **209** (1997), 281–314.

[94] A.F. Vasseur, *A new proof of partial regularity of solutions to Navier-Stokes equations*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), no. 5-6, 753–785.

[95] M. Vishik, *Incompressible flows of an ideal fluid with vorticity in borderline spaces of Besov type.* Ann. Sci. École Norm. Sup. (4) **32** (1999), no. 6, 769–812.

[96] W. Wang and Z. Zhang, *On the interior regularity criteria and the number of singular points to the Navier-Stokes equations*, J. D'Analyse Math. **123** (2014) no. 1, 139–170.

[97] J. Wolf, *A direct proof of the Caffarelli-Kohn-Nirenberg theorem*, Parabolic and Navier-Stokes equations. Part 2, Banach Center Publ., vol. 81, Polish Acad. Sci. Inst. Math., Warsaw, 2008, pp. 533–552.

[98] J. Wolf, *A new criterion for partial regularity of suitable weak solutions to the Navier-Stokes equations*, Advances in mathematical fluid mechanics, Springer, Berlin, 2010, pp. 613–630.

[99] J. Wu, *Inviscid limits and regularity estimates for the solutions of the 2-D dissipative quasi-geostrophic equations.* Indiana University Mathematics Journal, 46 (1997), No. 4, 1113–1124.

[100] J. Wu, *Global solutions of the 2D dissipative quasi-geostrophic equation in Besov spaces.* SIAM J. Math. Anal. 36(3) (2004), 1014–1030 (electronic).

[101] V. I. Yudovic, *Non-stationary flows of an ideal incompressible fluid.* Ž. Vyčisl. Mat. i Mat. Fiz. **3** (1963), 1032–1066.

[102] M. Ziane, *Regularity results for Stokes type systems related to climatology*, Appl. Math. Lett. **8** (1995), no. 1, 53–58.

[103] M. Ziane, *Regularity results for the stationary primitive equations of the atmosphere and the ocean*, Nonlinear Anal. **28** (1997), no. 2, 289–313.