Complex Analysis II

Spring 2015

These are notes for the graduate course Math 5293 (Complex Analysis II) taught by Dr. Anthony Kable at the Oklahoma State University (Spring 2015). The notes are taken by Pan Yan (pyan@okstate.edu), who is responsible for any mistakes. If you notice any mistakes or have any comments, please let me know.

Contents

1 ∂ and $\overline{\partial}$ Operations (01/12)

We define two operators ∂ and $\overline{\partial}$ on differentiable functions on $\mathbb{C}$ (In *Complex Made Simple*, these are called $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$). The definitions are

$$\partial = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}),$$

$$\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}).$$

We will write ∂_x, ∂_y for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ for brevity.

Suppose that $f = u + iv$ is a differentiable function on an open subset of $\mathbb{C}$ with u, v real-valued. Then

$$\begin{aligned}\overline{\partial} f &= \frac{1}{2}(\partial_x f + i\partial_y f) \\ &= \frac{1}{2}\left((u_x + iv_x) + i(u_y + iv_y)\right) \\ &= \frac{1}{2}\left((u_x - v_y) + i(u_y + v_x)\right).\end{aligned}$$

This tells us that

$$\overline{\partial} f = 0 \Leftrightarrow \begin{cases} u_x = v_y, \\ u_y = -v_x. \end{cases}$$

That is, $\overline{\partial} f = 0$ is equivalent to the Cauchy-Riemann equations.

Suppose that f satisfies the Cauchy-Riemann equiations, so that $\overline{\partial} f = 0$. What is $\partial f = 0$? Well,

$$\begin{aligned}\partial f &= \frac{1}{2}(\partial_x f - i\partial_y f) \\ &= \frac{1}{2}(u_x + iv_x - i(u_y + iv_y)) \\ &= \frac{1}{2}(u_x + iv_x - i(-v_x + iu_x)) \\ &= \frac{1}{2}(u_x + iv_x + iv_x + u_x) \\ &= u_x + iv_x.\end{aligned}$$

Recall that if f is complex differentiable (at a point or on a set) then

$$f' = u_x + iv_x.$$

That is, if $\overline{\partial} f = 0$ and f is complex differentiable, then $\partial f = f'$.

We observe that

$$\begin{cases} \partial x = \partial + \overline{\partial}, \\ \partial y = i(\partial - \overline{\partial}). \end{cases}$$

The operators ∂ and $\overline{\partial}$ are complex linear. They also satisfy the product rule.

$$\begin{aligned} \partial(fg) &= \frac{1}{2}(\partial_x(fg) - i\partial_y(fg)) \\ &= \frac{1}{2}(f\partial_x g + g\partial_x f - if\partial_y g - ig\partial_y f) \\ &= f \cdot \frac{1}{2}(\partial_x g - i\partial_y g) + g \cdot \frac{1}{2}(\partial_x f - \partial_y f) \\ &= f\partial g + g\partial f. \end{aligned}$$

Similarly, $\overline{\partial}(fg) = f\overline{\partial}g + g\overline{\partial}f$. We also have $\overline{\partial f} = \overline{\partial}\ \overline{f}$ (this follows straight from the definitions.) If f is a holomorphic function, then we have $\overline{\partial}f = 0$ and so $\partial\overline{f} = 0$.

∂ and $\overline{\partial}$ also satisfy a chain rule. It says

$$\partial(f \circ g)(p) = \partial f(g(p)) \cdot \partial g(p) + \overline{\partial} f(g(p)) \cdot \partial\overline{g}(p)$$

and

$$\overline{\partial}(f \circ g)(p) = \partial f(g(p)) \cdot \overline{\partial} g(p) + \overline{\partial} f(g(p)) \cdot \overline{\partial}\overline{g}(p).$$

To keep things simple, we can write the first one as

$$\partial(f \circ g) = \partial f \circ g \cdot \partial g + (\overline{\partial} f) \circ g \cdot \partial\overline{g}.$$

To verify, let $g = u + iv$, then

$$\begin{aligned} 2\partial(f \circ g) &= \partial_x(f \circ g) - i\partial_y(f \circ g) \\ &= (\partial_x f) \circ g \cdot u_x + (\partial_y f) \circ g \cdot v_x - i(\partial_x f) \circ g \cdot u_y - i(\partial_y f) \circ g \cdot v_y \\ &= (\partial_x f) \circ g \cdot (u_x - iu_y) + (\partial_y f) \circ g \cdot (v_x - iv_y) \\ &= (\partial_x f) \circ g \cdot (2\partial u) + (\partial_y f) \circ g \cdot (2\partial v) \\ &= (\partial_x f) \circ g \cdot \partial(g + \overline{g}) + (\partial_y f) \circ g \cdot \partial(g - \overline{g}) \cdot \frac{1}{i} \\ &= (\partial_x f) \circ g \cdot \partial g + (\partial_x f) \circ g \cdot \partial\overline{g} - i(\partial_y f) \circ g \cdot \partial g - i(\partial_y f) \circ g \cdot \partial\overline{g} \\ &= ((\partial_x f) \circ g - i(\partial_y f) \circ g)\partial g + ((\partial_x f) \circ g + i(\partial_y f) \circ g)\partial\overline{g} \\ &= 2(\partial f) \circ g \cdot \partial g + 2(\overline{\partial} f) \circ g \cdot \partial\overline{g}. \end{aligned}$$

So $\partial(f \circ g) = (\partial f) \circ g \cdot \partial g + (\overline{\partial} f) \circ g \cdot \partial\overline{g}$.

Next, note that ∂ and $\overline{\partial}$ commute and

$$\begin{aligned} \partial\overline{\partial} &= \frac{1}{2}(\partial_x - i\partial_y) \cdot \frac{1}{2}(\partial_x + i\partial_y) \\ &= \frac{1}{4}(\partial_x^2 + i\partial_x\partial_y - i\partial_y\partial_x + \partial_y^2) \\ &= \frac{1}{4}(\partial_x^2 + \partial_y^2). \end{aligned}$$

That is,

$$\Delta = 4\partial\overline{\partial} = 4\overline{\partial}\partial$$

where Δ is the Laplacian.

Proposition 1.1. *Let $U, V \subset \mathbb{C}$ be open sets, $g : U \to V$ a holomorphic function, and $f \in C^2(V)$. Then*

$$\Delta(f \circ g) = ((\Delta f) \circ g) \cdot |g'|^2.$$

Proof. We know that $\Delta = 4\partial\overline{\partial}$. By the chain rule for $\partial, \overline{\partial}$, we have

$$\begin{aligned}
\overline{\partial}(f \circ g) &= (\partial f) \circ g \cdot \overline{\partial} g + (\overline{\partial} f) \circ g \cdot \overline{\partial}\, \overline{g} \\
&= (\overline{\partial} f) \circ g \cdot \overline{\partial g} \text{ (because } \overline{\partial} g = 0 \text{ since } g \text{ is holomorphic)} \\
&= (\overline{\partial} f) \circ g \cdot \overline{g'} \text{ (because } g' = \partial g \text{ when } g \text{ is holomorphic).}
\end{aligned}$$

So

$$\begin{aligned}
\partial\overline{\partial}(f \circ g) &= \partial\left((\overline{\partial} f) \circ g \cdot \overline{g'}\right) \\
&= \partial\left((\overline{\partial} f) \circ g\right) \cdot \overline{g'} + (\overline{\partial} f) \circ g \cdot \partial\overline{g'} \text{ (product rule for } \partial) \\
&= \partial\left((\overline{\partial} f) \circ g\right) \cdot \overline{g'} \text{ (because } \partial\overline{g'} = \overline{\overline{\partial} g'} = 0 \text{ since } g' \text{ is holomorphic)} \\
&= [(\partial\overline{\partial} f) \circ g \cdot \partial g + (\overline{\partial}\overline{\partial} f) \circ g \cdot \partial\overline{g}] \cdot \overline{g'} \\
&= (\partial\overline{\partial} f) \circ g \cdot \partial g \cdot \overline{g'} \text{ (because } \partial\overline{g} = \overline{\overline{\partial} g} = 0 \text{ since } g \text{ is holomorphic)} \\
&= (\partial\overline{\partial} f) \circ g \cdot g' \cdot \overline{g'} \text{ (because } \partial g = g' \text{ when } g \text{ is holomorphic)} \\
&= (\partial\overline{\partial} f) \circ g \cdot |g'|^2.
\end{aligned}$$

Multiplying both sides by 4 to get the required equation. $\square$

2 Harmonic Functions I (01/14)

Definition 2.1. Let $V \subset \mathbb{C}$ be an open set. A function $f : V \to \mathbb{C}$ that is twice continuously differentiable is said to be *harmonic* if $\Delta f = 0$.

Remark 2.2. *(i) Since $\Delta \overline{f} = \overline{\Delta f}$, the real and imaginary parts of a harmonic function are also harmonic.*

(ii) Holomorphic functions are harmonic. If g is holomorphic, then g is infinitely differentiable and $\Delta g = 4\partial\overline{\partial} g = 0$ since $\overline{\partial} g = 0$.

(iii) Thus the real and imaginary parts of holomorphic functions are harmonic.

(iv) Suppose that u is a harmonic function. Then ∂u is a holomorphic function. Here is the reason. ∂u is C^1 and $\overline{\partial}(\partial u) = \overline{\partial}\partial u = \frac{1}{4}\Delta u = 0$ and so ∂u satisfies the Cauchy-Riemann equations. This implies that ∂u is holomorphic.

(v) As a consequence, a harmonic function is C^∞. This is because ∂u must be C^∞ (since ∂u is holomorphic) and so u is also C^∞.

Theorem 2.3. *Let $V \subset \mathbb{C}$ be a simply connected open set and u a real-valued harmonic function on V. Then there is some $F \in H(V)$ such that $u = Re(F)$.*

Proof. We know that $\partial u \in H(V)$. Since V is simply connected, there is some $G \subset H(V)$ such that $G' = \partial u$. Write $G = A + Bi$, where A, B are real-valued. Then $G' = A_x + B_x i = B_y - A_y i$. Also $\partial u = \frac{1}{2}(u_x - iu_y)$. Thus $A_x = \frac{1}{2}u_x$ (by comparing real parts) and $A_y = \frac{1}{2}u_y$ (by comparing imaginary parts). So $\nabla(u - 2A) = (u_x - 2A_x, u_y - 2A_y) = 0$. Let W be a connected component of V. Then $u - 2A$ is constant on W, say c_W. Then define $F : V \to \mathbb{C}$ by $F(z) = 2G(z) - c_W$ for $z \in W$. Then $F \subset H(V)$ and $\mathrm{Re}(F) = 2A + c_W = u$. □

Remark 2.4. *This theorem has a converse (see Complex Made Simple).*

This theorem has a lot of consequences for harmonic functions. One is that harmonic functions are actually real analytic.

Theorem 2.5 (Strong Maximum Principle)**.** *Let $V \subset \mathbb{C}$ be a connected open set. Let u be a real-valued harmonic function on V. If u has a local maximum point, then u is constant.*

Proof. Say $p \in V$ is a local maximum point and choose $\overline{D}(p, r) \subset V$. Choose $F \in H(D(p, r))$ such that $u = \mathrm{Re}(F)$. Consider $\exp(F) \in H(D(p, r))$. Well,

$$|\exp(F)| = \exp(\mathrm{Re}(F)) = \exp(u)$$

has a local maximum at p. Thus $\exp(F)$ is constant and so $|\exp(F)| = \exp(u)$ is constant. Thus u is constant on $D(p, r)$. It follows from the *Weak Identity Principle* (see homework 1) that u is constant. □

3 Harmonic Functions II (01/16)

Definition 3.1. Let $V \subset \mathbb{C}$ be an open set and $f : V \to \mathbb{C}$ a continuous function. Then f is said to have the *Mean Value Property* if for all $p \in V$ and all $r > 0$ such that $\overline{D(p,r)} \subset V$ we have

$$f(p) = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta.$$

Definition 3.2. Let $V \subset \mathbb{C}$ be an open set and $f : V \to \mathbb{C}$ a continuous function. Then f is said to have the *Local Mean Value Property* if for all $p \in V$ there is some $\rho > 0$ such that for all $0 < r < \rho$ we have $\overline{D(p,r)} \subset V$ and

$$f(p) = \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta.$$

Remark 3.3. *(i) Holomorphic functions have the Mean Value Property. This is a direct consequence of the Cauchy Integral Formula.*

(ii) If a function has the Mean Value Property then so do its real and imaginary parts.

Proposition 3.4. *Harmonic functions have the Mean Value Property.*

Proof. Let $V \subset \mathbb{C}$ be open, $p \in V$, and $r > 0$ be such that $\overline{D(p,r)} \subset V$. Choose $R > r$ such that $D(p,R) \subset V$. Let u be a real-valued harmonic function on V. Since $D(p,R)$ is simply connected, there exists $F \in H(D(p,R))$ such that $\mathrm{Re}(F) = u$. By applying the above remarks to F we know that F and hence u have the Mean Value Property on $D(p,R)$. This implies the required identity for $\overline{D(p,r)}$. This implies the general harmonic functions have the Mean Value Property too, by linearity. $\square$

Definition 3.5. A family $\mathcal{F}$ of continuous real-valued functions on a connected open set V is said to have the *Weak Maximum Principle* if whenever $f \in \mathcal{F}$ has a global maximum in V then this function f must be constant.

We formulate the *Strong Maximum Principle* by replacing "global maximum" with "local maximum".

We already know the family of real-valued harmonic functions on a connected open set has the Strong Maximum Principle.

Proposition 3.6. *Let V be a connected open set and $\mathcal{F}$ denote the family of all functions on V that are continuous, real-valued, and satisfy the Local Mean Value Property. Then $\mathcal{F}$ has the Weak Maximum Principle.*

Proof. Let $f \in \mathcal{F}$ and suppose that f achieves its global maximum value of M at some point in V. Let $S = \{z \in V : f(z) = M\}$. By hypothesis, S is non-empty and S is closed since f is continuous. I now wish to show that S is open. Suppose not and let $p \in S$ be a boundary point of S. Choose a $\rho > 0$ as in the Local Mean Value Property at p. Choose a point $q \in D(p, \rho)$ such that $f(q) < M$. Let $r = |p - q| < q$. By the Local Mean Value Property, we have

$$\begin{aligned} M &= f(p) \\ &= \frac{1}{2\pi} \int_0^{2\pi} f(p + re^{i\theta})\, d\theta \\ &< M \end{aligned}$$

because the integrand is $\leq M$, continuous, and less than M at least at one point, namely q. This is a contradiction. Thus S is open. By the connectedness of V, $S = V$ and so f is constant. □

Definition 3.7. Let $V \subset \mathbb{C}$ be a bounded, connected open set. Let $\mathcal{F}$ be a family of real-valued continuous functions on V. We say that $\mathcal{F}$ has the *Very Weak Maximum Principle* if whenever M is a number and $f \in \mathcal{F}$ such that

$$\liminf_{z \to q, z \in V} f(z) \leq M$$

for all $q \in \partial V$ then $f(z) \leq M$ for all $z \in V$.

Proposition 3.8. *Let $V \subset \mathbb{C}$ be a bounded, connected open set. Let $\mathcal{F}$ be a family of real-valued, continuous functions on V. Suppose that for all connected open sets $W \subset V$ the family $\{f|_W : f \in \mathcal{F}\}$ satisfies the Local Mean Value Property. Then $\mathcal{F}$ has the Very Weak Maximum Principle.*

4 Harmonic Functions III (01/21)

We want to exhibit an explicit solution to the Dirichlet problem for the unit disk. Our aim is that given $f \in C(\partial\mathbb{D})$, find a solution $u \in C(\overline{\mathbb{D}}) \cap C^2(\mathbb{D})$ such that $u|_{\partial\mathbb{D}} = f$ and $\Delta u = 0$.

For explicit solution, we need to start with some known solutions. How do we find these? One way is to use separation of variables. We will use polar coordinates for the separation. We know

$$\Delta = \partial_r^2 + \frac{1}{r}\partial r + \frac{1}{r^2}\partial_\theta^2.$$

Guess that $u = g(r)h(\theta)$. Then

$$\Delta u = g''h + \frac{1}{r}g'h + \frac{1}{r^2}gh'' = 0.$$

Divide by equation by gh, and we get

$$\begin{aligned}
&\frac{g''}{g} + \frac{1}{r}\frac{g'}{g} + \frac{1}{r^2}\frac{h'}{h} = 0 \\
\Rightarrow &\frac{g''}{g} + \frac{1}{r}\frac{g'}{g} = -\frac{1}{r^2}\frac{h'}{h} \\
\Rightarrow &\frac{r^2 g'' + r g'}{g} = -\frac{h''}{h}
\end{aligned}$$

This implies that both $\frac{r^2g''+rg'}{g}$ and $\frac{h''}{h}$ must be constant. Call the second constant $-k^2$. We have

$$\begin{aligned}
&\frac{h''}{h} = -k^2 \\
\Rightarrow &h'' + k^2 h = 0.
\end{aligned}$$

A solution to this equation is $e^{ik\theta}$. Another one is $e^{-ik\theta}$. All solutions are combinations of these two. To make $e^{\pm ik\theta}$ continuous functions on the circle, we must have $k \in \mathbb{Z}$. Since $-k \in \mathbb{Z}$ too, we can just consider $e^{ik\theta}$.

Now we must have

$$\frac{r^2 g'' + r g'}{g} = k^2,$$

i.e.,

$$r^2 g'' + r g' - k^2 g = 0$$

and this is an Euler equation. Try r^α as a solution for the Euler equation, and we get

$$\alpha(\alpha - 1) r^\alpha + \alpha r^\alpha - k^2 r^\alpha = 0.$$

So $\alpha^2 - \alpha + \alpha - k^2 = 0$ and hence

$$\alpha = \pm k.$$

This gives the separated solution $r^{\pm k} e^{ik\theta}$ for $k \in \mathbb{Z}$. Since we require continuous functions on $\mathbb{D}$, we must take only positive powers of r. This means only $r^{|k|} e^{ik\theta}$ are actually acceptable, where $k \in \mathbb{Z}$. Optimistically we form a combination of all these solutions with equal weights

$$P_r(\theta) = \sum_{n \in \mathbb{Z}} r^{|n|} e^{in\theta}.$$

This converges provided that $r < 1$. This function is called the *Poisson Kernel*. We hope that the Poisson kernel should be harmonic and we should be able to create a solution to the original Dirichlet problem by combining the Poisson kernel P_r and its translates in an integral.

We need a lemma that gives us many different expressions for P_r so that we can deduce its properties.

5 Harmonic Functions IV (01/23)

Recall
$$P_r(\theta) = \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta}.$$
By comparison with a geometric series, the series defining the Poisson kernel converges uniformly on the interval $[0, p]$ for any $p < 1$. We assume that $r \in [0, 1)$ when we write down the Poisson kernel. We conclude
$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta)d\theta = \sum_{n\in\mathbb{Z}} r^{|n|} \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\, d\theta = 1$$
for all r since
$$\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\, d\theta = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{unless } n = 0. \end{cases}$$

Next,
$$\begin{aligned}
P_r(\theta) &= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \\
&= 1 + \sum_{n=1}^{\infty} r^n (e^{in\theta} + e^{-in\theta}) \\
&= 1 + 2\mathrm{Re}\left(\sum_{n=1}^{\infty} r^n e^{in\theta}\right) \quad (\text{since } \overline{e^{in\theta}} = e^{-in\theta}) \\
&= 1 + 2\mathrm{Re}\left(\sum_{n=1}^{\infty} (re^{i\theta})^n\right) \\
&= 1 + 2\mathrm{Re}\left(\frac{re^{i\theta}}{1 - re^{i\theta}}\right) \\
&= \mathrm{Re}\left(1 + \frac{2re^{i\theta}}{1 - re^{i\theta}}\right) \\
&= \mathrm{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right) \\
&= \mathrm{Re}\left(\frac{(1 + re^{i\theta})(1 - re^{-i\theta})}{(1 - re^{i\theta})(1 - re^{-i\theta})}\right) \\
&= \mathrm{Re}\left(\frac{1 + 2ir\sin(\theta) - r^2}{1 - 2r\cos(\theta) + r^2}\right) \\
&= \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2} \\
&= \frac{1 - r^2}{(1 - r\cos(\theta))^2 + r^2\sin^2(\theta)}.
\end{aligned}$$

So $P_r(\theta)$ is real-valued.

$P_r(\theta) > 0$ for all r and θ (this follows from the last expression in the string). If $\delta > 0$, then $P_r(\theta) \to 0$ as $r \to 1^-$ uniformly for $\theta \in [-\pi, -\delta] \cup [\delta, \pi]$. Indeed for $\theta \in [-\pi, -\delta] \cup [\delta, \pi]$, we have

$$0 < P_r(\theta) \leq \frac{1 - r^2}{(1 - r\cos(\delta))^2} \leq \frac{1 - r^2}{(1 - \cos(\delta))^2}.$$

This implies that uniform convergence claim.

Families of functions are called "approximate identities" if some property as in 5.1 holds.

Proposition 5.1. *Suppose that $g_r : [-\pi, \pi] \to \mathbb{R}$ is a family of continuous, 2π-periodic, positive functions depending on a parameter $r \in [0, 1)$. Suppose that*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta)\, d\theta = 1$$

for all r, and for any $\delta > 0$, $g_r \to 0$ uniformly as $r \to 1^-$ for $\theta \in [-\pi, -\delta] \cup [\delta, \pi]$. For any f a continuous 2π-periodic function on $[-\pi, \pi]$, define

$$(g_r * f)(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta - \psi) f(\psi)\, d\psi$$

(with g_r and f extended periodically to $\mathbb{R}$). Then $g_r * f \to f$ uniformly as $r \to 1^-$.

Proof. Well,

$$\begin{aligned}
|(g_r * f)(\theta) - f(\theta)| =& \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\theta - \psi)[f(\psi) - f(\theta)] d\psi \right| \\
=& \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} g_r(\psi)[f(\theta - \psi) - f(\theta)] d\psi \right| \\
\leq& \frac{1}{2\pi} \int_{-\pi}^{-\delta} g_r(\psi) \left| f(\theta - \psi) - f(\theta) \right| d\psi + \frac{1}{2\pi} \int_{-\delta}^{\delta} g_r(\psi) \left| f(\theta - \psi) - f(\theta) \right| d\psi \\
&+ \frac{1}{2\pi} \int_{\delta}^{\pi} g_r(\psi) \left| f(\theta - \psi) - f(\theta) \right| d\psi \\
=& I_1 + I_2 + I_3.
\end{aligned}$$

We have

$$I_1 \leq \frac{1}{2\pi} \int_{-\pi}^{-\delta} \eta \cdot 2M d\psi \leq 2M\eta$$

and similarly,

$$I_3 \leq 2M\eta,$$

$$
\begin{aligned}
I_2 &\leq \frac{1}{2\pi}\int_{-\delta}^{\delta} g_r(\psi)\eta \, d\psi \\
&\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} g_r(\psi)\eta \, d\psi \\
&= \eta.
\end{aligned}
$$

Thus $|(g_r * f)(\theta) - f(\theta)| \leq (4M+1)\eta$ when $r > r_0$ for all θ. Take $\eta = \frac{\varepsilon}{4M+2}$ to complete the argument. $\square$

6 Harmonic Functions V (01/26)

Say $f \in C(\partial\mathbb{D})$. For $0 \leq r < 1$ we define

$$
g_r(\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \psi) f(e^{i\psi}) \, d\psi.
$$

We can regard $g_r(\theta)$ as a function on $\partial\mathbb{D}$ because g_r is 2π-periodic. This is true because P_r is 2π-periodic. Also, $P_0(\theta) = 1$ for all θ. Thus we may define the *Poisson integral*

$$
P[f](z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \psi) f(e^{i\psi}) \, d\psi
$$

where $z \in \mathbb{D}$ and $z = re^{i\theta}$ with $r \in [0,1)$. This is well-defined by the previous observations. We know that $P[f](re^{i\theta}) \to f(e^{i\theta})$ as $r \to 1^-$ uniformly in θ. Also, if f is a real-valued function, let $z = re^{i\theta}$, then we have

$$
\begin{aligned}
P[f](z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{Re}\left(\frac{1 + re^{i(\theta-\psi)}}{1 - re^{i(\theta-\psi)}}\right) f(e^{i\psi}) \, d\psi \\
&= \operatorname{Re}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + ze^{-i\psi}}{1 - ze^{-i\psi}} f(e^{i\psi}) \, d\psi\right) \\
&= \operatorname{Re}(F(z)).
\end{aligned}
$$

For fixed ψ,

$$
z \mapsto \frac{1 + ze^{-i\psi}}{1 - ze^{-i\psi}} f(e^{i\psi})
$$

is a holomorphic function on $\mathbb{D}$. Also, it is continuous as a function of ψ and z. By a previous lemma from Complex Analysis I, it follows that F is holomorphic. (Recall

$$
\begin{aligned}
\int_\Delta F(z) \, dz &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_\Delta \frac{1 + ze^{-i\psi}}{1 - ze^{-i\psi}} f(e^{i\psi}) \, dz \, d\psi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} o \, d\psi \\
&= 0
\end{aligned}
$$

and then Morera's Theorem applies.) It follows that $P[f]$ is a harmonic function in $\mathbb{D}$. If f is not real valued, $P[f]$ is also harmonic. This follows because

$$P[f + ig] = P[f] + iP[g].$$

Theorem 6.1. *Let $f \in C(\partial\mathbb{D})$ and define $u : \overline{\mathbb{D}} \to \mathbb{C}$ by*

$$u(z) = \begin{cases} P[f](z) & \text{if } z \in \mathbb{D}, \\ f(z) & \text{if } z \in \partial\mathbb{D}. \end{cases}$$

Then $u \in C(\overline{\mathbb{D}})$, $u|_{\mathbb{D}}$ is harmonic, and $u|_{\partial\mathbb{D}} = f$. (Thus u solves the Dirichlet Problem for the disk with f as the boundary data.)

Proof. We know $u|_{\mathbb{D}}$ is harmonic. We know $u|_{\partial\mathbb{D}}$ is f. We don't know that u is continuous. We do know $u|_{\mathbb{D}}$ is continuous and that $u|_{\partial\mathbb{D}}$ is continuous. We have to verify that u is continuous at a point $p \in \partial\mathbb{D}$.

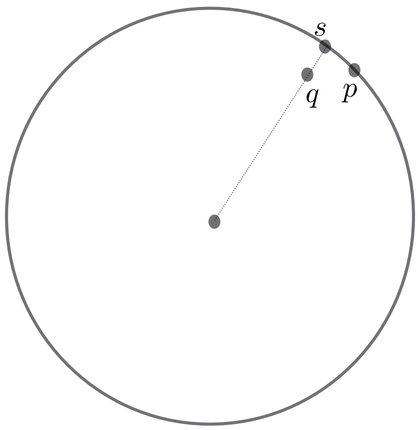


Figure 1:

Let $\varepsilon > 0$. We may find $\delta_1 > 0$ such that if $s \in \partial\mathbb{D}$ and $|s-p| < \delta_1$ then $|u(s)-u(p)| < \frac{\varepsilon}{2}$. We may find $\delta_2 > 0$ such that if $r \in (1-\delta_2, 1)$ then $|u(re^{i\alpha} - u(e^{i\alpha})| < \frac{\delta}{2}$ for all α. By

the picture and the triangle inequality, we will be done if we can choose $\delta_3 > 0$ such that if $|p-q| < \delta_3$ then $|q| \in (1-\delta_2, 1)$. We can do this because $|q| + |p-q| \le |p| = 1$ and so if $\delta_3 < \delta_2$ then
$$|q| \ge 1 - |p-q| > 1 - \delta_3 > 1 - \delta_2.$$
This completes the proof. □

For uniqueness, let's start with an example on a unbounded domain.

Example 6.2. The function $u(x,y) = y$ is harmonic on π^+. It vanishes identically on $\partial\pi^+$ (real axis). However, $u \not\equiv 0$. So the Dirichlet problem in π^+ with zero boundary data has at least two solutions u and 0.

Theorem 6.3. *Let $V \subset \mathbb{C}$ be an open, connected, bounded set. Let $u_1, u_2 \in C(\overline{V})$ be such that $u_1|_V$ and $u_2|_V$ are harmonic and $u_1|_{\partial V} = u_2|_{\partial V}$. Then $u_1 = u_2$.*

Proof. We may assume u_1, u_2 are real-valued. Let $v = u_1 - u_2$. Then v is harmonic and so v has the Mean Value Property in V. Thus v satisfies the Very Weak Maximum Principle. Now if $q \in \partial V$ then
$$\lim_{z \to q, z \in V} \sup v(z) = 0$$
since v is continuous on $\overline{v}$ and zero on ∂V. We conclude that $v(z) \le 0$ for all $z \in V$. Thus $u_1(z) \le u_2(z)$ for all $z \in V$. By the same argument, $u_2(z) \le u_1(z)$ for all $z \in V$ and so $u_1 = u_2$ on V. This completes the proof. □

Remark 6.4. *This theorem applies to $\mathbb{D}$ and so $P[f]$ is the unique solution to the Dirichlet problem.*

7 Harmonic Functions VI (01/28)

Proposition 7.1. *Let $p \in \mathbb{C}$ and $R > 0$. Then the Dirichlet problem with continuous boundary data has one and only one solution on $\overline{D}(p, R)$.*

Proof. We know that the Dirichlet problem with continuous boundary data has at most one solution on the closure of any bounded, connected open set. If $u \in C(\overline{D}(p,R))$, then v defined by $v(z) = u(p + Rz)$ is an element of $C(\overline{D}(0,1))$. We can reverse this by
$$u(w) = v\left(\frac{w-p}{R}\right).$$
We also know that if ϕ is harmonic and h is holomorphic, then $\phi \circ h$ is harmonic. (Recall $\Delta(\phi \circ h) = ((\Delta\phi) \circ h) \cdot |h'|^2$ when h is holomorphic.) In particular, if u is harmonic then so is v and vice versa.

Given continuous boundary data on $\overline{D}(p, R)$, say f, we define
$$g(z) = f(p + Rz).$$

This gives us $g \in C(\partial\mathbb{D})$. Then solve the Dirichlet problem to get v harmonic on $\mathbb{D}$ continuous on $\overline{\mathbb{D}}$ with g as boundary data. Then define $u : \overline{D}(p, R) \to \mathbb{C}$ by $u(w) = v\left(\frac{w-p}{R}\right)$. Then u solves the original problem. $\square$

Theorem 7.2. *Let V be an open set in $\mathbb{C}$ and ψ be a continuous function on V that satisfies the Local Mean Value Property. Then ψ is harmonic on V.*

Proof. It suffices to show that for all $p \in V$ there is a disk $D(p, R) \subset V$ such that $\psi|_{D(p,R)}$ is harmonic. Fix p we choose a disk $D(p, R)$ such that $\overline{D}(p, R) \subset V$. We know that the Dirichlet problem with continuous boundary data can be solved for $\overline{D}(p, R)$. Let $u : \overline{D}(p, R) \to \mathbb{C}$ be the solution for the boundary data $\psi|_{\partial D(p,R)}$. By hypothesis, ψ has the Local Mean Value Property in $D(p, R)$. Also, u has the Local Mean Value Property in $D(p, R)$ because $u|_{D(p,R)}$ is harmonic. Thus $u - \psi$ has the Local Mean Value Property on $D(p, R)$. Furthermore, $u - \psi \in C(\overline{D}(p, R))$ and $u - \psi \equiv 0$ on $\partial D(p, R)$. It follows from the Very Weak Maximum Principle that $u - \psi \leq 0$ on $D(p, R)$. Similarly, $\psi - u \leq 0$ on $D(p, R)$. Thus $u = \psi$ on $D(p, R)$ and so ψ is harmonic. $\square$

Let $f \in C(\partial\mathbb{D})$. Then for $0 \leq r < 1$, we have

$$\begin{aligned}
P[f](re^{i\theta}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - \psi) f(e^{i\psi})\, d\psi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{n\in\mathbb{Z}} r^{|n|} e^{in(\theta-\psi)}\right) f(e^{i\psi})\, d\psi \\
&= \sum_{n\in\mathbb{Z}} r^{|n|} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in(\theta-\psi)} f(e^{i\psi})\, d\psi \\
&= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\psi}) e^{-in\psi}\, d\psi \\
&= \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \hat{f}(n)
\end{aligned}$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\psi})e^{-in\psi}\, d\psi$ is by definition the n-th Fourier coefficient. The Fourier series of f is $\sum_{n\in\mathbb{Z}} \hat{f}(n) \cdot e^{in\theta}$. From our previous work we know

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|n|} e^{in\theta} \hat{f}(n) = f(e^{i\theta})$$

and average is uniform on $\mathbb{D}$.

8 Harmonic Functions VII (01/30)

The Poisson integral is an $\mathrm{Aut}(\mathbb{D})$-intertwining operator. What this means is that we have an operator $p : C(\partial\mathbb{D}) \to \mathrm{Harm}(\mathbb{D})$ (here $\mathrm{Harm}(\mathbb{D})$ means harmonic functions on $\mathbb{D}$). We know that the group $\text{Aut}(\mathbb{D})$ acts on the disk by holomorphic functions. If $u \in \text{Harm}(\mathbb{D})$ and $\psi \in \text{Aut}(\mathbb{D})$ then $u \circ \psi \in \text{Harm}(\mathbb{D})$. We deduce from what we know that $\text{Aut}(\mathbb{D})$ also preserves $\partial\mathbb{D}$. This means that if $f \in C(\partial\mathbb{D})$ and $\psi \in \text{Aut}(\mathbb{D})$ then $f \circ \psi \in C(\partial\mathbb{D})$. The *intertwining property* says concretely that if $f \in C(\partial\mathbb{D})$ and $\psi \in \text{Aut}(\mathbb{D})$, then

$$P[f \circ \psi] = P[f] \circ \psi.$$

This equation is true! The simplest reason is that $P[f \circ \psi]$ is a harmonic function on $\mathbb{D}$ with $f \circ \psi$ as its boundary data. $P[f] \circ \psi$ is harmonic on $\mathbb{D}$ with boundary data $f \circ \psi$ (because ψ is uniformly continuous on $\overline{\mathbb{D}}$). Now uniqueness implies that the two are equal.

Now we come to the *Schwarz Reflection Principle*.

Let D be a connected open set in $\mathbb{C}$. Define

$$\begin{aligned} D^+ &= D \cap \pi^+, \\ D^0 &= D \cap \{\text{real axis}\}, \\ D^- &= D \cap \pi^-. \end{aligned}$$

We say that D is symmetric about the real axis if $\{\overline{z} | z \in D\} = D$. This means that D^- is the reflection of D^+ in the real axis.

Theorem 8.1 (Most Basic Version of Schwarz Reflection Principle)**.** *Suppose that $D \subset \mathbb{C}$ is a connected open set that is symmetric about the real axis. Let $f \in C(D^+ \cup D^0) \cap H(D^+)$ and suppose that f takes real values on D^0. Then there is a function $F \in H(D)$ such that $F|_{D^+ \cup D^0} = f$ and $F(\overline{z}) = \overline{F(z)}$ for all $z \in D$.*

Proof. We define $F : D \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z), & z \in D^+, \\ f(z), & z \in D^0, \\ \overline{f(\overline{z})}, & z \in D^-. \end{cases}$$

If $p \in D^0$ and (z_n) is a sequence in D^- such that $z_n \to p$ then $\overline{z_n} \to \overline{p} = p$ and so $\overline{f(\overline{z_n})} \to \overline{f(p)}$ by the continuity of f on $D^+ \cup D^0$. Since $f(p) \in \mathbb{R}$, $\overline{f(\overline{z_n})} \to f(p)$. It follows that F is continuous at p. Certainly, F is continuous on D^+ and D^-. Thus $F \in C(D)$.

If $q \in D^-$ then $\overline{q} \in D$ and we may represent f as a power series centered at $\overline{q}$ with a positive radius of convergence. Say $f(z) = \sum_{n=0}^{\infty} a_n (z - \overline{q})^n$ on a small disk around $\overline{q}$. Then for z in a small disk about q, we have

$$\overline{f(\overline{z})} = \overline{\sum_{n=0}^{\infty} a_n (\overline{z} - \overline{q})^n} = \sum_{n=0}^{\infty} \overline{a_n} (z - q)^n.$$

Thus F is holomorphic on this disk around q. We claim that F is holomorphic on D. This is done by Morera's Theorem. Since F is continuous, we only need to verify that

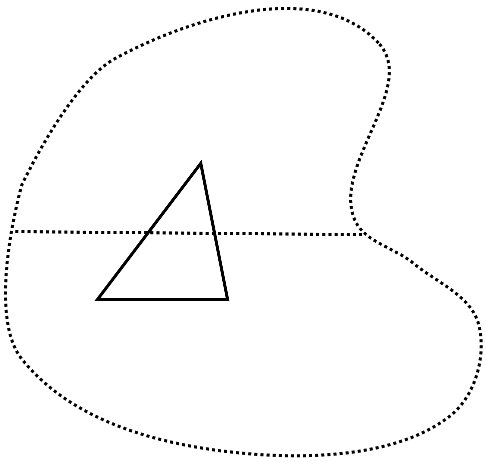


Figure 2: Triangular Touches The Boundary

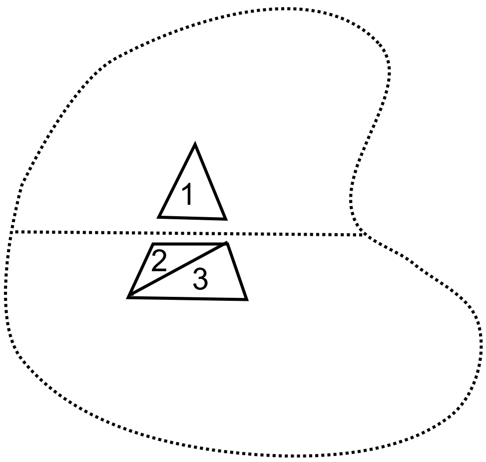


Figure 3: Separate The Triangular

$\int_\Delta F(z)dz = 0$ for all triangles Δ in D. If $\Delta \subset D^+$ or $\Delta \subset D^-$, we already know that $\int_\Delta F(z)dz = 0$ because $F \in H(D^+)$ and $F \in H(D^-)$. The remaining case is that Δ touches D^0. The proof is to separate the triangles. Then

$$\begin{aligned}\int_\Delta F(z)dz &= \lim_{\varepsilon\to 0^+}\int_{\Delta_1+\Delta_2+\Delta_3} F(z)dz\\ &= \lim_{\varepsilon\to 0^+}\int_{\Delta_1} F(z)dz + \lim_{\varepsilon\to 0^+}\int_{\Delta_2+\Delta_3} F(z)dz\\ &= 0.\end{aligned}$$

This shows $F \in H(D)$. The equation $\overline{F(z)} = F(\overline{z})$ follows from the definition of F. $\square$

9 Harmonic Functions VIII (02/02)

Example 9.1. Let $S = \{x + iy" - 1 < x < 1\}$ and $f : \overline{S} \to \mathbb{C}$ be continuous such that f is holomorphic on S and real valued on ∂S. Show that f extends to an entire function F such that $F(z) = F(z + 4)$ for all $z \in \mathbb{C}$.

Reflect through the line $\mathrm{Re}(z) = 1$. Do this by using the map $z \to 2 - \overline{z}$. Let $S' = \{x + iy : 1 < x < 3\}$ and define $F : \overline{S} \cup \overline{S'} \to \mathbb{C}$ by

$$F(z) = \begin{cases} f(z) \text{ if } z \in \overline{S}, \\ \overline{f(2 - \overline{z})} \text{ if } z \in \overline{S'}. \end{cases}$$

The main point about this definition is that F is continuous. The sets in the piecewise definition are closed so we just have to make sure that the two definitions agree on $\overline{S} \cap \overline{S'} = \{1 + iy : y \in \mathbb{R}\}$.

$$\overline{f(2 - \overline{1 + iy})} = \overline{f(2 - (1 - iy))} = \overline{f(1 + iy)} = f(1 + iy)$$

because f is real on ∂S. This verifies the requirement, so F is continuous. On $\mathrm{int}(\overline{S} \cup \overline{S'})$, F is holomorphic by Schwarz Reflection Principle (or by Morera's Theorem). Next,

$$\begin{aligned} F(3 + iy) &= \overline{f(2 - \overline{3 + iy})} = \overline{f(-1 + iy)} \\ &= f(-1 + iy) \text{ (because } f \text{ is real on } \partial S) \\ &= F(-1 + iy). \end{aligned}$$

This means we can define F on $\{x + iy : -1 < x < 7\}$ by

$$F(z) = \begin{cases} F(z) & \text{if } -1 \le \mathrm{Re}(z) \le 3, \\ F(z - 4) & \text{if } 3 \le \mathrm{Re}(z) \le 7. \end{cases}$$

We can continue in this way to extend F to the entire plane. The Identity Principle implies that the resulting function satisfies $F(z) = F(z + 4)$ for all z.

Theorem 9.2 (Schwarz Reflection Principle for Harmonic Functions)**.** *Let D be a connected open set in $\mathbb{C}$ that is symmetric about the real axis. Let $v : D^+ \cup D^0 \to \mathbb{R}$ be a harmonic function on D^+ and continuous on $D^+ \cup D^0$ and assume that $v|_{D^0} = 0$. Then there is a harmonic function $V : D \to \mathbb{R}$ such that $V|_{D^+} = v$ and $V(\overline{z}) = -V(z)$ for all $z \in D$.*

Proof. We define $V : D \to \mathbb{R}$ by

$$V(z) = \begin{cases} v(z) & \text{if } z \in D^+ \cup D^0, \\ -v(\overline{z}) & \text{if } z \in D^- \cup D^0. \end{cases}$$

We need to verify that V is continuous. Both $D^+ \cup D^0$ and $D^- \cup D^0$ are closed in D. Thus we just have to show that the two definitions agree on the overlap, which is D^0. This is true because $v|_{D^0} \equiv 0$ and $-0 = 0$.

To verify that V is harmonic, we will verify that V has the Local Mean Value Property. If $p \in D^+$ then we can take $\rho > 0$ small enough that $\overline{D}(p, \rho) \subset D^+$. If $r \in [0, \rho)$ then

$$\begin{aligned}
&\frac{1}{2\pi} \int_0^{2\pi} V(p + re^{i\theta})\, d\theta \\
=&\frac{1}{2\pi} \int_0^{2\pi} v(p + re^{i\theta})\, d\theta \\
=&v(p) \quad \text{(because } v \text{ is harmonic)} \\
=&V(p).
\end{aligned}$$

If $p \in D^-$ then $\overline{p} \in D^+$ and so we have some $\rho > 0$ such that if $r \in [0, \rho)$ then

$$\begin{aligned}
v(\overline{p}) &= \frac{1}{2\pi} \int_0^{2\pi} v(\overline{p} + re^{i\theta})\, d\theta \\
&= \frac{1}{2\pi} \int_0^{2\pi} v(\overline{p} + re^{-i\theta})\, d\theta \quad \text{(change of variable)} \\
&= \frac{1}{2\pi} \int_0^{2\pi} v(\overline{p + re^{i\theta}})\, d\theta.
\end{aligned}$$

Now take negative of both sides to get

$$V(p) = \frac{1}{2\pi} \int_0^{2\pi} V(p + re^{i\theta})\, d\theta.$$

If $p \in D^0$ then $V(p) = 0$. Also

$$\frac{1}{2\pi} \int_0^{2\pi} V(p + re^{i\theta})\, d\theta = 0$$

for small enough r because the values on the lower half circle are the negatives of the value on the upper half circle. Thus V has the Local Mean Value Property and so V is harmonic. $\square$

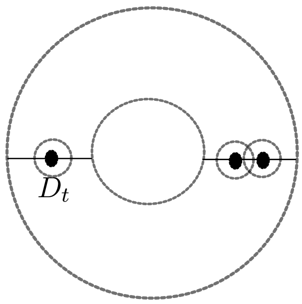
10 Harmonic Functions IX (02/04)

Recall that to use Schwarz Reflection Principle we always have a domain D that is symmetric with respect to the real axis. If $f \in H(D^+)$ extends continuously to $D^+ \cup D^0$, and is real on D^0 then f extends by reflection to an element of $H(D)$. If v is a real-valued harmonic function on D^+ that extends continuously to D^0 with value 0 on D^0 then v extends by reflection to an function harmonic on D. Now we prove another version of Schwarz Reflection Principle.

Theorem 10.1 (Schwarz Reflection Principle)**.** *Let D be a domain symmetric about the real axis. Let $f \in H(D^+)$ and suppose that Im(f) extends continuously to $D^+ \cup D^0$ with value 0 on D^0. Then f extends to an element $F \in H(D)$ such that $F(\overline{z}) = \overline{F(z)}$ for all $z \in D$. (In particular, the real part of f extends continuously to D).*

Proof. All we have to do is to confirm that f extends continuously to $D^+ \cup D^0$. If it does, then the theorem follows by applying the first version that we proved.

Let $v = \mathrm{Im}(f)$. Then v is harmonic on D^+ and extends continuously to $D^+ \cup D^0$ with value 0 on D^0. By the second version of Schwarz Reflection, v extends to a harmonic function V on D.



Next we choose a disk D_t centered at $t \in D^0$ small enough that $D_t \subset D$ for each $t \in D^0$. On each D_t, we have the harmonic function V defined. Also, D_t is simply connected. Thus we may choose $\tilde{f}_t \in H(D_t)$ such that $\mathrm{Im}(\tilde{f}_t) = V|_{D_t}$. Note that $\tilde{f}_t$ is not unique – we may add or subtract a real constant to each one, and this is the only freedom in choosing $\tilde{f}_t$.

Let $t \in D^0$. We know that $\mathrm{Im}(\tilde{f}_t) = V = v = \mathrm{Im}(f)$ on the set $D_t \cap D^+$. This means that there is a real constant c_t such that $f - \tilde{f}_t = c_t$ on $D_t \cap D^+$. Let $f_t = \tilde{f}_t + c_t$ for all $t \in D^0$. Then $f_t \in H(D_t)$ and $f|_{D_t \cap D^+} = f_t|_{D_t \cap D^+}$.

Define $F : D^+ \cup (\cup_{t \in D^0} D_t) \to \mathbb{C}$ by

$$f(z) = \begin{cases} f(z), & z \in D^+, \\ f_t(z), & z \in D_t. \end{cases}$$

F will be a continuous extension of f to $D^+ \cup (\cup_{t\in D^0} D_t) \supset D^+ \cup D^0$, provided that we can show that the different definitions of the value of f agree on the overlaps. We already know that $f|_{D_t\cap D^+} = f_t|_{D_t\cap D^+}$ for all $t \in D^0$. We also have to check that if $s, t \in D^0$, then $f_t|_{D_t\cap D_s} = f_s|_{D_t\cap D_s}$. Note that $D_t \cap D_s$ is always connected and $D_t \cap D_s \cap D^+$ is never empty if $D_t \cap D_s \neq \emptyset$. We know that

$$f_t|_{D_t\cap D_s\cap D^+} = f|_{D_t\cap D_s\cap D^+} = f_s|_{D_t\cap D_s\cap D^+}$$

and now the facts that f_s, f_t are holomorphic, $D_t \cap D_s \cap D^+$ has limit point in $D_t \cap D_s$, and $D_t \cap D_s$ is connected imply that

$$f_t|_{D_t\cap D_s} = f_s|_{D_t\cap D_s}.$$

This verifies that F is continuous and hence f extends continuously to $D^+ \cap D^0$. □

11 Harmonic Functions X (02/06)

Recall

$$P_r(\theta) = \frac{1-r^2}{1-2r\cos(\theta)+r^2}.$$

Thus

$$\frac{1-r}{1+r} = \frac{1-r^2}{1+2r+r^2} \le P_r(\theta) \le \frac{1-r^2}{1-2r+r^2} = \frac{1+r}{1-r}.$$

So we get

$$\frac{1-r}{1+r} \le P_r(\theta) \le \frac{1+r}{1-r}, \ \forall\theta, \forall r \in [0,1). \tag{11.1}$$

If we have u harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$ then $u = P[u]$. (The reason is that both u and $P[u]$ are harmonic, and they have the same boundary values.)

Assume that u is continuous on $\overline{\mathbb{D}}$, harmonic on $\mathbb{D}$, and $u(z) \ge 0$ for all $z \in \overline{\mathbb{D}}$. Then for $re^{i\theta} \in \mathbb{D}$ we have

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-\psi)u(e^{i\psi})\,d\psi.$$

We apply the inequality in (11.1) multiplied through by $u(e^{i\psi})$ and we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r}{1+r}u(e^{i\psi})\,d\psi \le u(re^{i\theta}) \le \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1+r}{1-r}u(e^{i\psi})\,d\psi,$$

Hence

$$\frac{1-r}{1+r}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\psi})\,d\psi \le u(re^{i\theta}) \le \frac{1+r}{1-r}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\psi})\,d\psi.$$

Notice that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}u(e^{i\psi})\,d\psi = P[u](0) = u(0).$$

Therefore,

$$\frac{1-r}{1+r}u(0) \leq u(re^{i\theta}) \leq \frac{1+r}{1-r}u(0).$$

This is called the *Harnack's Inequality*.

First improvement of this inequality is that if $u \geq 0$ is harmonic in $\mathbb{D}$ then the inequality still holds. (We are dropping $u \in C(\overline{\mathbb{D}})$.) Let $0 < \rho < 1$. Define $u_\rho(z) = u(\rho z)$. Certainly $u_\rho \geq 0$, u_ρ is harmonic on $D(0, \frac{1}{\rho})$ and in particular it is harmonic on $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. Thus

$$\frac{1-r}{1+r}u_\rho(0) \leq u_\rho(re^{i\theta}) \leq \frac{1+r}{1-r}u_\rho(0)$$

for all $\rho \in (0,1), r \in [0,1)$, and all θ. So

$$\frac{1-r}{1+r}u(0) \leq u(\rho re^{i\theta}) \leq \frac{1+r}{1-r}u(0).$$

Now take the limit as $\rho \to 1^-$ to get the required inequality for u.

A consequence of this inequality is that if $q_1, q_2 \in \mathbb{D}$ then there are positive constants c_1, c_2 such that

$$c_1 u(q_1) \leq u(q_2) \leq c_2 u(q_1)$$

for all non-negative harmonic functions in $\mathbb{D}$. This follows from

$$\frac{1-|q_1|}{1+|q_1|}u(0) \leq u(q_1) \leq \frac{1+|q_1|}{1-|q_1|}u(0),$$

$$\frac{1-|q_2|}{1+|q_2|}u(0) \leq u(q_2) \leq \frac{1+|q_2|}{1-|q_2|}u(0),$$

Moreover, if q_1, q_2 are restricted to lie in a compact set, then the constants c_1, c_2 may be taken uniform for all q_1, q_2 in this compact set.

There is a version of Harnack's Inequality for $D(a, R)$. If $u \geq 0$ is harmonic on $D(a, R)$, then $v : \mathbb{D} \to \mathbb{R}$ given by

$$v(w) = u(a + Rw)$$

is harmonic and non-negative. The inverse relationship is

$$u(z) = v\left(\frac{z-a}{R}\right).$$

We know that

$$\frac{1-r}{1+r}v(0) \leq v(re^{i\theta}) \leq \frac{1+r}{1-r}v(0)$$

for all $r \in [0,1)$ and all θ. Then

$$\frac{1-r}{1+r}u(a) \leq u(a + Rre^{i\theta}) \leq \frac{1+r}{1-r}u(a),$$

So

$$\frac{1-\frac{\rho}{R}}{1+\frac{\rho}{R}}u(a) \le u(a+\rho e^{i\theta}) \le \frac{1+\frac{\rho}{R}}{1-\frac{\rho}{R}}u(a),$$

and so

$$\frac{R-\rho}{R+\rho}u(a) \le u(a+\rho e^{i\theta}) \le \frac{R+\rho}{R-\rho}u(a).$$

This is the *Harnack's Inequality for* $D(a, R)$, which would be very useful.

If also follows that if $K \subset D(a, R)$ is a compact set, then there are positive constants c_1, c_2 such that

$$c_1 u(q_1) \le u(q_2) \le c_2 u(q_1)$$

for all $q_1, q_2 \in K$ and all non-negative harmonic functions u in $D(a, R)$.

Theorem 11.1 (Harnack's Inequality)**.** *Let $V \subset \mathbb{C}$ be a connected open set and $K \subset V$ a compact set. Then there are positive constants c_1, c_2 such that*

$$c_1 u(q_1) \le u(q_2) \le c_2 u(q_1)$$

for all $q_1, q_2 \in K$ and and all non-negative harmonic functions u on V.

The proof will be given in the next lecture.

12 Harmonic Functions XI (02/09)

Here is the proof for Theorem 11.1.

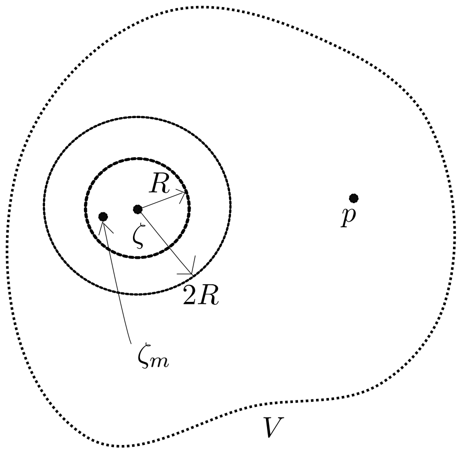
Proof. Let $p \in V$. Define

$$\begin{aligned} S = \{z \in V |\ & \text{there is a disk } D(z, r) \in V \text{ and constants } k_1, k_2 > 0 \text{ such that} \\ & k_1 u(p) \le u(q) \le k_2 u(p) \text{ for all } q \in D(z, r) \text{ and all non-negative} \\ & \text{harmonic functions } u\} \end{aligned}$$

First, $S \neq \emptyset$ because $p \in S$ by our previous Harnack's Inequality for a disk.

Second, S is open. If $z \in S$, then we choose $D(z, r)$ as in the definition. If $w \in D(z, r)$ then we choose $D(w, \rho) \subset D(z, r)$. The disk $D(w, \rho)$ has the required property, so $w \in S$. Thus $D(z, r) \subset S$. This shows S is open.

Third, we want to show that S is closed in V. Suppose that $\zeta \in \mathrm{cl}_V(S)$ and choose a sequence (ζ_n) in S such that $\zeta_n \to \zeta$. Now $\zeta \in V$ and so I may choose $R > 0$ such that $D(\zeta, 2R) \subset V$. Choose m such that $\zeta_m \in D(\zeta, R)$. I claim that there is a two-sided estimate for $u(z)$ in terms of $u(p)$ for all $z \in D(\zeta, R)$. If so then $\zeta \in S$ and so S is closed in V. By definition, the fact that $\zeta_m \in S$ implies that there is a two-sided estimate for $u(\zeta_m)$ in terms of $u(p)$. By the version of Harnack's Inequality for disks, applied to $\overline{D}(\zeta, R) \subset D(\zeta, 2R)$ there is a uniform two-sided estimate for $u(z)$ in terms of $u(\zeta_m)$ for all



$z \in \overline{D}(\zeta, R)$. By transitivity of inequality, we have a uniform two-sided estimate for $u(z)$ in terms of $u(p)$ for all $z \in \overline{D}(\zeta, R)$. This verifies that f is closed in V. By connectedness, $S = V$.

Next, let $K \subset V$ be a compact set. For each $z \in K$ let $D(z, r_z)$ be a disk as in the previous step. Then $\{D(z, r_z)\}$ is an open cover of K, so we can choose a finite subcover $\{D(z, r_z) | z \in F\}$. For each $z \in F$, we have an estimate

$$A_z u(p) \leq u(w) \leq B_z u(p)$$

for all $w \in D(z, r_z)$ and all non-negative harmonic functions u. Let

$$A = \min\{A_z | z \in F\}, B = \max\{B_z | z \in F\}.$$

Then

$$Au(p) \leq u(w) \leq Bu(p)$$

for all $w \in K$. If w_1, w_2 are in K, then

$$u(w_1) \leq Bu(p) \leq A^{-1}Bu(w_2).$$

Similarly,

$$u(w_2) \leq A^{-1}Bu(w_1).$$

So

$$AB^{-1}u(w_2) \leq u(w_1).$$

This completes the proof. $\square$

Theorem 12.1. *Let V be a connected open set in $\mathbb{C}$. The set of harmonic functions is a closed subset of $C(V)$ with the topology of uniform convergence on compact subsets.*

Proof. Let (u_n) be a sequence of harmonic functions of V and suppose that $u_n \to u$ uniformly on compact subsets of V, with $u \in C(V)$. We have to show that u is harmonic. We check that u has the Mean Value Property. Suppose that $\overline{D}(z,r) \subset V$. Then $\partial D(z,r)$ is a compact subset of V and so $u_n|_{\partial \overline{D}(z,r)} \to u|_{\partial \overline{D}(z,r)}$ uniformly. Also, $u_n(z) \to u(z)$. Thus,

$$\begin{aligned}
u(z) &= \lim_{n\to\infty} u_n(z) \\
&= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} u_n(z + re^{i\theta})\, d\theta \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \lim_{n\to\infty} u_n(z + re^{i\theta})\, d\theta \quad \text{(since convergence is unifrom)} \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z + re^{i\theta})\, d\theta
\end{aligned}$$

and so u has the Mean Value Property. $\square$

Theorem 12.2 (Harnack's Theorem)**.** *Let V be a connected open set and suppose that (u_n) is a pointwise increasing sequence of harmonic functions in V. Then either (u_n) converges uniformly to ∞ on compact subsets of V or (u_n) converges uniformly to a harmonic function u on compact subsets.*

Proof. By replacing u_n by $u_n - u_1$ we may assume that $u_n \geq 0$ on V. Suppose that $u_n(z_0) \to \infty$ for some $z_0 \in V$. Let $K \subset V$ be compact then $K \cup \{z_0\}$ is also compact and so there are A, B positive constants such that

$$Au_n(z_0) \leq u_n(z) \leq Bu_n(z_0)$$

for all $z \in K$. This shows that $u_n \to \infty$ uniformly on K.

Otherwise, $(u_n(z))$ is bounded above for all $z \in V$. Choose $z_0 \in V$. Let $K \subset V$ be a compact set and note that

$$u_m(z) - u_n(z) \leq B(u_m(z_0) - u_n(z_0))$$

for all $z \in K$ and all $m \geq n$. This shows that (u_n) is uniformly Cauchy on K. The limit u is continuous and harmonic by the previous theorem. $\square$

13 Hermitian Metrics on Plane Domains I (02/11)

Let $U \subset \mathbb{C}$ be domain (a connected open set).

Definition 13.1. A *Hermitian metric* on U is a continuous function $\rho : U \to [0, \infty)$ such that the restriction of ρ to the set $\{z \in U | \rho(z) \neq 0\}$ is C^2.

Definition 13.2. The Hermitian metric ρ is said to be *non-degenerate* if it does not take the value 0.

Say we have a Hermitian metric on a domain U and $\gamma : [a, b] \to U$ is a piecewise smooth path. Then we define the *ρ-length of* γ to be

$$\mathcal{L}_\rho(\gamma) = \int_a^b |\gamma'(t)| \rho(\gamma(t))\, dt.$$

We also define a function $d_\rho : U \times U \to [0, \infty)$ by

$$d_\rho(p, q) = \inf\{\mathcal{L}_\rho(\gamma)| \text{ all piecewise smooth } \gamma : [0, 1] \to U \text{ such that } \gamma(0) = p, \gamma(1) = q\}.$$

This function always satisfies
(1) $d_\rho(p, q) \geq 0$,
(2) $d_\rho(p, q) = d_\rho(q, p)$,
(3) $d_\rho(p, s) \leq d_\rho(p, q) + d_\rho(q, s)$. If ρ is non-degenerate, then d_ρ is a metric on U. Generally, it is a pseudometric.

Example 13.3. $\rho : \mathbb{C} \to [0, \infty)$ given by $\rho(z) = 1$ for all $z \in \mathbb{C}$. In this case, $\mathcal{L}_\rho(\gamma)$ is the usual length of γ and d_ρ is the usual distance on $\mathbb{C}$.

How do we know? Let $p, q \in \mathbb{C}$. Translation of $\mathbb{C}$ does not change $\mathcal{L}_\rho(\gamma)$ (because ρ is constant). Rotation of $\mathbb{C}$ also does not change $\mathcal{L}_\rho(\gamma)$. Thus, we may assume that $p = 0$ and $q \in [0, \infty)$. Let $\gamma : [0, 1] \to \mathbb{C}$ be a path from 0 to q. Then $\gamma(t) = (x(t), y(t))$ for some $x, y : [0, 1] \to \mathbb{R}$. Define $\sigma : [0, 1] \to \mathbb{C}$ be $\sigma(t) = (x(t), 0)$. σ is piecewise smooth. $\gamma(0) = (x(0), y(0)) = 0$ and so $y(0) = 0$. $\gamma(1) = (x(1), y(1)) = q$ and so $y(1) = 0$. Thus $\sigma(0) = (x(0), 0) = 0$ and $\sigma(1) = (x(1), 0) = q$. Now

$$\begin{aligned}
\mathcal{L}_\sigma(\gamma) &= \int_0^1 |\gamma'(t)| dt = \int_0^1 \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\, dt \\
&\geq \int_0^1 |\dot{x}(t)|\, dt \\
&\geq \int_0^1 \dot{x}(t)\, dt \\
&= x(1) - x(0) \\
&= q.
\end{aligned}$$

This tells us that $d_\rho(p, q) \geq q$. The line segment $[0, q]$ realizes the length q. Thus $d_\rho(0, q) = q = |0 - q|$. Since the normal distance is invariant under translation and rotation, d_ρ coincides with the normal distance.

Example 13.4. The *Poincare metric* on $D(o, R)$ is

$$\rho_R(z) = \frac{R}{R^2 - |z|^2}.$$

(Note that a lot of people use $\frac{2R}{R^2-|z|^2}$.)

Example 13.5. The *spherical metric* on $\mathbb{C}$ is

$$\sigma(z) = \frac{z}{1 + |z|^2}.$$

Next time I want to justify the name of spherical metric. After that, I will define pull-back of metrics and reformulate the Schwarz-Pick Lemma.

14 Hermitian Metrics on Plane Domains II (02/13)

Recall

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}.$$

There are two metrics on S^2.

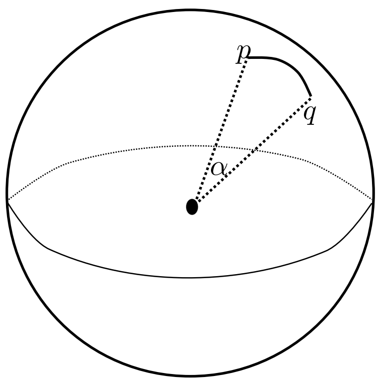


Figure 4: Spherical metric

The first one is the *chordal metric*. The distance between $p, q \in S^2$ is

$$d_{\text{chordal}}(p, q) = \|p - q\|.$$

The second one is the *spherical metric*. The distance between $p, q \in S^2$ is

$$d_{\text{spherical}}(p, q) = \arccos(p \cdot q).$$

d_{chordal} and $d_{\text{spherical}}$ are comparable:

$$\frac{2}{\pi} d_{\text{spherical}}(p, q) \leq d_{\text{chordal}}(p, q) \leq d_{\text{spherical}}(p, q).$$

Remark 14.1. *Why is $d_{spherical}$ actually a metric? To show that $d_{spherical}$ is actually a metric, we will show that*

$$d_{spherical}(p, q) = \min\{\mathcal{L}(\gamma) | \gamma : [0, 1] \to S^2 \textit{is a piecewise smooth path with} \\ \gamma(0) = p, \gamma(1) = q, \textit{ where } \mathcal{L}(\gamma) = \int_0^1 \|\gamma'(t)\| \, dt.\}$$

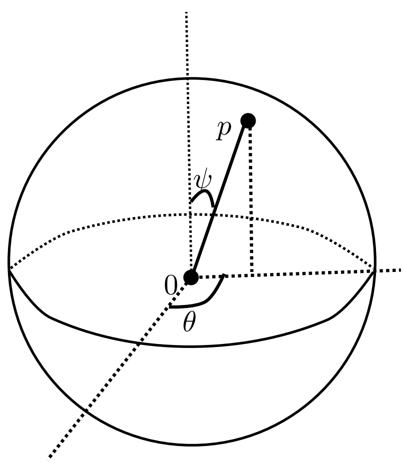


Figure 5: Spherical coordinates

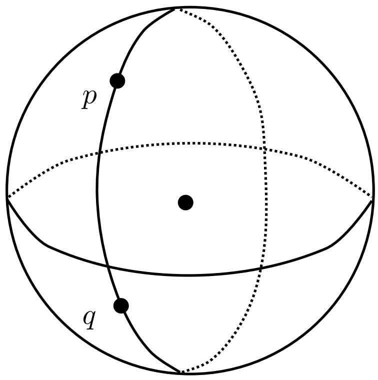


Figure 6:

On the sphere, I will use spherical coordinates (see Figure 5) with θ as the azimuth and ψ as the zenith:

$$\begin{cases} x = \cos(\theta)\sin(\psi), \\ y = \sin(\theta)\sin(\psi), \\ z = \cos(\psi). \end{cases}$$

Rotation of the sphere about 0 does not change $d_{\text{spherical}}$. It also does not change $\mathcal{L}(\gamma)$. Thus I may assume that p, q both lie on the great circle where $\theta = 0$ (see the above figure). Let $\gamma : [0,1] \to S^2$ be a piecewise smooth path. Then we may find functions $\theta, \psi : [0,1] \to \mathbb{R}$ such that they are piecewise smooth and

$$\gamma(t) = \begin{pmatrix} \cos(\theta(t))\sin(\psi(t)) \\ \sin(\theta(t))\sin(\psi(t)) \\ \cos(\psi(t)) \end{pmatrix}.$$

Then

$$\gamma'(t) = \begin{pmatrix} -\sin(\theta)\sin(\psi)\cdot\dot{\theta} + \cos(\theta)\cos(\psi)\cdot\dot{\psi} \\ \cos(\theta)\sin(\psi)\cdot\dot{\theta} + \sin(\theta)\cos(\psi)\cdot\dot{\psi} \\ -\sin(\psi)\cdot\dot{\psi} \end{pmatrix},$$

$$\|\gamma'(t)\|^2 = \sin^2(\psi)(\dot{\theta})^2 + (\dot{\psi})^2,$$

$$
\begin{aligned}
\mathcal{L}(\gamma) &= \int_0^1 \sqrt{\sin^2(\psi)(\dot{\theta})^2 + (\dot{\psi})^2}\, dt \\
&\geq \int_0^1 |\dot{\psi}|\, dt \\
&\geq |\psi(1) - \psi(0)| \\
&= d_{\text{spherical}}(p, q).
\end{aligned}
$$

This verifies that

$$
\begin{aligned}
d_{\text{spherical}}(p,q) \leq \inf\{\mathcal{L}(\gamma) | \gamma : [0,1] \to S^2 &\text{is a piecewise smooth path with} \\
\gamma(0) = p, \gamma(1) = q, \text{ where } \mathcal{L}(\gamma) &= \int_0^1 \|\gamma'(t)\|\, dt.\}
\end{aligned}
$$

Actually

$$
\begin{aligned}
d_{\text{spherical}}(p,q) = \min\{\mathcal{L}(\gamma) | \gamma : [0,1] \to S^2 &\text{is a piecewise smooth path with} \\
\gamma(0) = p, \gamma(1) = q, \text{ where } \mathcal{L}(\gamma) &= \int_0^1 \|\gamma'(t)\|\, dt.\}
\end{aligned}
$$

because the great arc achieves the minimum.

Let $p : \mathbb{C} \to S^2$ be the inverse of stereographic projection. Then

$$
p(x+iy) = \left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, \frac{x^2+y^2-1}{1+x^2+y^2} \right).
$$

We want to know what Hermitian metric σ on $\mathbb{C}$ is such that

$$
\mathcal{L}_\sigma(\gamma) = \mathcal{L}(p \circ \gamma).
$$

Note that if we find such a σ then

$$
d_\sigma(z,w) = d_{\text{spherical}}(p(z), p(w)).
$$

Say we have a path $\gamma : [0,1] \to \mathbb{C}$. We have to compute $\|(p \circ \gamma)'(t)\|$. This is because

$$
\mathcal{L}(p \circ \gamma) = \int_0^1 \|(p \circ \gamma)'\| dt.
$$

We want

$$
\begin{aligned}
\mathcal{L}(p \circ \gamma) &= \int_0^1 \|(p \circ \gamma)'\| dt \\
&= \int_0^1 |\gamma'(t)| \sigma(\gamma(t))\, dt \\
&= \mathcal{L}_\sigma(\gamma).
\end{aligned}
$$

Let $\gamma(t) = x(t) + y(t)i$, then

$$
\|(p \circ \gamma)'(t)\| = \frac{2}{1+x^2+y^2} |\gamma'(t)| = \frac{2}{1+|\gamma(t)|^2} |\gamma'(t)|
$$

This tells us that $\sigma(z) = \frac{2}{1+|z|^2}$ works. That is, $d_\sigma(z,w) = d_{\text{spherical}}(p(z), p(w))$.

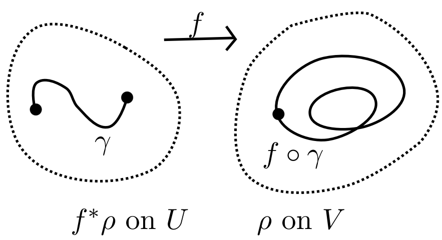
15 Hermitian Metrics on Plane Domains III (02/18)

We now come to the pullback of Hermitian metrics.

Say $f : U \to V$ is a holomorphic function and ρ is a Hermitian metric on V. Then we define a metric on U by

$$(f^*\rho)(z) = \rho(f(z))|f'(z)|.$$

$(f^*\rho)(z)$ can be zero in two ways – either $\rho(f(z)) = 0$ or $f'(z) = 0$. Certainly $f^*\rho$ is a continuous function into $[0, \infty)$. If we pick a point in $\tilde{U} = \{z \in U | (f^*\rho)(z) \neq 0\}$, then $f^*\rho$ is C^2 near this point.



Here is the explanation for why the pullback is defined this way.

$$\begin{aligned}
\mathcal{L}_\rho(f \circ \gamma) &= \int_0^1 \left|(f \circ \gamma)'(t)\right| \rho((f \circ \gamma(t)))\, dt \\
&= \int_0^1 \left|f'(\gamma(t)) \cdot \gamma'(t)\right| \rho(f(\gamma(t)))\, dt \\
&= \int_0^1 \left|\gamma'(t)\right| \rho(f(\gamma(t))) \left|f'(\gamma(t))\right|\, dt \\
&= \int_0^1 \left|\gamma'(t)\right| (f^*\rho)(\gamma(t))\, dt \\
&= \mathcal{L}_{f^*\rho}(\gamma).
\end{aligned}$$

If γ is a path from p to q in U then $f \circ \gamma$ is a path from $f(p)$ to $f(q)$ in V. We have just seen that these two paths have the same length. It follows that

$$d_\rho(f(p), f(q)) \leq d_{f^*\rho}(p, q).$$

This inequality is not generally an equality, but it is if f is conformal equivalence. To see this, we would like to apply the inequality to $f^{-1} : V \to U$. To do this, we should verify that the pullback is functorial.

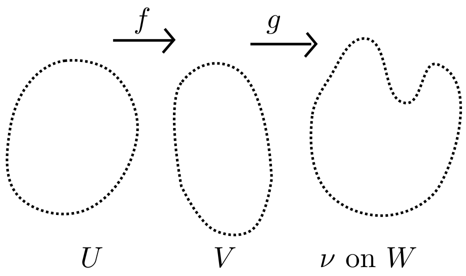


Figure 7:

Functorial says

$$(g \circ f)^*\nu = f^* g^* \nu.$$

Now

$$\begin{aligned} f^* g^* \nu(z) &= g^*\nu(f(z)) \left|f'(z)\right| \\ &= \nu(g(f(z))) \cdot \left|g'(f(z))\right| \cdot \left|f'(z)\right| \\ &= \nu(g \circ f(z)) \left|(g \circ f)'(z)\right| \\ &= (g \circ f)^*\nu(z). \end{aligned}$$

In particular, $(f^{-1})^*(f)^*\rho = \rho$ and so the previous argument works to show that $d_\rho(f(p), f(q)) = d_{f^*\rho}(p, q)$ if f is a conformal equivalence. That is, a conformal equivalence becomes an isometry (of pseudometrics) between U and V.

We have actually seen pullback of Hermitian metrics in disguise. If $f : \mathbb{D} \to \mathbb{D}$ is holomorphic then the Schwarz-Pick Lemma says

$$\frac{|f'(z)|}{1-|f(z)|^2} \leq \frac{1}{1-|z|^2}$$

for all $z \in \mathbb{D}$. $\frac{1}{1-|z|^2}$ is the Poincare metric ρ_1 on $D(0,1)$. Also,

$$(f^*\rho_1)(z) = \rho_1(f(z))|f'(z)| = \frac{|f'(z)|}{1-|f(z)|^2}.$$

So

$$f^*\rho_1 \leq \rho_1.$$

The idea is to generalize this inequality. The key is to introduce the curvature of a metric.

16 Curvature I (02/23)

Let $U \subset \mathbb{C}$ be a domain. Let $\rho : U \to [0, \infty)$ be a Hermitian metric. We define the *curvature* of ρ to be $k_\rho : \{z \in U | \rho(z) \neq 0\} \to \mathbb{R}$ by

$$k_\rho(z) = \frac{-\Delta \log(\rho)(z)}{\rho^2(z)}.$$

The curvature is a conformal invariant and this is what makes it so important.

Say $A > 0$. Then $A\rho$ is also a Hermitian metric on U. Moreover,

$$\begin{aligned} k_{A\rho}(z) &= \frac{-\Delta \log(A\rho)(z)}{(A\rho(z))^2} \\ &= \frac{-\Delta[\log(A) + \log(\rho)](z)}{A^2\rho^2(z)} \\ &= \frac{-\Delta \log(\rho)(z)}{A^2\rho^2(z)} \quad \text{(since } \Delta \log(A) = 0\text{)} \\ &= \frac{1}{A^2} k_\rho(z). \end{aligned}$$

Theorem 16.1 (Conformal Invariance of Curvature)**.** *Let U, V be domains, $f : U \to V$ a holomorphic function and ν a Hermitian metric on V. Then*

$$k_{f^*\nu}(w) = k_\nu(f(w))$$

for all w such that both sides are defined.

Proof. On the set where $k_{f^*\nu}$ is defined, we have

$$\begin{aligned}
k_{f^*\nu} &= -\frac{\Delta \log(f^*\nu)}{(f^*\nu)^2} \\
&= -\frac{\Delta \log\left((\nu \circ f) \cdot |f'|\right)}{(\nu \circ f)^2 \cdot |f'|^2} \\
&= -\frac{\Delta \log(\nu \circ f) + \Delta \log |f'|}{(\nu \circ f)^2 \cdot |f'|^2} \\
&= -\frac{\Delta \log(\nu \circ f)}{(\nu \circ f)^2 \cdot |f'|^2} \\
&\quad \text{(because at a point where } k_{f^*\nu} \text{ is defined, } f' \text{ is not 0. It is therefore} \\
&\quad \text{not 0 near this point and so } \log|f'| \text{ is harmonic.)} \\
&= -\frac{\Delta\left[(\log \circ \nu) \circ f\right]}{(\nu \circ f)^2 |f'|^2} \\
&= -\frac{(\Delta(\log \circ \nu) \circ f) \cdot |f'|^2}{(\nu \circ f)^2 |f'|^2} \\
&\quad \text{(recall if } g \text{ is holomorphic and } \psi \text{ is } C^2 \text{ then } \Delta(\psi \circ g) = (\Delta\psi \circ g) \cdot |g'|^2\text{.)} \\
&= -\frac{\Delta(\log \circ \nu) \circ f}{(\nu \circ f)^2} \\
&= k_\nu \circ f.
\end{aligned}$$

This is equivalent to the required property. □

Remark 16.2. *$k_\nu(f(w))$ is defined when $\nu(f(w)) \neq 0$. $k_{f^*\nu}(w)$ is defined when $f^*\nu(w) \neq 0$ which is equivalent to $\nu(f(w)) \neq 0$ and $f'(w) \neq 0$.*

Example 16.3. Let's compute k_{ρ_R} where $\rho_R : D(0, R) \to [0, \infty)$ is

$$\rho_R(z) = \frac{R}{R^2 - |z|^2}.$$

I will use $\Delta = 4\partial\overline{\partial}$. I start with

$$\begin{aligned}
\overline{\partial}\log(\rho_R) &= \overline{\partial}\log\left(\frac{R}{R^2-|z|^2}\right)\\
&= \overline{\partial}\left(\log(R)-\log(R^2-|z|^2)\right)\\
&= -\overline{\partial}\log(R^2-|z|^2)\\
&= -\overline{\partial}\log(R^2-x^2-y^2)\\
&= -\frac{1}{2}(\partial_x+i\partial_y)\log(R^2-x^2-y^2)\\
&= -\frac{1}{2}\left(\frac{-2x}{R^2-x^2-y^2}+i\frac{-2y}{R^2-x^2-y^2}\right)\\
&= \frac{x+iy}{R^2-x^2-y^2}\\
&= \frac{z}{R^2-z\overline{z}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\partial\overline{\partial}\log(\rho_R) &= \partial\left(\frac{z}{R^2-z\overline{z}}\right)\\
&= \frac{(R^2-z\overline{z})\partial z - z\partial(R^2-z\overline{z})}{(R^2-z\overline{z})^2}\\
&= \frac{R^2-z\overline{z}-z(-\overline{z})}{(R^2-z\overline{z})^2} \quad (\text{since } \partial(z)=1, \partial\overline{z}=0)\\
&= \frac{R^2}{(R^2-z\overline{z})^2}\\
&= \frac{R^2}{(R^2-|z|)^2}.
\end{aligned}$$

Therefore, $\Delta\log(\rho_R) = \frac{4R^2}{(R^2-|z|)^2}$. Thus,

$$k_{\rho_R} = -\frac{\frac{4R^2}{(R^2-|z|)^2}}{\left(\frac{R}{R^2-|z|}\right)^2} = -4.$$

What we are aiming for is a generalization of the Schwarz-Pick Lemma. Recall that the Schwarz-Pick Lemma can be rephrased as saying that if $f:\mathbb{D}\to\mathbb{D}$ is holomorphic then $f^*\rho_1 \leq \rho_1$. We know that $f^*\rho_1$ has curvature -4 at every point where it is defined. In general, we will show that if ν is a Hermitian metric on $\mathbb{D}$ with $k_\nu \leq 4$ at every point where it is defined, then $\nu \leq \rho_1$.

17 Curvature II (02/25)

Lemma 17.1. *Let $U \subset \mathbb{C}$ be an open set and $\varphi : U \to \mathbb{R}$ a C^2 function that has a local maximum at $q \in U$. Then $\Delta\varphi(q) \leq 0$.*

Proof. Suppose not. Then either $\varphi_{xx}(q) > 0$ or $\varphi_{yy}(q) > 0$. Assume that $\varphi_{xx}(q) > 0$. Consider the function $x \mapsto \varphi(x, q_2)$ where $q = (q_1, q_2)$. The second derivative of this function is positive at q_1, and so positive at all x near enough to q_1, since the second derivative is continuous. We know that q is a critical point of φ since it is a local maximum and so the derivative of $x \mapsto \varphi(x, q_2)$ is zero at q_1. By the Mean Value Theorem, the derivative is strictly positive to the right of q_1 and so there are points as close to q_1 as desired such that $\varphi(x, q_2) > \varphi(q_1, q_2)$. This contradicts the fact that q is a local maximum point for φ. □

Theorem 17.2 (Ahlfors-Schwarz-Pick Lemma)**.** *Let ν be a Hermitian metric on $D(0, R)$ for some $R > 0$. Suppose that $k_\nu(z) \leq -4$ for all z where k_ν is defined. Then $\nu(z) \leq \rho_R(z)$ for all $z \in D(0, R)$.*

Proof. The inequality $k_\nu(z) \leq -4$ says

$$\frac{-\Delta(\log(\nu))(z)}{(\nu(z))^2} \leq -4,$$

which is equivalent to

$$\Delta(\log(\nu)) \geq 4\nu^2.$$

We know that $k_{\rho_R} = -4$ for all z and so

$$\Delta(\log(\rho_R)) = 4\rho_R^2.$$

Actually, $\Delta(\log(\rho_r)) = 4\rho_r^2$ on $D(0, r)$ for all $0 < r < R$. Choose $r \in (0, R)$. Define $\psi : D(0, r) \to \mathbb{R}$ by

$$\psi(z) = \frac{\nu(z)}{\rho_r(z)}.$$

Note that $\psi(z) \geq 0$ for all z and that $\lim_{|z| \to r} \psi(z) = 0$. (Note that ν is bounded on $\overline{D}(0, r)$ and $\lim_{|z| \to r^-} \rho_r(z) = \infty$.) The function ψ must achieve a maximum value somewhere on $D(0, r)$. Call this maximum M_r and let q_r be a point in $D(0, r)$ where the maximum is achieved. We aim to show that $M_r \leq 1$. If $M_r = 0$ then we are done, so assume not. Then $M_r > 0$ and so $\nu(q_r) \neq 0$. We may find an open set $U \subset D(0, r), q_r \in U$ such that ν never vanished on U. By Lemma 17.1,

$$\begin{aligned}
0 &\geq \Delta \log(\psi) \\
&\quad \text{(note, log is monotone increasing, so } \log(\psi) \text{ also has a maximum at } q_r\text{)} \\
&= \Delta \left(\log(\frac{\nu}{\rho_r}) \right) (q_r) \\
&= \Delta(\log(\nu) - \log(\rho_r))(q_r) \\
&= (\Delta(\log(\nu)) - \Delta(\rho_r)))\,(q_r) \\
&\geq 4\nu^2(q_r) - 4\rho_r^2(q_r) \quad (\text{note } \Delta(\log(\rho_r)) = 4\rho_r^2) \\
&= 4\rho_r^2(q_r) \cdot \left(\frac{\nu^2(q_r)}{\rho_r^2(q_r)} - 1 \right) \\
&= 4\rho_r^2(q_r) \left(\psi^2(q_r) - 1 \right) \\
&= 4\rho_r^2(q_r)(M_r^2 - 1).
\end{aligned}$$

This implies that $M_r^2 - 1 \leq 0$ and so $M_r \leq 1$. We have now proved that $\frac{\nu(z)}{\rho_r(z)} \leq 1$ for all $z \in D(0, r)$. That is, $\nu(z) \leq \rho_r(z)$ for all $z \in D(0, r)$. We now let $r \to R^{-1}$ to conclude that $\nu(z) \leq \rho_R(z)$ for all $z \in D(0, R)$. □

Corollary 17.3 (Ahlfors-Schwarz-Pick Lemma)**.** *Let $U \subset \mathbb{C}$ be a domain with a metric ν on it that satisfies $k_\nu(z) \leq -c$ for all $z \in U$ where the curvature is defined. Here $c > 0$ is a constant. Let $f : D(0, R) \to U$ be a holomorphic function. Then*

$$(f^*\nu)(w) \leq \frac{2}{\sqrt{c}} \rho_R(w)$$

for all $w \in D(0, R)$.

Proof. Let $\tilde{\nu} = \frac{\sqrt{c}}{2}\nu$. Then

$$k_{\tilde{\nu}}(z) = \frac{k_\nu(z)}{c/4} = \frac{4k_\nu(z)}{c} \leq -4.$$

Thus $k_{f^*\tilde{\nu}}(w) \leq -4$ for all $w \in D(0, R)$ where it is defined. (This is because $k_{f^*\tilde{\nu}}(w) = k_{\tilde{\nu}}(f(w))$ whenever both are defined.) By Theorem 17.2, it follows that

$$k_{f^*\tilde{\nu}}(w) \leq \rho_R(w)$$

for all $w \in D(0, R)$. But

$$f^*\tilde{\nu} = \frac{\sqrt{c}}{2} f^*\nu$$

and the required inequality follows. □

Remark 17.4. *Liouville's Theorem follows from this. Say $f : \mathbb{C} \to \mathbb{C}$ is bounded and entire. Choose K such that $f(\mathbb{C}) \subset D(0, K)$. We know that ρ_R on $D(0, K)$ has curvature -4 and so $(f^*\rho_K)(w) \leq \rho_R(w)$ for any $R > 0$. But $\lim_{R\to\infty} \rho_R(w) = 0$ (because $\rho_R(w) = \frac{R}{R^2-|w|^2}$) and so $(f^*\rho_K)(w) = 0$ for all w. Recall that $(f^*\rho_K)(w) = \rho_K(f(w))|f'(w)|$. Also $\rho_K(f(w)) \neq 0$ and so $f'(w) = 0$ for all w. Thus f is constant.*

18 Curvature III (02/27)

Our main aim now is to find a non-degenerate metric μ on $\mathbb{C}\backslash\{0, 1\}$ such that $k_\mu(z) \leq -c < 0$ for all $z \in \mathbb{C}\backslash\{0, 1\}$.

We will construct μ as a product $\mu = \mu_0\mu_1$ where μ_0 is a metric on $\mathbb{C}\backslash\{0\}$ and μ_1 is a metric on $\mathbb{C}\backslash\{1\}$. We will have

$$k_\mu(z) = \frac{k_{\mu_0}(z)}{\mu_1^2(z)} + \frac{k_{\mu_1}(z)}{\mu_0^2(z)}.$$

I will take

$$\mu_0(z) = \frac{1 + |z|^{z\alpha}}{|z|^{2\beta}}$$

where α and β are constants that will be chosen later. I will take

$$\mu_1(z) = \frac{1 + |z - 1|^{2\alpha}}{|z - 1|^{2\beta}}.$$

Note that

$$\log(\mu_0(z)) = \log(1 + |z|^{2\alpha}) - \log(|z|^{2\beta})$$

and

$$\Delta \log(|z|^{2\beta}) = 0$$

and so

$$\Delta \log(\mu_0(z)) = \Delta \log(1 + |z|^{2\alpha}).$$

Use $\Delta = 4\partial\overline{\partial}$ to compute this.

$$\begin{aligned}
\overline{\partial} \log(1 + |z|^{2\alpha}) &= \overline{\partial} \log(1 + (x^2 + y^2)^\alpha) \\
&= \frac{1}{2}\left(\frac{2\alpha(x^2 + y^2)^{\alpha-1}x}{1 + (x^2 + y^2)^\alpha} + i\frac{2\alpha(x^2 + y^2)^{\alpha-1}y}{1 + (x^2 + y^2)^\alpha}\right) \\
&= \frac{\alpha(x + iy)(x^2 + y^2)^{\alpha-1}}{1 + (x^2 + y^2)^\alpha} \\
&= \frac{\alpha z|z|^{2\alpha-2}}{1 + |z|^{2\alpha}} \\
&= \frac{\alpha z \cdot z^{\alpha-1} \cdot \overline{z}^{\alpha-1}}{1 + z^\alpha\overline{z}^\alpha} \quad \text{(locally on } \mathbb{C}\backslash\{0\}) \\
&= \frac{\alpha z^\alpha\overline{z}^{\alpha-1}}{1 + z^\alpha\overline{z}^\alpha}.
\end{aligned}$$

Thus,

$$\begin{aligned}\partial\overline{\partial}\log(1+|z|^{2\alpha}) &= \partial\left(\frac{\alpha z^{\alpha}\overline{z}^{\alpha-1}}{1+z^{\alpha}\overline{z}^{\alpha}}\right)\\ &= \frac{(1+z^{\alpha}\overline{z}^{\alpha})\alpha^2 z^{\alpha-1}\overline{z}^{\alpha-1} - \alpha z^{\alpha}\overline{z}^{\alpha-1}\cdot\alpha z^{\alpha-1}\overline{z}^{\alpha}}{(1+z^{\alpha}\overline{z}^{\alpha})^2}\\ &= \frac{\alpha^2|z|^{2\alpha-2}}{(1+|z|^{2\alpha})^2}.\end{aligned}$$

Thus

$$\Delta\log(1+|z|^{2\alpha}) = \frac{4\alpha^2|z|^{2\alpha-2}}{(1+|z|^{2\alpha})^2}.$$

This gives

$$k_{\mu_0}(z) = -\frac{4\alpha^2|z|^{2\alpha-2}}{(1+|z|^{2\alpha})^2}\cdot\frac{|z|^{4\beta}}{(1+|z|^{2\alpha})^2} = -\frac{4\alpha^2|z|^{2\alpha+4\beta-2}}{(1+|z|^{2\alpha})^4}.$$

We assume now that

$$\alpha+2\beta=1$$

so that $2\alpha+4\beta=2$ and

$$k_{\mu_0}(z) = \frac{-4\alpha^2}{(1+|z|^{2\alpha})^4}.$$

Let

$$\mu_1(z) = \frac{1+|z-1|^{2\alpha}}{|z-1|^{2\beta}}.$$

Then

$$k_{\mu_1}(z) = \frac{-4\alpha^2}{(1+|z-1|^{2\alpha})^4}.$$

With $u=u_0u_1$, we get

$$k_{\mu}(z) = \frac{-4\alpha^2}{(1+|z|^{2\alpha})^4}\cdot\frac{|z-1|^{4\beta}}{(1+|z-1|^{2\alpha})^2} + \frac{-4\alpha^2}{(1+|z-1|^{2\alpha})^4}\cdot\frac{|z|^{4\beta}}{(1+|z|^{2\alpha})^2}.$$

We need $\lim_{z\to\infty}k_{\mu}(z)$ to be some negative value or $-\infty$. If we impose the condition

$$4\beta = 12\alpha,$$

then $\lim_{z\to\infty}k_{\mu}(z) = -8\alpha^2$. We need $\alpha+2\beta=1$ and $\beta=3\alpha$, and so $\alpha=\frac{1}{7}$ and $\beta=\frac{3}{7}$.

Note that

$$\mu(z) = \frac{1+|z|^{2/7}}{|z|^{6/7}}\cdot\frac{1+|z-1|^{2/7}}{|z-1|^{6/7}}.$$

With this choice, we have

$$\lim_{z\to 0}k_{\mu}(z) = -\frac{1}{49},$$

$$\lim_{z \to 1} k_\mu(z) = -\frac{1}{49},$$
$$\lim_{z \to \infty} k_\mu(z) = -\frac{8}{49}.$$

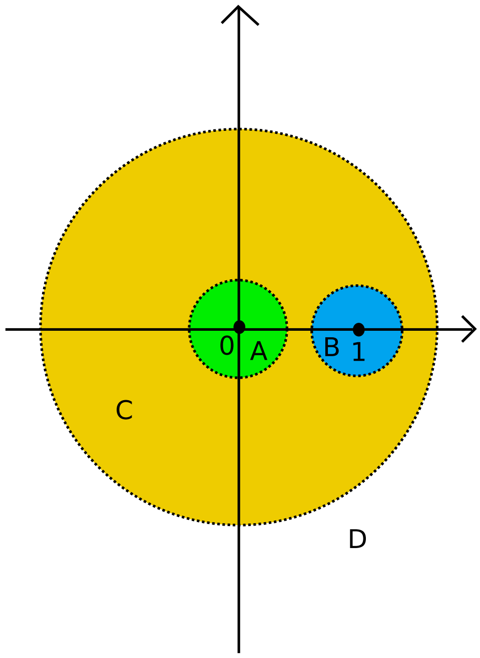


Figure 8:

We use a picture (see the above figure) to illustrate the value of k_μ. $k_\mu < -\frac{1}{98}$ on A. $k_\mu < -\frac{1}{98}$ on B. C is a compact set, and on here k_μ achieves a negative maximum. On D $k_\mu < -\frac{4}{49}$. It follows that there exists a constant $c > 0$ such that

$$k_\mu(z) \leq -c$$

for all $z \in \mathbb{C} \backslash \{0, 1\}$. Also note that μ is non-degenerate.

Theorem 18.1 (Litter Picard Theorem)**.** *If f is an entire function that omits two values from its image, then f is constant.*

Proof. Suppose that f omits two values. By composition f with a linear map, we may assume that the omitted values are 0 and 1. Thus $f : \mathbb{C} \to \mathbb{C} \backslash \{0, 1\}$. Let $R > 0$. By the Ahlfors-Schwarz-Pick Lemma, we have

$$(f^* \mu)(z) \leq \frac{2}{\sqrt{c}} \cdot \frac{R}{R^2 - |z|^2}$$

for all $z \in D(0,R)$. Fix $z \in \mathbb{C}$ and take $R > |z|$, then let $R \to \infty$ in the inequality. We obtain

$$\mu(f(z))|f'(z)| \leq 0.$$

Thus

$$\mu(f(z))|f'(z)| = 0.$$

We know that $\mu(f(z)) \neq 0$ and so $f'(z) = 0$. But z was arbitrary, and so $f' \equiv 0$. Thus f is constant. $\square$

19 Curvature IV (03/02)

On S^2 we have the spherical metric $d_{\text{spherical}}$. We showed that $d_{\text{spherical}}$ derives from the construction of minimizing the length of paths joining two points. The metric space $(S^2, d_{\text{spherical}})$ is compact.

Let $U \subset \mathbb{C}$ be a domain. Then the space $C(U, S^2)$ is a metric space in such a way that $(f_n) \subset C(U, S^2)$ converges to a point $f \in C(U, S^2)$ if and only if $f_n|_K \to f|_K$ uniformly for all compact set $K \subset U$. It is a consequence of the Arzela-Ascoli Theorem that subset $\mathcal{F} \subset C(U, S^2)$ is precompact if and only if it is equicontinuous. (Note that any subset of S^2 is precompact, so the other condition in Arzela-Ascoli Theorem is automatic.)

We know that $\mathcal{M}(U)$ (the set of all meromorphic functions on U) is a subset of $C(U, S^2)$. Recall that meromorphic is the same as holomorphic from U to S^2 (=$\mathbb{C}_\infty$) and not constantly ∞. More explicitly, $f : U \to S^2$ is holomorphic if
(1) $f : U \to S^2$ is continuous,
(2) f is holomorphic on $f^{-1}(S^2 \backslash \{0, 0, 1\})$,
(3) $\frac{1}{f}$ is holomorphic on $f^{-1}(S^2 \backslash \{(0, 0, -1)\})$.

The key to understanding when a family is equicontinuous inside $C(U, S^2)$ is the "spherical derivative".

Say that we have a holomorphic function $f : U \to S^2$ whose image does not include ∞. Then we can think of f as a map $F : U \to \mathbb{C}$ composed with $p : \mathbb{C} \to S^2$. We want to pullback the Hermitian metric on S^2 that leads to $d_{\text{spherical}}$ under f. We already calculated the pullback of this metric under p. We got the spherical metric $\sigma(z) = \frac{2}{1+|z|^2}$. Thus the pullback of the Hermitian metric on S^2 by f is

$$(F^*\sigma)(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}.$$

By definition,

$$f^\#(z) = \frac{2|f'(z)|}{1 + |f(z)|^2}$$

is the *spherical derivative* of f at z.

Lemma 19.1. *Let $f : U \to S^2$ be a holomorphic function whose image includes neither 0 nor ∞. Then*

$$f^\#(z) = \left(\frac{1}{f}\right)^\# (z)$$

for all $z \in U$.

Proof. We know $(\frac{1}{f})'(z) = \frac{-f'(z)}{f(z)}$ and so

$$\begin{aligned}\left(\frac{1}{f}\right)^{\#}(z) &= \frac{2\left|\frac{-f'(z)}{f(z)}\right|}{1+\left|\frac{1}{f(z)}\right|^2} \\ &= \frac{2|f'(z)|}{1+|f(z)|^2} \\ &= f^{\#}(z).\end{aligned}$$

□

We can use this to extend the spherical derivative to functions that do include ∞ in their image. We can make this more concrete.

First, suppose f has a simple pole at p and $f(z) = \frac{c_{-1}}{z-p} + h(z)$, h holomorphic at p. Then

$$f(z) = \frac{c_{-1} + (z-p)h(z)}{z-p}$$

and so

$$\frac{1}{f(z)} = \frac{z-p}{c_{-1} + (z-p)h(z)}.$$

This gives

$$\left(\frac{1}{f(z)}\right)' = \frac{c_{-1} + (z-p)h(z) - (z-p)k(z)}{(c_{-1} + (z-p)h(z))^2}$$

where $k(z)$ is the derivative of the button. So

$$f^{\#}(z) = \left(\frac{1}{f}\right)^{\#}(z) = \frac{2\frac{c_{-1}+(z-p)h(z)-(z-p)k(z)}{(c_{-1}+(z-p)h(z))^2}}{1+\left|\frac{z-p}{c_{-1}+(z-p)h(z)}\right|^2}.$$

Now

$$f^{\#}(p) = \lim_{z\to p}\left(\frac{1}{f}\right)^{\#}(z) = 2\cdot\left|\frac{1}{c_{-1}}\right| = \frac{2}{|c_{-1}|}.$$

Second, if f has a double or higher pole at p then $f^{\#}(p) = 0$.

The sequence of constant functions $(n) \subset C(\mathbb{C}, S^2)$ converges to the constant function at ∞. This tells us that $\mathcal{M}(U)$ is not a closed subset of $C(U, S^2)$. In fact, $\mathrm{cl}(\mathcal{M}(U)) = \mathcal{M}(U) \cup \{\infty\}$.

If (f_n) is a sequence of holomorphic functions on U and $f_n \to \infty$ in $C(U, S^2)$ then we say that (f_n) is *compactly divergent*. Concretely, (f_n) is compactly divergent if given $K \subset \mathbb{C}$ compact and B a number, there is a N such that $|f_n(z)| \geq B$ for all $z \in K$ and all $n \geq N$.

20 Curvature V (03/06)

Theorem 20.1 (Marty's Theorem, 1931)**.** *Let $U \subset \mathbb{C}$ be a domain. Let $\mathcal{F}$ be a family of functions in $\mathcal{M}(U)$. Then $\mathcal{F}$ is precompact in $C(U, S^2)$ if and only if for all compact sets $K \subset U$, there is a constant $M_K > 0$ such that $f^{\#}(z) \leq M_K$ for all $z \in K$ and all $f \in \mathcal{F}$.*

Proof. I will only show that the bound on the spherical derivatives implies that $\mathcal{F}$ is precompact. Let $p \in U$. I will show that $\mathcal{F}$ is equicontinuous at p. Choose a disk $D(p,r) \subset U$. Let $q \in D(p,r)$. I need to estimate $d_\sigma(f(p), f(q))$ for $f \in \mathcal{F}$. Let $\sigma : [0,1] \to D(p,r)$ be the line segment from p to q. Then

$$\begin{aligned} d_\sigma(f(p), f(q)) &\leq \mathcal{L}_\sigma(f \circ \sigma) \\ &= \mathcal{L}_{f^*\sigma}(\gamma) \\ &= \int_0^1 |\gamma'(t)|(f^*\sigma)(\gamma(t))\, dt \\ &= \int_0^1 |\gamma'(t)| f^{\#}\sigma(\gamma(t))\, dt \\ &\leq M_{\overline{D}(p,r)} \int_0^1 |\gamma'(t)| dt \\ &= M_{\overline{D}(p,r)} |p - q|. \end{aligned}$$

That is, $\mathcal{F}$ is uniformly Lipschitz at p. Thus $\mathcal{F}$ is equicontinuous at p. Thus, plus Arzela-Ascoli Theorem, implies the theorem. □

Corollary 20.2. *Let $\mathcal{F} \subset H(U)$ where $U \subset \mathbb{C}$ is a domain and suppose that for every compact $K \subset U$, $f^{\#}(z) \leq M_K$ for some constant M_K, all $f \in \mathcal{F}$, and all $z \in K$. Then given $(f_n) \in \mathcal{F}$, we can find a subsequence (f_{n_k}) that either converges uniformly on compact subsets of U to $f \in H(U)$ or is compactly divergent.*

Theorem 20.3 (Montel's Great Theorem/Fundamental Theorem on Normality)**.** *Let $U \subset \mathbb{C}$ be a domain. Let $\mathcal{F} \in \mathcal{M}(U)$ be a family of functions and suppose that there are three distinct points $A, B, C \in S^2$ such that $f(U) \in S^2 \backslash \{A, B, C\}$ for all $f \in \mathcal{F}$. Then $\mathcal{F}$ is precompact.*

We need a lemma.

Lemma 20.4. *Let*

$$\mu(z) = \frac{1 + |z|^{2/7}}{|z|^{6/7}} \cdot \frac{1 + |z-1|^{2/7}}{|z-1|^{6/7}}$$

be the metric that we previously constructed on $\mathbb{C}\backslash\{0,1\}$. Then there is a constant $L > 0$ such that $\sigma(z) \leq L \cdot \mu(z)$ for all $z \in \mathbb{C}\backslash\{0,1\}$.

Proof. Consider $\frac{\sigma(z)}{\mu(z)}$ on $\mathbb{C}\backslash\{0,1\}$. We have $\lim_{z\to 0}\mu(z)=\infty$ and $\lim_{z\to 1}\mu(z)=\infty$ and so $\lim_{z\to 0}\frac{\sigma(z)}{\mu(z)}=0$ and $\lim_{z\to 1}\frac{\sigma(z)}{\mu(z)}=0$. Also,

$$\lim_{z\to\infty}\frac{\sigma(z)}{\mu(z)}=\lim_{z\to\infty}\frac{2}{1+|z|^2}\cdot\frac{|z|^{6/7}}{1+|z|^{2/7}}\cdot\frac{|z-1|^{6/7}}{1+|z-1|^{2/7}}=\lim_{z\to\infty}\frac{|z|^{8/7}}{|z|^2}=0$$

because $8/7<2$. It follows that $\frac{\sigma(z)}{\mu(z)}$ is bounded above on $\mathbb{C}\backslash\{0,1\}$, as required. □

Now we are ready to prove the Fundamental Theorem on Normality.

Proof of Theorem 20.3. Without loss of generality, I may assume that the three points omitted by $\mathcal{F}$ are $0,1,\infty$. In order to verify that the spherical derivatives of $\mathcal{F}$ are bounded on compact subsets, it suffices to verify that they are bounded on disks around each point. Choose $p\in U$ and a disk $D(p,2R)\subset U$. We may assume that $p=0$ by composing the family with $z\mapsto z-p$. By the Ahlfors-Schwarz-Pick Lemma, there is a constant $N>0$ such that $f^*\mu\leq N\rho_{2R}$ for any holomorphic function $f:D(0,2R)\to\mathbb{C}\backslash\{0,1\}$. Thus, for any holomorphic function $f:D(0,2R)\to\mathbb{C}\backslash\{0,1\}$, we have

$$f^*\sigma\leq L\cdot f^*\mu\leq L\cdot N\cdot\rho_{2R}.$$

Thus, if $z\in D(0,R)$, then

$$(f^*\sigma)(z)\leq L\cdot N\cdot\frac{2R}{4R^2-|z|^2}\leq L\cdot N\cdot\frac{2R}{4R^2-R^2}=\frac{2LN}{3R}.$$

Thus, we have

$$f^{\#}(z)\leq\frac{2LN}{3R}$$

for all $z\in D(0,R)$. This verifies the hypothesis of Marty's Theorem for the family of all holomorphic functions $U\to\mathbb{C}\backslash\{0,1\}$. □

21 Curvature VI (03/09)

Theorem 21.1 (Great Picard Theorem)**.** *Suppose that $f\in H(D'(0,r))$ with $r>0$ has an essential singularity at 0. Then $f(D'(0,r))$ is either $\mathbb{C}$ or $\mathbb{C}\backslash\{p\}$ for some $p\in\mathbb{C}$.*

Proof. Suppose to the contrary that $f(D'(0,r))\subset\mathbb{C}\backslash\{p,q\}$ for some $p,q\in\mathbb{C}$, $p\neq q$. By composing with a linear map if necessary, we may assume that $f(D'(0,r))\subset\mathbb{C}\backslash\{0,1\}$. For $n\geq 1$, we define $f_n\in H(D'(0,r))$ by

$$f_n(z)=f\left(\frac{z}{n}\right).$$

Note that $f_n(D'(0,r))\subset f(D'(0,r))\subset\mathbb{C}\backslash\{0,1\}$. It follows from the Fundamental Theorem on Normality that $\{f_n|n\geq 1\}$ is normal (precompact in $C(D'(0,r),S^2)$) (when regarded as a family in $\mathcal{M}(D'(0,r), S^2)$). Hence we may find a subsequence (f_{n_k}) such that either $f_{n_k} \to g$ uniformly on compact subsets of $D'(0,q)$ for some $g \in H(D'(0,q))$ or (f_{n_k}) is compactly divergent.

Assume first that $f_{n_k} \to g$. This means that $f_{n_k} \to g$ uniformly on the set $K = \{z||z| = \frac{r}{2}\}$. Now g is bounded on K and so there is some $M > 0$ such that $|f_{n_k}(z)| \leq M$ for all $k \geq 1$ and all $z \in K$. It follows that $|f(z)| \leq M$ for all z such that $|z| = \frac{r}{2n_k}$ for any $k \geq 1$. It follows from the Maximum Modulus Principle that $|f(z)| \leq M$ for all z such that $\frac{r}{2n_{k+1}} \leq |z| \leq \frac{r}{2n_k}$. It follows that $|f(z)| \leq M$ for all $z \in D'(0, \frac{r}{2n_1})$. By Riemann's Removable Singularity Theorem, f has a removable singularity at 0, a contradiction.

We conclude that (f_{n_k}) must be compactly divergent. Because we have assumed that the image of f (and hence of f_{n_k}) does not contain 0, $\left(\frac{1}{f_{n_k}}\right)$ is a sequence of holomorphic functions on $D'(0,r)$. They are constructed from $\frac{1}{f} \in H(D'(0,r))$ by the same method as $f_{n_k} \to 0$ uniformly on compact subsets of $D'(0,r)$. We conclude that $\frac{1}{f}$ has a removable singularity at 0 (by rerunning the previous argument) and $\frac{1}{f}(0) = 0$. Thus f has a pole at 0, a contradiction. This means that f admits at most one value from its image. $\square$

22 Runge's Theorem and the Mittag-Leffler Theorem I (03/11)

We begin this chapter with a problem from comprehensive exam.

Comprehensive Exam Problem: Let $f \in H(\mathbb{D}) \cap C(\overline{\mathbb{D}})$. Show that f can be uniformly approximated on compact subsets of $\mathbb{D}$ by rational functions of the form

$$r(z) = \sum_{k=1}^{n} \frac{a_k}{z - z_k}, \ a_k \in \mathbb{C}, |z_k| = 1 \text{ for } 1 \leq k \leq n.$$

(i.e., $\forall K \subset \mathbb{D}$ compact, $\forall \epsilon > 0$, $\exists r(z)$ of the given type such that $|r(z) - f(z)| < \epsilon, \forall z \in K$.)

Solution. Start with Cauchy's Integral Formula for the circle $|z| = r < 1$. It says

$$f(z) = \frac{1}{2\pi i} \int_{|w|=r} \frac{f(w)}{w - z} dw$$

valid for all z such that $|z| < r$. Next, because f extends continuously to $\overline{D}$ we may let $r \to 1^-$ to conclude that

$$f(z) = \frac{1}{2\pi i} \int_{|w|=1} \frac{f(w)}{w - z} dw$$

for all $z \in \mathbb{D}$. So

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z} ie^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{e^{i\theta} - z} e^{i\theta}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} f(e^{i\theta})}{e^{i\theta} - z}\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} g(\theta, z) d\theta \end{aligned}$$

where

$$g(\theta, z) = \frac{e^{i\theta} f(e^{i\theta})}{e^{i\theta} - z}.$$

A Riemann sum for this integral looks like

$$\begin{aligned} \sum_{k=1} n g(\theta_k^*, z) \Delta\theta_k &= \sum_{k=1}^{n} \frac{e^{i\theta_k^*} f(e^{i\theta_k^*})}{e^{i\theta_k^*} - z} \Delta\theta_k \\ &= \sum_{k=1}^{n} \frac{a_k}{z - z_k} \end{aligned}$$

if $z_k = e^{i\theta_k^*}$ and $a_k = -e^{i\theta_k^*} f(e^{i\theta_k^*}) \Delta\theta_k$.

Recall that a tagged partition P of $[a, b]$ is a choice of numbers

$$a = t_0 < t_1 < t_2 \cdots < t_n = b$$

and a choice of $s_j \in [t_{j-1}, t_j]$ for all $1 \leq j \leq n$. The mesh of P is

$$\|P\| = \max_{1 \leq j \leq n} (t_j - t_{j-1}).$$

The Riemann sum of g for P is

$$R(P, g) = \sum_{j=1}^{n} g(s_j)(t_j - t_{j-1}).$$

Lemma 22.1. *Let $a < b$ and K be a compact metric space. Let $g : [a, b] \times K \to \mathbb{C}$ be a continuous function. Let $\varepsilon > 0$. There is some $\delta > 0$ such that if P is a tagged partition of $[a, b]$ with $\|p\| < \delta$ then*

$$\left| \int_a^b g(t, z)dt - R(P, g(\cdot, z)) \right| < \varepsilon$$

for all $z \in K$ (where R is the Riemann sum for P).

Proof. We begin by writing

$$\begin{aligned}
\left| \int_a^b g(t, z)dt - R(P, g(\cdot, z)) \right| &= \left| \int_a^b g(t, z)dt - \sum_{j=1}^{n} g(s_j, z)(t_j - t_{j-1}) \right| \\
&= \left| \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} g(t, z)dz - \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} g(s_j, z)dz \right| \\
&\leq \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} |g(t, z) - g(s_j, z)| \, dz.
\end{aligned}$$

Let $\eta > 0$ and choose $\delta > 0$ such that if $d_\infty ((t, z), (s, w)) < \delta$ then

$$|g(t, z) - g(s, w)| < \eta.$$

(Here we are using the fact that $[a, b] \times K$ is compact and that g is uniformly continuous.) If $\|P\| < \delta$ then $d_\infty ((t, z), (s_j, z)) < \delta$ for all $t \in [t_{j-1}, t_j]$. Thus if $\|P\| < \delta$ then

$$\begin{aligned}
\left| \int_a^b g(t, z)dt - R(P, g(\cdot, z)) \right| &< \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \eta dt = \sum_{j=1}^{n} \eta(t_j - t_{j-1}) \\
&= \eta(b - a).
\end{aligned}$$

If we choose $\eta = \frac{\varepsilon}{b-a}$, then we can get the required conclusion. $\square$

Lemma 22.2. *Let $V \subset \mathbb{C}$ be open and $K \subset V$ be compact. Then there is a cycle Γ such that $\Gamma^* \subset V \backslash K$, $Ind(\Gamma, z) = 0$ for all $z \in \mathbb{C} \backslash V$, $Ind(\Gamma, z) = 0$ or 1 for all $z \in V \backslash \Gamma^*$, and $Ind(\Gamma, z) = 1$ for all $z \in K$.*

For a proof of Lemma 22.2, see Chapter 10 of Complex Made Simple, page 196-199.

Based on Lemma 22.1, Lemma 22.2, and the Homology Version of Cauchy's Integral Formula, we now know that if $V \subset \mathbb{C}$ is an open set, $K \subset \mathbb{C}$ is compact, and $f \in H(V)$, then f can be approximated uniformly on K by a rational function whose poles lie in $\mathbb{C} \backslash K$. (They will lie on Γ^* where Γ is a cycle as in Lemma 22.2.)

23 Runge's Theorem and the Mittag-Leffler Theorem II (03/13)

We denote $\mathcal{R}$ to be the space of all rational functions on $\mathbb{C}$. Note that if $R \in \mathcal{R}$, then we can think of R as a function from $\mathbb{C}_\infty$ to $\mathbb{C}_\infty$.

$\mathbb{R}_A$ is the space of all rational functions whose poles lie in A. Note that if $R \in \mathcal{R}_A$ then R may be written as a sum

$$R = \sum_{a \in A} R_a$$

where $R_a \in \mathcal{R}_{\{a\}}$. This is the partial fractions expression of R.

$$R(z) = P(z) + \sum_{a \in A \backslash \{\infty\}} \sum_{j=1}^{na} \frac{c_{a,j}}{(z-a)^j}.$$

We have $c_{a,j} = 0$ for all but finitely many a. Note that $P \in \mathcal{R}_\infty$. In fact, $\mathcal{R}_\infty$ is exactly the space of all polynomials.

Lemma 23.1. *Let $K \subset \mathbb{C}$ be a compact set and $D(p, r) \subset \mathbb{C} \backslash K$. Let $\varepsilon > 0$ and R be a rational function with all its poles lying in $D(p, r)$. Let $q \in D(p, r)$. Then there is $S \in \mathcal{R}_{\{q\}}$ such that $|R(z) - S(z)| < \varepsilon$ for all $z \in K$.*

Proof. It is sufficient to show that if $\alpha \in D(p, r)$ and $\beta \in D(p, r)$, then $\frac{1}{z-\beta}$ can be uniformly approximated on K by an element of $\mathcal{R}_{\{\alpha\}}$. Well,

$$\frac{1}{z-\beta} = \sum_{n=0}^{\infty} \frac{(\beta-\alpha)^n}{(z-\alpha)^{n+1}}. \tag{23.1}$$

We can ignore the polynomial part since it would not have poles in the disk and thus, we can approximate it by itself and kill it off. We have to switch back and forth between thinking of this in $\mathbb{C}$ and $\mathbb{C}_\infty$.

If $z \in K$, then we can find a disk centered at β such that if α lies in this disk, then $|\alpha - \beta| < \frac{1}{2}|z - \alpha|$ for all $z \in K$. Then the common ratio in Equation 23.1 is less than or equal to $\frac{1}{2}$ for all $z \in K$ and so the series converges uniformly on this disk. This shows that we can push poles from β to any sufficiently close by point. $\square$

Lemma 23.2. *Let $K \subset \mathbb{C}$ be a compact set. Suppose that $A(0, N, \infty) \subset \mathbb{C}\backslash K$. Then we can approximate a rational function with poles in $A(0, N, \infty) \cup \{\infty\}$ uniformly on K by a rational function with poles at specified $q \in A(0, N, \infty) \cup \{\infty\}$.*

Proof. Critical formulas are the following:

$$\frac{1}{z-\beta} = \sum_{n=0}^{\infty} -\frac{z^n}{\beta^{n+1}}, \tag{23.2}$$

$$z = \beta - \beta \sum_{n=0}^{\infty} \left(\frac{z}{z-\beta}\right)^n. \tag{23.3}$$

Formula 23.2 allows us to approximate $\frac{1}{z-\beta}$ uniformly on K by polynomials provided that $|\beta|$ is large enough. For example, $|\beta| \geq 2N$ would work. Formula 23.3 allows us to approximate z uniformly on K by rational functions with poles at β provided that $|\beta|$ is sufficiently large. To handle the case where $|\beta|$ is smaller, use Lemma 23.1. □

Lemma 23.3. *Let $K \subset \mathbb{C}$ be compact. Let $A \subset \mathbb{C}_\infty$ be a set such that every connected component of $\mathbb{C}_\infty\backslash K$ contains some point of A. Then any rational function whose poles (in $\mathbb{C}_\infty$) do not lie in K may approximated uniformly on K by an element of $\mathcal{R}_A$.*

24 Runge's Theorem and the Mittag-Leffler Theorem III (03/23)

If $q \in \mathbb{C}_\infty$ then a standard neighborhood of q is $D(q, r)$ if $q \neq \infty$ and $A(0, k, \infty) \cup \{\infty\}$ if $q = \infty$. We showed previously that if p, q belongs to a standard neighborhood of $w \in \mathbb{C}_\infty$, $R \in \mathcal{R}_{\{p\}}$, $\varepsilon > 0$, K is a compact set disjoint from the neighborhood then there is $\tilde{R} \in \mathcal{R}_{\{q\}}$ such that $|R(z) - \tilde{R}(z)| < \varepsilon$ for all $z \in K$.

Lemma 24.1. *Let $U \subset \mathbb{C}$ be an open set, $K \subset U$ a compact set, $A \subset \mathbb{C}_\infty\backslash K$ a set that meets each component of $\mathbb{C}_\infty\backslash K$, $R \in \mathcal{R}_{\mathbb{C}_\infty\backslash K}$, $\varepsilon > 0$. Then there is $\tilde{R} \in \mathcal{R}_A$ such that $|R(z) - \tilde{R}(z)| < \varepsilon$ for all $z \in K$.*

Proof. By the Partial Fractional Expression, it suffices to assume that $R \in \mathcal{R}_{\{q\}}$ with $q \in \mathbb{C}_\infty\backslash K$. Let C be the component of $\mathbb{C}_\infty\backslash K$ that contains q. Define

$$\begin{aligned} S = \{w \in C | &R \text{ can be uniformly approximated as closely as desired} \\ &\text{on } K \text{ by an element of } \mathcal{R}_{\{w\}}\}. \end{aligned}$$

Well, we know $q \in S$ and so $S \neq \emptyset$. If $w \in S$ then we may choose a standard neighborhood B of w that is contained in C. (This is because C is open in $\mathbb{C}_\infty$.) By the previous work, since R can be uniformly approximated on K by an element of $\mathcal{R}_{\{w\}}$, it can also be uniformly approximated on K by an element of $\mathcal{R}_{\{z\}}$ for any $z \in B$. Thus $B \subset S$ and so S is open. Now suppose that $w \in \mathrm{cl}_C(S)$. We may choose a standard neighborhood $B \subset C$ of w. Then $B \cap S \neq \emptyset$. Choose $z \in B \cap S$. Then R can be approximated uniformly on K by an element of $\mathcal{R}_{\{z\}}$. It follows that R may be uniformly approximated on K by an element of $\mathcal{R}_{\{w\}}$. Thus $w \in S$. Thus S is closed in C. It follows that $S = C$. In particular, there is some $a \in A$ such that $a \in C = S$. Thus R may be uniformly approximated on K by an element of $\mathcal{R}_{\{a\}} \subset \mathcal{R}_A$. □

Theorem 24.2 (Runge's Theorem, Version 1). *Let $U \subset \mathbb{C}$ be open, $K \subset U$ compact, $A \subset \mathbb{C}_\infty \backslash K$ a set that meets every component of $\mathbb{C}_\infty \backslash K$, $f \in H(U)$, $\varepsilon > 0$. Then there is $R \in \mathcal{R}_A$ such that $|f(z) - R(z)| < \varepsilon$ for all $z \in K$.*

Proof. We know that there is a cycle Γ such that $\Gamma^* \subset U \backslash K$, $\mathrm{Ind}(\Gamma, w) = 0$ for all $w \notin U$, $\mathrm{Ind}(\Gamma, z) = 1$ for all $z \in K$. This implies that

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z} dw$$

for all $z \in K$ by the Homology version of Cauchy's Integral Formula. We may approximate the integral (and hence f) uniformly on K by a Riemann sum. This Riemann sum is an element of $\mathcal{R}_{\Gamma^*}$. Now $\Gamma^* \subset \mathbb{C}_\infty \backslash K$ and so this is an element of $\mathbb{R}_{\mathbb{C}_\infty \backslash K}$. By Lemma 24.1, this may be uniformly approximated by an element of $\mathcal{R}_A$. □

25 Runge's Theorem and the Mittag-Leffler Theorem IV (03/25)

Let $U \subset \mathbb{C}$ be open. We obtained a compact exhaustion by defining

$$\begin{aligned} K_n &= \overline{D}(0, n) \cap \{z \in U |\ d(z, \mathbb{C} \backslash U) \geq \frac{1}{n}\} \\ &= \overline{D}(0, n) \cap \bigcap_{p \in \mathbb{C} \backslash U} \{z |\ |z - p| \geq \frac{1}{n}\}. \end{aligned}$$

Then

$$\mathbb{C}_\infty \backslash K_n = (A(0, n, \infty) \cup \{\infty\}) \cap \bigcup_{p \in \mathbb{C} \backslash U} D(p, \frac{1}{n}).$$

We want to verify that every component of $\mathbb{C}_\infty \backslash K_n$ contains a component of $\mathbb{C}_\infty \backslash U$. The only way this could fail is if there were some component C of $\mathbb{C}_\infty \backslash K_n$ such that $C \cap (\mathbb{C}_\infty \backslash U) = \emptyset$. Since $C \subset \mathbb{C}_\infty \backslash K_n$, either C contains some point in $D(p, \frac{1}{n})$ with $p \in \mathbb{C} \backslash U$ or C contains some point in $A(0, n, \infty) \cup \{\infty\}$. But note that $D(p, \frac{1}{n})$ and $A(0, n, \infty) \cup \{\infty\}$ are all connected sets. Thus C contains either $D(p, \frac{1}{n})$ for some $p \in \mathbb{C} \backslash U$ or $A(0, n, \infty) \cup \{\infty\}$. Therefore, C does not contain some point of $\mathbb{C} \backslash U$.

Now we come to the second version of Runge's Theorem, and this is the version that people usually refer to.

Theorem 25.1 (Runge's Theorem, Version 2)**.** *Let $U \subset \mathbb{C}$ be an open set. Let $A \subset \mathbb{C}_\infty \backslash U$ be a set that meets every component of $\mathbb{C}_\infty \backslash U$. Let $f \in H(U)$. Then there is a sequence (g_n) such that $g_n \in \mathcal{R}_A$ and $g_n \to f$ in $H(U)$.*

Proof. We choose a compact exhaustion (K_n) of U such that every component of $\mathbb{C}_\infty \backslash K_n$ contains a component of $\mathbb{C}_\infty \backslash U$. Note that A meets every component of $\mathbb{C}_\infty \backslash K_n$ for all $n \geq 1$. By Theorem 24.2, for each $n \geq 1$ we may choose $g_n \in \mathcal{R}_A$ such that $|f(z) - g(z)| < \frac{1}{n}$ for all $z \in K_n$. Let $K \subset U$ be a compact set. Then $\{\text{int}(K_n) | n \geq 1\}$ are an open cover of K. Thus there is a finite subcover and, since $\text{int}(K_n)$ increases with n, there is some $m \geq 1$ such that $K \subset \text{int}(K_m)$. In fact, $K \subset K_n$ for all $n \geq m$. Thus, $|f(z) - g(z)| < \frac{1}{n}$ for all $z \in K$ and all $n \geq m$. This shows that $g_n \to f$ uniformly on K, as required. □

Theorem 25.2. *Let $U \subset \mathbb{C}$ be open. Then U is simply connected if and only if $\mathbb{C}_\infty \backslash U$ is connected.*

Remark 25.3. *In Theorem 25.2, we are not assuming U is connected, so U is simply connected means every component of U is simply connected in the sense of the topology.*

Proof of Theorem 25.2. Suppose $\mathbb{C}_\infty \backslash U$ is not connected. Then it has a disconnection $\mathbb{C}_\infty \backslash U = K_1 \cup K_2$. Since $\mathbb{C}_\infty \backslash U$ is closed, it is compact. Thus K_1, K_2 (closed subsets of $\mathbb{C}_\infty \backslash U$) are both compact. One of them contains ∞; renumber if necessary so that $\infty \in K_1$. Then K_2 is a compact subset of $\mathbb{C}$. Consider

$$V = \mathbb{C}_\infty \backslash K_1 = U \cup K_2.$$

This is an open set in $\mathbb{C}$ and it contains the compact set K_2. This means that we can find a cycle Γ such that $\Gamma^* \subset V \backslash K_2 = U$, $\text{Ind}(\Gamma, w) = 0$ for all $w \notin V$ and $\text{Ind}(\Gamma, z) = 1$ for all $z \in K_2$. This gives us a cycle Γ in U such that the index of Γ about some points in the component of U is non-zero. This tells us that U is not simply connected. This verifies that if U is simply connected, then $\mathbb{C}_\infty \backslash U$ is connected.

Now suppose that $\mathbb{C}_\infty \backslash U$ is connected. Then the set $A = \{\infty\}$ meets every component of $\mathbb{C}_\infty \backslash U$. Recall that $\mathcal{R}_A$ is exactly the set of polynomials. Let $f \in H(U)$ be non-vanishing, so that $\frac{f'}{f} \in H(U)$. By Runge's Theorem with $A = \{\infty\}$, we may find a sequence (g_n) of polynomials such that $g_n \to \frac{f'}{f}$ in $H(U)$. Let γ be a loop in U. Then

$$\int_\gamma \frac{f'(w)}{f(w)} dw = \lim_{n \to \infty} g_n(w) dw = 0$$

because polynomials always have antiderivatives. This implies that f has a logarithm on U. So U is simply connected (either $U = \mathbb{C}$ or each component of U is conformal equivalent to $\mathbb{D}$.) □

26 Runge's Theorem and the Mittag-Leffler Theorem V (03/27)

Suppose that a function f that is holomorphic on $D'(p,r)$ and has a pole at p. Then f has a Laurent expansion

$$\begin{aligned} f(z) &= \sum_{n=-m}^{\infty} c_n(z-p)^n \\ &= \frac{c_{-m}}{(z-p)^m} + \cdots + \frac{c_{-1}}{z-p} + c_0 + c_1(z-p) + c_2(z-p)^2 + \cdots . \end{aligned}$$

The rational function

$$\frac{c_{-m}}{(z-p)^m} + \cdots + \frac{c_{-1}}{z-p}$$

is called the *principal part* of f at p. We will write $P(f,p)$ for the principal part of f at p. If $f \in \mathcal{M}(U)$ then f has a set of poles, say $E \subset U$. We know that E is discrete in U. That is, U is closed in U and every point of E is isolated in E. E might still be infinite, by if so any limit points must lie on ∂U.

Example 26.1. (1) E might be $\mathbb{Z}$ where $U = \mathbb{C}$, for example, the function $\csc(\pi z)$.
(2) E might be $\{1 - \frac{1}{n} | n \geq 1\}$ where $U = \mathbb{D}$.

You can try to reverse this. That is, start with U and a plausible E and ask for a function $f \in \mathcal{M}(U)$ with poles at the points of E. More precisely, we could ask for $f \in \mathcal{M}(U)$ such that E is its set of poles and $P(f,p)$ is chosen ahead of time. Mittag-Leffler's Theorem says that this is always possible.

Theorem 26.2 (Mittag-Leffler's Theorem)**.** *Let $U \subset \mathbb{C}$ be an open set, $E \subset U$ be a set that has no limit points in U, and for each $p \in E$, let R_p be a rational function of the form*

$$\frac{c_{-m}}{(z-p)^m} + \cdots + \frac{c_{-1}}{z-p}$$

where $m \geq 1$. Then there is $f \in \mathcal{M}(U)$ such that f has poles only at the points of E and $P(f,p) = R_p$ for all $p \in E$.

Proof. Choose a compact exhaustion of U, say (K_n), such that it has the extra property (every component of $\mathbb{C}_\infty \backslash K_n$ contains a component of $\mathbb{C}_\infty \backslash U$). Note that $E \cap K_n$ is finite for all $n \geq 1$. Write $E_1 = E \cap K_1$ and $E_n = E \cap (K_n \backslash K_{n-1})$. Let $g_n = \sum_{p \in E_n} R_p$. Let $A = \mathbb{C}_\infty \backslash U$. By Runge's Theorem, we may choose $h_n \in \mathcal{R}_A$ such that $h_1 = 0$ and

$$|g_n(z) - h_n(z)| < \frac{1}{2^n}$$

for all $z \in K_{n-1}$ when $n \geq 2$. Let us define

$$f = \sum_{n=1}^{\infty} (g_n - h_n).$$

We have to check that this converges appropriately and has the right properties. Fix $l \geq 1$ and consider the sum defining f on K_l. We can write

$$f(z) = \sum_{n=1}^{l} (g_n(z) - h_n(z)) + \sum_{n=l+1}^{\infty} (g_n(z) - h_n(z)).$$

Note that $K_l \subset K_{n-1}$ for all $n \geq l+1$. Thus $|g_n(z) - h_n(z)| < \frac{1}{2^n}$ for all $z \in K_l$ and all $n \geq l+1$. So $\sum_{n=l+1}^{\infty}(g_n(z)-h_n(z))$ converges uniformly on K_l by the Weierstrauss M-test. The limit is holomorphic on $\operatorname{int}(K_l)$. The first sum is finite and defines a meromorphic function on $\operatorname{int}(K_l)$. This shows (since $\bigcup \operatorname{int}(K_l) = U$) that $f \in \mathcal{M}(U)$.

Say $q \in U$. Choose l such that $q \in \operatorname{int}(K_{l-1})$. Then we have

$$f(z) = \sum_{n=1}^{l} (g_n(z) - h_n(z)) + \sum_{n=l+1}^{\infty} (g_n(z) - h_n(z))$$

on $\operatorname{int}(K_l)$. The second term is holomorphic at every point of $\operatorname{int}(K_{l-1})$. Also, (h_n) are all holomorphic on U. Thus

$$P(f, q) = P\left(\sum_{n=1}^{l} g_n, q\right) = \sum_{n=1}^{l} P(g_n, q).$$

This expression shows firstly, if $q \notin E$ then f does not have a pole at q; secondly, if $q \in E$ then f does not have a pole at q and the principal part is the correct thing. □

27 The Weierstrass Factorization Theorem I (03/30)

The aim is that given a sequence (ζ_n) of complex numbers we attempt to find an entire function having a zero at each ζ_n (with multiplicity) and no other zeroes.

We definitely need $\zeta_n \to \infty$ for this to be possible. We could assume that $\zeta_n \neq 0$ (deal with 0 separately) and try $\prod_{n=1}^{\infty}(1-\frac{z}{\zeta_n})$. The only problem is convergence. From before, the product will converge provided that the sum $\sum_{n=1}^{\infty}\left|\frac{1}{\zeta_n}\right|$ converges. In fact, if this condition holds then the product converges uniformly on compact sets and solves the problem. So we have solved the problem for $\zeta_n = n^2$ or $\zeta_n = 2^n$, $\cdots$. The idea is to modify the product by introducing an exponential factor in each term. Doing this avoids any new zeroes and may help with convergence.

To find the appropriate exponential factor, we really only need to work on $\mathbb{D}$ because $\frac{z}{\zeta_n} \in \mathbb{D}$ for all large enough n. Try a factor $(1-z)\exp(\cdot)$. We know $(1-z)\exp(-\log(1-z)) = 1$ on $\mathbb{D}$. We know that

$$P_n(z) = \sum_{j=1}^{n} \frac{z^j}{j}$$

is the Maclaurin polynomial of order n for $-\log(1-z)$, so we try

$$E_n(z) = (1-z)\exp(P_n(z)).$$

We hope that E_n is pretty close to 1 on $\mathbb{D}$.

To estimate $|1-E_n(z)|$ on $\mathbb{D}$, we start with calculating $E_n'(z)$.

$$\begin{aligned}
E_n'(z) &= (1-z)P_n'(z)\exp(P_n(z)) - \exp(P_n(z)) \\
&= [(1-z)(\sum_{j=1}^{n} z^{j-1}) - 1]\exp(P_n(z)) \\
&= [(1-z^n) - 1]\exp(P_n(z)) \\
&= -z^n \exp(P_n(z)).
\end{aligned}$$

So

$$(1-E_n(z))' = z^n \exp(P_n(z)).$$

Also

$$E_n(0) = (1-0)\exp(P_n(0)) = 1.$$

Thus $1-E_n(z)$ vanishes at 0. Now we conclude that $1-E_n(z)$ has a zero of order exactly $(n+1)$ at 0. It follows that $\frac{1-E_n(z)}{z^{n+1}}$ has a removable singularity at 0 and so we may (and do) extend it to be entire. We know that

$$1 - E_n(z) = \sum_{k=0}^{\infty} c_k z^{n+k+1}$$

and for some complex numbers c_k. So

$$(1 - E_n(z))' = \sum_{k=0}^{\infty} c_k \cdot (n+k+1) \cdot z^{n+k} = z^n \exp(P_n(z)).$$

Hence

$$\exp(P_n(z)) = \sum_{k=0}^{\infty} c_k \cdot (n+k+1) \cdot z^k.$$

On the other hand,

$$\begin{aligned}
\exp(P_n(z)) &= \sum_{l=0}^{\infty} \frac{1}{l!} (P_n(z))^l \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} \left(\sum_{j=1}^{n} \frac{z^j}{j} \right)^l \\
&= \sum_{l=0}^{\infty} \frac{1}{l!} z^l \left(\sum_{j=1}^{n} \frac{z^{j-1}}{j} \right)^l .
\end{aligned}$$

We conclude that

$$\begin{aligned}
c_k \cdot (n+k+1) &= \text{ coefficients of } z^k \text{ in } \sum_{l=0}^{k} \frac{1}{l!} z^l \left(\sum_{j=1}^{n} \frac{z^{j-1}}{j} \right)^l \\
&\geq \frac{1}{k!}
\end{aligned}$$

and so

$$c_k \geq \frac{1}{k!(n+k+1)} > 0.$$

Go back to

$$\frac{1 - E_n(z)}{z^{n+1}} = \sum_{k=0}^{\infty} c_k z^k$$

armed with knowledge that $c_k > 0$ for all k. It follows that if $|z| \leq 1$ then

$$\begin{aligned}
\left|\frac{1-E_n(z)}{z^{n+1}}\right| &\leq \sum_{k=0}^{\infty} |c_k| \cdot |z|^k \\
&= \sum_{k=0}^{\infty} c_k \cdot |z|^k \\
&\leq \sum_{k=0}^{\infty} c_k \cdot 1^k \\
&= \frac{1-E_n(1)}{1^{n+1}} \\
&= \frac{1-0}{1} = 1.
\end{aligned}$$

We have proved the following lemma.

Lemma 27.1. *If* $|z| \leq 1$ *then* $|1 - E_n(z)| \leq |z|^{n+1}$.

What do we need for $\prod_{k=1}^{\infty} E_{n_k}(\frac{z}{\zeta_k})$ to converge uniformly on compact subsets of $\mathbb{C}$?

$$\prod_{k=1}^{\infty} E_{n_k}(\frac{z}{\zeta_k}) = \prod_{k=1}^{\infty}\left(1 + (E_{n_k}(\frac{z}{\zeta_k}) - 1)\right)$$

will converge uniformly on compact subsets provided that

$$\sum_{k=1}^{\infty} |E_{n_k}(\frac{z}{\zeta_k}) - 1|$$

converges uniformly on compact subsets. Recall $\zeta_k \to \infty$, so for z in compact set $\frac{z}{\zeta_k} \to 0$ uniformly. Except for a finite number of terms, we will have

$$\begin{aligned}
\sum_{k=k_0}^{\infty} |E_{n_k}(\frac{z}{\zeta_k}) - 1| &\leq \sum_{k=k_0}^{\infty} |\frac{z}{\zeta_k}|^{n_k+1} \quad \text{(by the lemma)} \\
&\leq \sum_{k=k_0}^{\infty} (\frac{R}{|\zeta_k|})^{n_k+1}
\end{aligned}$$

provided $|z| \leq R$ on the compact set. Can we make $\sum_{k=1}^{\infty}(\frac{R}{|\zeta_k|})^{n_k+1}$ converge by choosing n_k (based on $|\zeta_k|$) Actually $n_k = k$ always works. If $\zeta_k \to \infty$ then $\sum_{k=1}^{\infty}(\frac{R}{|\zeta_k|})^{n_k+1}$ always converges. Why? Because the tail is directly comparable to a geometric series.

28 The Weierstrass Factorization Theorem II (04/01)

Last time we defined

$$E_n(z) = (1-z)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right).$$

Note $E_0(z) = 1 - z, E_1(z) = (1-z)\exp(z), \cdots$.

Lemma 28.1. $|1 - E_n(z)| \leq |z|^{n+1}$ *for all* $z \in \mathbb{D}$.

Lemma 28.2. *If* (ζ_m) *is a sequence of complex numbers such that* $\zeta_m \to \infty$ *and*

$$\sum_{m=1}^{\infty} (\frac{R}{|\zeta_m|})^{n_m+1} < \infty$$

for all $R > 0$ *then* $\prod_{m=1}^{\infty} E_{n_m}(\frac{z}{\zeta_m})$ *converges uniformly on compact subsets of* $\mathbb{C}$ *to an entire function with zeroes at the elements of* $\{\zeta_m | m \geq\}$ *with multiplicities at* α *equal to* $\#\{m | \zeta_m = \alpha\}$.

Theorem 28.3 (Weierstrass Factorization Theorem)**.** *Let* $f \in H(\mathbb{C})\backslash\{0\}$. *Let* $(\zeta_m)_{m=1}^{M}$ *be the sequence of non-zeroes of* f *listed with multiplicity. (Here* $0 \leq M \leq \infty$*). Then there is an integer* $N \geq 0$ *and an entire function* g *such that*

$$f(z) = z^N e^{g(z)} \prod_{m=1}^{N} E_m(\frac{z}{\zeta_m})$$

for all $z \in \mathbb{C}$.

Proof. We already observed that $\zeta_m \to \infty$ if $M = \infty$. We also observed that if $M = \infty$, then $\sum_{m=1}^{\infty} (\frac{R}{|\zeta_m|})^{m+1}$ is convergent for all $R > 0$. Let N be the order of 0 as a zero of f. This means that $P(z) = z^N \prod_{m=1}^{M} E_m(\frac{z}{\zeta_m})$ is an entire function whose zeroes math the non-zero zeroes of f in both location and multiplicity. Let

$$h(z) = \frac{f(z)}{P(z)}.$$

This function extends to be entire (the zeroes of P are removable singularity). Moreover, h is non-vanishing. So there exists $g \in H(\mathbb{C})$ such that $h(z) = e^{g(z)}$ since $\mathbb{C}$ is simply connected. □

Theorem 28.4. *Let* $f \in \mathcal{M}(\mathbb{C})$. *Then there are* $g, h \in H(\mathbb{C})$ *such that* $f = \frac{g}{h}$ *and* h *is not identically zero.*

Proof. If $f \equiv 0$ then take $g \equiv 0$ and $h \equiv 1$. Now assume $f \not\equiv 0$. The poles of f are at most countable and my be arranged in a sequence $(\zeta_m)_{m=1}^M$ of the non-zero poles together with 0 a pole of multiplicity N. Let $h(z) = z^N \prod_{m=1}^M E_m(\frac{z}{\zeta_m})$. Note that h has a zero at each pole of f with equal multiplicity. The function $g = fh$ extends to be an entire function and so we have $f = \frac{g}{h}$. □

What about sets that aren't $\mathbb{C}$? Say $U \subset \mathbb{C}$ is an open set. Firstly, can you find a holomorphic function on U with prescribed zeroes? The answer is yes. Secondly, can you get a Weierstrass factorization theorem? The answer is not exactly. Thirdly, is $\mathcal{M}(U)$ the field of fractions of $H(U)$? The answer is yes provided U is connected (this follows once we had an affirmative answer to the first question).

How do we get the affirmative answer to the first question? It is easiest to explain when (ζ_m) is bounded. If it is, then we can find a sequence of points $(\mu_m) \subset \mathbb{C}\backslash U$ such that $|\zeta_m - \mu_m| \to 0$ as $m \to \infty$. Then we use

$$\prod_{m=1}^{\infty} E_m\left(\frac{\zeta_m - \mu_m}{z - \mu_m}\right).$$

29 The Weierstrass Factorization Theorem III (04/03)

Lemma 29.1. *Let $U \subset \mathbb{C}$ be an open set and (ζ_m) be a sequence in U with no limit points in U and finite multiplicities. Assume that (ζ_m) is bounded. Then disc$(\zeta_m, \mathbb{C}\backslash U) \to 0$.*

Proof. We know that if we define

$$K_n = \{z \in U | d(0,z) \le n, \operatorname{disc}(z, \mathbb{C}\backslash U) \ge \frac{1}{n}\}.$$

Then (K_n) is a compact exhaustion of U. Note that since (ζ_m) is bounded, we have $d(0, \zeta_m) \le n$ for all m and all sufficiently large n. Choose N such that this holds if $n \ge N$. If $n \ge N$, then

$$\zeta_m \in K_n \Leftrightarrow \operatorname{disc}(\zeta_m, \mathbb{C}\backslash U) \ge \frac{1}{n}.$$

The set $\{m : \zeta_m \in K_n\}$ is finite and so the number of m such that $\operatorname{disc}(\zeta_m, \mathbb{C}\backslash U) \ge \frac{1}{n}$ is finite for all n. This implies that $\operatorname{disc}(\zeta_m, \mathbb{C}\backslash U) \to 0$. □

This allows us to show that given a bounded sequence (ζ_m) with finite multiplicities in an open set U, we can find $f \in H(U)$ whose zeroes are precisely the ζ_m with multiplicities. By the lemma, we can find $\zeta_m \in \mathbb{C}\backslash U$ such that $|\zeta_m - \mu_m| \to 0$. The product $\prod_{m=1}^{\infty} E_m\left(\frac{\zeta_m - \mu_m}{z - \mu_m}\right)$ will work. Suppose $K \subset U$ is a compact set. Then $\operatorname{disc}(K, \mathbb{C}\backslash U) > 0$ and so there is $\eta > 0$ such that $|\mu_m - z| \ge \eta$ for all $m \ge 1$ and all $z \in K$. Thus

$$\left|\frac{\zeta_m - \mu_m}{z - \mu_m}\right| \le \frac{1}{\eta} |\mu_m - \zeta_m|$$

for all $z \in K$ and all $m \geq 1$. Thus $\left(\frac{\zeta_m - \mu_m}{z - \mu_m}\right)$ converges to 0 uniformly on K. It follows from the inequality

$$|1 - E_m(w)| \leq |w|^{m+1}$$

for all $w \in \mathbb{D}$ that the product is uniformly convergent on K. Note that $\prod_{m=1}^{\infty} E_m\left(\frac{\zeta_m - \mu_m}{z - \mu_m}\right)$ has a simple zero at $z = \zeta_m$, no other zeroes, and is holomorphic on U. This expression solves our problem.

The general case of the result has one more reduction. Let $U \subset \mathbb{C}$ be open, (ζ_m) a finite multiplicity sequence in U that has no limit points in U. Choose $p \in U$, $p \neq \zeta_m$ for any m. Since p is not a limit point of the sequence, $|\zeta_m - p| \geq \eta$ for some positive η and all m. Thus the sequence $\left(\frac{1}{\zeta_m - p}\right)$ is bounded. We find $\mu_m \in \mathbb{C} \backslash \tilde{U}$ where $\tilde{U} = \{\frac{1}{z-p} : z \in U\}$ such that $\left|\frac{1}{\zeta_m - p} - \mu_m\right| \to 0$ and we use the product

$$\prod_{m=1}^{\infty} E_m\left(\frac{\mu_m - \frac{1}{\mu - p}}{\mu_m - \frac{1}{z-p}}\right).$$

You have to check that the singularity at p is removable.

Theorem 29.2. *Let $U \subset \mathbb{C}$ be an open set and (ζ_m) a sequence with finite multiplicity and no limit points in U. Then there exists $f \in H(U)$ whose zeroes are precisely the ζ_m counted with multiplicity.*

One application: Let $U \subset \mathbb{C}$ be a simply connected open set and let $f \in H(U)$ such that all the zeroes of f are of even order. Then there exists $g \in H(U)$ such that $f = g^2$. Note that the statement implies that $f \not\equiv 0$. Let (ζ_m) be the sequence (finite or infinite) of zeroes of f listed without multiplicity and let N_m be the order of ζ_m as a zero of f. Note the assumption implies that $2|N_m$ for all m. By the theorem, we may find $h \in H(U)$ such that the zeroes of h are also at ζ_m but the order of the zero at ζ_m is $\frac{1}{2}N_m$. Define $\psi = \frac{f}{h^2}$. This extends to be holomorphic on U and non-vanishing. Since $\psi \in H(U)$ is non-vanishing, there exists $\varphi \in H(U)$ such that $\psi = \varphi^2$ because U is simply connected. Thus $\varphi^2 = \frac{f}{h^2}$. So $f = (\varphi h)^2 = g^2$ with $g = \varphi h \in H(U)$.

Next time I want to apply Mittag-Leffler and this theorem to establish holomorphic interpolation. The basic version is the following.

Theorem 29.3 (Holomorphic Interpolation, Basic Version)**.** *Suppose $U \subset \mathbb{C}$ is open, (ζ_m) is a sequence without repetition in U with no limit points, (c_m) is a sequence of complex numbers. Then there exists $f \in H(U)$ such that $f(\zeta_m) = c_m$.*

30 The Weierstrass Factorization Theorem IV (04/06)

Definition 30.1. An open set $U \subset \mathbb{C}$ is called a *domain of holomorphy* if it is connected and for all connected open sets $V \supset U, V \neq U$, there is some $f \in H(U)$ such that f is not the restriction to U of any element of $H(V)$.

Lemma 30.2. *Let $U \subset \mathbb{C}$ be an open set. Then we may find a set $S \subset U$ such that S has no limit points in U but every point of ∂U is a limit point of S.*

Proof. For all $m \geq 1$, the set $\partial U \cap \overline{D}(0, m)$ is compact. Thus there is a countable set $A_n \subset \partial U \cap \overline{D}(0, m)$ that is dense in $\partial U \cap \overline{D}(0, m)$. Let $A = \cup_{m=1}^{\infty} A_m$. Then A is countable and dense in ∂U. For each $a \in A$, we may choose a sequence $(z_n(a))$ in U such that $z_n(a) \to a$. Note that we may assume that the map $(a, n) \mapsto z_n(a)$ is one-to-one, because there are always uncountably many choices for $z_n(a)$.

Choose a compact exhaustion (K_m) of U. Enumerate the elements of A as (a_l). Define S to contain all the elements of the sequence $(z_n(a_1))$, all the elements of the sequence $(z_n(a_2))$ that do not lie in K_1, all the elements of the sequence $(z_n(a_3))$ that do not lie in K_2, and so on, i.e.,

$$S = \{z_n(a_l) | z_n(a_l) \notin K_{l-1} \text{ for } l \geq 2\}.$$

Note that only a finite number of the terms in the sequence $(z_n(a_l))$ lie in any particular K_m. It follows that $S \cap K_m$ is finite for every m. It follows that S has no limit points in U. It also follows that S contains the tail of each sequence $(z_n(a_l))$. Thus $a_l \in \mathrm{cl}(S)$ for all l. Thus $\partial U \subset \mathrm{cl}(S)$ since $\{a_l | l \geq 1\}$ is dense in ∂U. It follows from $S \cap \partial U = \emptyset$ that every point of ∂U is a limit point of S. $\square$

Theorem 30.3. *Every non-empty connected open subset of $\mathbb{C}$ is a domain of holomorphy.*

Proof. Let U be a connected open subset of $\mathbb{C}$. If $U = \mathbb{C}$, then the definition is vacuously satisfied. Otherwise, $\partial U \neq \emptyset$. Let $S \subset U$ be a set as in Lemma 30.2. We may find $f \in H(U)$ such that f has a simple zero at every element of S, but no other zeroes. Let $V \supsetneq U$ be a connected open set. Suppose that $g \in H(V)$ and $g|_U = f$. Since V is connected, V must contain at least one boundary point of U, say p. Then g is zero at every element of S and so the zero set of g has a limit point (p) in V. By the Identity Principle, $g \equiv 0$. Thus $f \equiv 0$, contrary to our construction. $\square$

31 The Weierstrass Factorization Theorem V (04/08)

Let $U \subset \mathbb{C}$ be an open set. Let $S \subset U$ be a set with no limit points in U. Suppose that we are given a polynomial $P_s(z) \in \mathbb{C}[z - s] \backslash \{0\}$ for each $s \in S$. We aim to show that there is $f \in H(U)$ such that the power series expansion of f centered at $s \in S$ is

$$P_s(z) + \text{ higher order terms.}$$

This is a very strong type of interpolation, since it is equivalent to specifying the value of f and a finite number of derivatives at each point of S.

Let $n_s = \deg(P_s)$ for each $s \in S$. We know already that we can choose a $g \in H(U)$ such that g has a zero of order exactly $n_s + 1$ at each point of S and no other zeroes. For each $s \in S$, we have a power series

$$g(z) = k_s(z - s)^{n_s+1} + \text{ higher order terms}$$

where $k_s \neq 0$ for all $s \in S$. This means that

$$\frac{P_s(z)}{g(z)} = R_s(z) + \text{ a power series}$$

where

$$R_s(z) = \frac{a_{s,n_s+1}}{(z-s)^{n_s+1}} + \frac{a_{s,n_s}}{(z-s)^{n_s}} + \cdots + \frac{a_{s,1}}{z-s}, \quad a_{s,j} \in \mathbb{C}.$$

We know from Runge's Theorem that there is $h \in \mathcal{M}(U)$ such that h has poles only at the points in S and the principal part of h at $s \in S$ is $R_s(z)$. Now define $f = gh$. Initially, $f \in \mathcal{M}(U)$. However, the only possible poles of f are at points of S. Actually, these are removable singularities because the zero of g at $s \in S$ has order exactly $n_s + 1$, whereas the pole of h has order at most $n_s + 1$. Thus f extends to an element of $H(U)$. Fix $s \in S$. Then we have

$$\begin{aligned} f(z) =& g(z)\left(R_s(z) + \text{ power series }\right) \text{ (on a delected disk } D'(s,r) \text{ for some } r > 0) \\ =& g(z)R_s(z) + g(z) \cdot \text{(power series)} \\ =& P_s(z) + \text{terms higher than } (z-s)^{n_s} + \text{terms higher than } (z-s)^{n_s}. \end{aligned}$$

Thus f works to solve the problem. This completes the proof of the "Holomorphic Interpolation Theorem".

32 Maximum Modulus Principle Revisited I (04/10)

We begin this chapter with a problem from comprehensive exam.

Comprehensive Exam, August 2009, Problem 5: Let

$$S = \{z \in \mathbb{C} | 0 < \operatorname{Re}(z) < 1, \operatorname{Im}(z) > 0\}.$$

Let $f \in H(S) \cap C(\overline{S})$. Suppose that $|f(z)| \leq 1$ for all z in the boundary of S. Also suppose that $|f(z)| \leq |z|$ for all $z \in S$. Show that $|f(z)| \leq 1$ for all $z \in S$.

We use Phragmèn-Lindelöf's idea to solve this problem.

The first ingredient is the following lemma.

Lemma 32.1. *Let $U \subset \mathbb{C}$ be open. Let $f \in H(U) \cap C(\overline{U})$. Suppose that $|f(z)| \leq M$ for all $z \in \partial U$ and $\lim_{z \to \infty} |f(z)| = 0$. Then $|f(z)| \leq M$ for all $z \in U$.*

Proof. Let $\varepsilon > 0$ and $w \in U$. Choose $R > 0$ such that $|w| < R$ and if $|z| \geq R$, then $|f(z)| \leq M + \varepsilon$. Consider $V = U \cap D(0, R)$. Then V is a bounded open set, and $\partial V \subset \partial U \cup \partial D(0, R)$. It follows that $|f(z)| \leq M + \varepsilon$ for $z \in \partial V$. Thus $|f(w)| \leq M + \varepsilon$. Since $\varepsilon > 0$ was arbitrary, $|f(w)| \leq M$, since $w \in U$ was arbitrary, we are done. □

The second ingredient in Phragmèn-Lindelöf results is a family of auxiliary functions that you have to discover. Back to the problem, the hint says to consider

$$g_\varepsilon(z) = f(z) e^{i\varepsilon z},$$

then let $\varepsilon \to 0^+$. Note

$$e^{i\varepsilon(x+iy)} = e^{i\varepsilon x} \cdot e^{-\varepsilon y}$$

and so

$$|e^{i\varepsilon(x+iy)}| = e^{-\varepsilon y}.$$

Thus $|e^{i\varepsilon z}| \leq 1$ for all $\varepsilon > 0$ and all $z \in \overline{S}$. Also, for z large, $|e^{i\varepsilon z}| \to 0$ rather quickly. In fact,

$$\begin{aligned} |g_\varepsilon(z)| &\leq |z| \cdot e^{-\varepsilon y} \\ &\leq (1+y) e^{-\varepsilon y} \\ &\to 0 \text{ as } |z| < \infty \text{ in } S^2. \end{aligned}$$

Note the first ingredient implies

$$|g_\varepsilon(z)| \leq 1, \forall z \in S.$$

Let $\varepsilon \to 0^+$ to conclude

$$|f(z)| \leq 1, \forall z \in S.$$

So we have solved the problem.

Let $\sigma > 0$ and define

$$S_\sigma = \{x + iy | -\sigma < x < \sigma\}.$$

The aim is to find out what the hypothesis for a Phragmèn-Lindelöf Theorem on S_σ should be. We look for a function that is bounded on ∂S_σ and grows quickly as $z \to \infty$ inside the strip.

Define $g \in H(S_\sigma) \cap C(\overline{S}_\sigma)$ by

$$g(z) = \cos(\frac{\pi}{2\sigma}z).$$

Then

$$\begin{aligned} g(x + iy) &= \cos(\frac{\pi x}{2\sigma} + i\frac{\pi y}{2\sigma}) \\ &= \cos(\frac{\pi x}{2\sigma}) \cdot \cos(i\frac{\pi y}{2\sigma}) - \sin(\frac{\pi x}{2\sigma}) \cdot \sin(i\frac{\pi y}{2\sigma}) \\ &= \cos(\frac{\pi x}{2\sigma}) \cdot \cosh(\frac{\pi y}{2\sigma}) - i \sin(\frac{\pi x}{2\sigma}) \cdot \sinh(\frac{\pi y}{2\sigma}) \end{aligned}$$

and so

$$\mathrm{Re}(g(x + iy)) = \cos(\frac{\pi x}{2\sigma}) \cdot \cosh(\frac{\pi y}{2\sigma}).$$

Let

$$h(z) = \exp(g(z)) = \exp(\cos(\frac{\pi}{2\sigma}z)).$$

Then

$$|h(z)| = |\exp(g(z))| = \exp(\mathrm{Re}(g(x + iy)))$$

and so

$$|h(x + iy)| = \exp(\cos(\frac{\pi x}{2\sigma}) \cdot \cosh(\frac{\pi y}{2\sigma})).$$

It follows that $|h(z)| \leq 1$ for $z \in \partial S_\sigma$, but $|h(z)| \to \infty$ fairly quickly as $|y| \to \infty$ for any $z = x + iy \in S_\sigma$.

Theorem 32.2 (Phragmèn-Lindelöf Theorem)**.** *Let $\sigma > 0$ and*

$$S_\sigma = \{x + iy | -\sigma < x < \sigma\}.$$

Let $f \in H(S_\sigma) \cap C(\overline{S}_\sigma)$ and suppose that $|f(z)| \leq M$ for all $z \in \partial S_\sigma$. Suppose that there are constants $c, K > 0$ and $\alpha \in (0, \frac{\pi}{2\sigma})$ such that

$$|f(z)| \leq K \exp(ce^{\alpha|z|})$$

for all $z \in S_\sigma$. Then $|f(z)| \leq M$ for all $z \in S_\sigma$.

Proof. Choose $\beta \in (\alpha, \frac{\pi}{2\sigma})$. Let $\varepsilon > 0$ and consider

$$h_\varepsilon(z) = \exp(-\varepsilon \cos(\beta z)).$$

Then

$$|h_\varepsilon(z)| = \exp(-\varepsilon \mathrm{Re}(\beta z)) = \exp(-\varepsilon \cos(\beta x)\cosh(\beta y)) \text{ (where } z = x+iy).$$

So

$$|h_\varepsilon(z)| \le \exp(-\varepsilon \cos(\beta\sigma)\cosh(\beta y)) \text{ (where } z = x+iy)$$

for all $z = x+iy \in \overline{S}_\sigma$. It follows that $|h_\varepsilon(z)| \le 1$ for all $z \in \overline{S}_\sigma$. Thus

$$|(fh_\varepsilon)(z)| \le M, \forall z \in \partial S_\sigma.$$

Moreover,

$$\begin{aligned}
|(fh_\varepsilon)(z)| \le& K\exp(ce^{\alpha|z|})\exp(-\varepsilon\cos(\beta\sigma)\cosh(\beta y)) \\
=& \exp\left(\ln(K) + ce^{\alpha|z|} - \frac{\varepsilon\cos(\beta\sigma)}{2}\cdot(e^{\beta y}+e^{-\beta y})\right) \\
\le& \exp\left(\ln(K) + ce^{\alpha\sigma}e^{|y|} - \frac{\varepsilon\cos(\beta\sigma)}{2}\cdot(e^{\beta y}+e^{-\beta y})\right) \\
& \text{(because } |z| \le |y| + \sigma \text{ on } \overline{S}_\sigma.)
\end{aligned} \tag{32.1}$$

for all $z \in S_\sigma$. As $y \to \infty$, the exponent in 32.1 is going to $-\infty$ because $\beta > \alpha$ and so $e^{\beta y} + e^{-\beta y}$ eventually dominates $e^{\alpha|y|}$. Thus

$$\lim_{z\to\infty}(fh_\varepsilon)(z) = 0.$$

From Lemma 32.1 we conclude that

$$|(fh_\varepsilon)(z)| \le M$$

for all $z \in S_\sigma$. However,

$$\lim_{\varepsilon\to 0^+} h_\varepsilon(z) = 1$$

for all z. Thus,

$$|f(z)| = \lim_{\varepsilon\to 0^+}|(fh_\varepsilon)(z)| \le M$$

for all $z \in S_\sigma$. □

Remark 32.3. *You will often see this Phragmèn-Lindelöf Theorem stated with the estimate*

$$|f(z)| \le K\exp(c|z|^A)$$

for some $c, K, A > 0$. Actually, this is weaker.

Remark 32.4. *An entire function f is "of exponential type" if*

$$|f(z)| \le K\exp(\tau|z|)$$

for some $K, \tau > 0$.

33 Maximum Modulus Principle Revisited II (04/13)

Here is another proof of the corollary to the Maximum Modulus Principle (Lemma 32.1), using "Landau trick".

Theorem 33.1. *Let $U \subset \mathbb{C}$ be a bounded open set and $f : U \to \mathbb{C}$ a holomorphic function. Suppose that there is a constant $M > 0$ such that*

$$\limsup_{z\to\zeta, z\in U}|f(z)| \leq M$$

for all $\zeta \in \partial U$. Then $|f(w)| \leq M$ for all $w \in U$.

Proof. Let $\varepsilon > 0$. We may cover the boundary of U by open disks such that $|f(w)| \leq M+\varepsilon$ for w in any of the intersections of these disks with U. Since ∂U is compact (U is bounded), I may choose a finite subcover from these disks. Let $w \in U$ and choose a compact subset K of U such that $w \in K$ and the complement of K in U is contained in the union of the disks. Choose a cycle Γ such that $\Gamma^* \subset U\backslash K$, $\mathrm{Ind}(\Gamma, z) = 0$ for all $z \notin U$, and $\mathrm{Ind}(\Gamma, p) = 1$ for all $p \in K$. Choose $m \geq 1$. Apply the Homology Version of Cauchy's Integral Formula to f^m and Γ. We get

$$f^m(w) = \frac{1}{2\pi i}\int_\Gamma \frac{f^m(\zeta)}{\zeta - w}\, d\zeta$$

since $f^m \in H(U)$. Now I will estimate using the ML-inequality. Let $\delta = \mathrm{disc}(w, \Gamma^*)$. I get

$$|f^m(w)| \leq \frac{1}{2\pi}\frac{(M+\varepsilon)^m}{\delta} \cdot \mathcal{L}(\Gamma)$$

for all $m \geq 1$. Thus by taking m-th roots,

$$|f(w)| \leq \frac{M+\varepsilon}{(2\pi\delta)^{1/m}} \cdot (\mathcal{L}(\Gamma))^{1/m}.$$

Now take the limit as $m \to \infty$, we get

$$|f(w)| \leq \frac{M+\varepsilon}{1} \cdot 1 = M + \varepsilon.$$

Now $\varepsilon > 0$ was arbitrary and so $|f(w)| \leq M$. □

Next, we combine Landau's trick with the Phragmèn-Lindelöf Method.

Theorem 33.2. *Let $U \subset \mathbb{C}$ be an open set with $U \neq \mathbb{C}$. Let $f \in H(U)$ and assume that f is bounded. Suppose that there is a constant $M > 0$ such that*

$$\limsup_{z\to\zeta, z\in U}|f(z)| \leq M$$

for all $\zeta \in \partial U$. Then $|f(w)| \leq M$ for all $w \in U$.

Proof. Let $\varepsilon > 0$. Note that $\partial U \neq \emptyset$. Choose $p \in \partial U$. I may find a disk centered at p such that if $z \in \text{disk} \cap U$ then $|f(z)| \leq M + \varepsilon$. This means that I may find $q \in U$ and $r > 0$ such that $|f(z)| \leq M_\varepsilon$ for all $z \in \overline{D}(q, r)$. Let $m \geq 1$ and consider the function

$$g_m(w) = \frac{r}{w - q} f^m(w).$$

Let $V = U \backslash \overline{D}(q, r)$. Then $\partial V \subset \partial U \cup \partial D(q, r)$. On ∂U, I have the estimate

$$\limsup_{z \to \zeta, z \in U} |g_m(z)| \leq \frac{r}{\text{disc}(q, \partial U)} \cdot (M + \varepsilon)^m.$$

On $\partial D(q, r)$, I have the estimate

$$|g_m(z)| \leq (M + \varepsilon)^m.$$

Moreover,

$$\lim_{w \to \infty, w \in V} |g_m(w)| = 0.$$

It follows from Lemma 32.1 that

$$|g_m(w)| \leq K(M + \varepsilon)^m$$

where

$$K = \max\{1, \frac{r}{\text{disc}(q, \partial U)}\}$$

for all $w \in V$. It follows that

$$\left|\frac{r}{w - q}\right|^{1/m} \cdot |f(w)| \leq K^{1/m}(M + \varepsilon)$$

for all $w \in V$. Let $m \to \infty$ to conclude that

$$|f(w)| \leq M + \varepsilon, \forall w \in V.$$

We also have $|f(w)| \leq M + \varepsilon$ for all $w \in \overline{D}(q, r)$. Thus $|f(w)| \leq M + \varepsilon, \forall w \in U$. Now $\varepsilon > 0$ was arbitrary and so $|f(w)| \leq M, \forall w \in U$. $\square$

Remark 33.3. *We considered $g_m(w) = \frac{r}{w-q} f^m(w)$. Here $f^m(w)$ is the setting up for Laudau trick, and $\frac{r}{w-q}$ is an auxiliary function for Phragmèn-Lindelöf Method.*

34 The Gamma Function I (04/15)

Definition 34.1. We define the *Gamma function* $\Gamma : \pi^{\mathrm{R}} \to \mathbb{C}$ by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt.$$

To see that this makes sense, first think about t^{z-1}. We define

$$t^w = e^{w\ln(t)}$$

for $t > 0$ and $w \in \mathbb{C}$. Note that $w \mapsto t^w$ is entire. We have

$$t^{w_1+w_2} = e^{(w_1+w_2)\ln(t)} = e^{w_1\ln(t)} \cdot e^{w_2\ln(t)} = t^{w_1} \cdot t^{w_2}.$$

Also,

$$|t^w| = |e^{w\ln(t)}| = e^{\mathrm{Re}(w\ln(t))} = e^{\mathrm{Re}(w)\cdot\ln(t)} = t^{\mathrm{Re}(w)}.$$

Next, the integral

$$\int_0^\infty t^{z-1}e^{-t}\,dt = \int_0^\infty \mathrm{Re}(t^{z-1})e^{-t}\,dt + i\int_0^\infty \mathrm{Im}(t^{z-1})e^{-t}\,dt$$

is the sum of two ordinary improper Riemann integral. What about convergence? The function $t^{z-1}e^{-t}$ is uniformly integrable on $(0,\infty)$ and the uniform integrability is uniform in z provided that you have $\varepsilon > 0$, $M > \varepsilon$ such that $\varepsilon \le \mathrm{Re}(z) \le M$. What this means is that given $\eta > 0$ there are $r > 0$ and $R > 0$ such that if $\varepsilon \le \mathrm{Re}(z) \le M$ then

$$\left|\int_{r_1}^{r_2} t^{z-1}e^{-t}\,dt\right| < \eta, \tag{34.1}$$

$$\left|\int_{R_1}^{R_2} t^{z-1}e^{-t}\,dt\right| < \eta, \tag{34.2}$$

for all $0 < r_1 < r_2 < r$ and $R < R_1 < R_2$.

For 34.1,

$$\begin{aligned}
\left|\int_{r_1}^{r_2} t^{z-1}e^{-t}\,dt\right| &\le \int_{r_1}^{r_2} t^{\mathrm{Re}(z)-1}e^{-t}\,dt \\
&\le \int_{r_1}^{r_2} t^{\mathrm{Re}(z)-1}\,dt \\
&\le \left.\frac{t^{\mathrm{Re}(z)}}{\mathrm{Re}(z)}\right|_{r_1}^{r_2} \\
&= \frac{r_2^{\mathrm{Re}(z)}}{\mathrm{Re}(z)} - \frac{r_1^{\mathrm{Re}(z)}}{\mathrm{Re}(z)} \\
&< \frac{r^{\mathrm{Re}(z)}}{\mathrm{Re}(z)}.
\end{aligned}$$

We can choose $r < 1$. Then

$$\left|\int_{r_1}^{r_2} t^{z-1}e^{-t}\,dt\right| \le \frac{r^\varepsilon}{\varepsilon} \to 0 \text{ as } r \to 0^+$$

because $\varepsilon > 0$. A note about 34.2:

$$t^{z-1}e^{-t} = (t^{z-1}e^{-t/2}) \cdot e^{-t/2}$$

where $t^{z-1}e^{-t/2}$ is bounded for $t \ge 1$.

In particular, the integral defining $\Gamma(z)$ always converges for $z \in \pi^{\mathrm{R}}$. Also, Γ is a holomorphic function.

Recall that the basic method for showing that a function defined by an integral is holomorphic is to use Morera's Theorem. This means verifying that $\Gamma(z)$ is continuous as a function of z and $\int_\Delta \Gamma(w)\,dw = 0$ for all triangle Δ such that the boundary of the triangle as well as its interior is a subset of the right half plane π^{R}. For continuity, it suffices to show that we have the continuity on $\varepsilon \le \mathrm{Re}(z) \le M$ for all $\varepsilon > 0, M > \varepsilon$. Suppose z_1, z_2 lie in this set. Then

$$\begin{aligned}
\Gamma(z_1) - \Gamma(z_2) &= \int_0^\infty (t^{z_1-1} - t^{z_2-1})e^{-t}\,dt \\
&= \left(\int_0^r + \int_r^R + \int_R^\infty\right)(t^{z_1-1} - t^{z_2-1})e^{-t}\,dt \\
&= \text{ something uniformly small } + \int_r^R (t^{z_1-1} - t^{z_2-1})e^{-t}\,dt + \text{ something uniformly small.}
\end{aligned}$$

Uniform continuity implies that the middle integral can be made small by making z_1 close to z_2. This gives continuity of Γ.

For the other part of Morera's Theorem, we want to say

$$\begin{aligned}
\int_\Delta \Gamma(w)\,dw &= \int_\Delta \left(\int_0^\infty t^{w-1}e^{-t}\,dt\right)\,dw \\
&= \int_0^\infty \left(\int_\Delta t^{w-1}e^{-t}\,dw\right)\,dt \\
&= \int_0^\infty 0dt \\
&\text{(since } w \mapsto t^{w-1} \text{ is holomorphic, } \int_\Delta t^{w-1}e^{-t}\,dw = 0) \\
&= 0.
\end{aligned}$$

The interchange (of $\int_0^\infty$ and $\int_\Delta$) can be justified for $\int_r^R t^{w-1}e^{-t}\,dt$ by Fubini from Advanced Calculus. $|\int_\Delta \int_0^r t^{w-1}e^{-t}\,dtdw|$ and $|\int_\Delta \int_R^\infty t^{w-1}e^{-t}dtdw|$ are less than any positive number by a suitable choice of r and R.

In fact, Γ extends to be an element of $\mathcal{M}(\mathbb{C})$. Start with a $z \in \pi^{\mathrm{R}}$. Then

$$\begin{aligned}
\Gamma(z) &= \int_0^\infty t^{z-1}e^{-t}\,dt \\
&= \int_0^1 t^{z-1}e^{-t}\,dt + \int_1^\infty t^{z-1}e^{-t}\,dt \\
&= \int_0^1 t^{z-1}\left(\sum_{m=0}^\infty \frac{(-1)^m}{m!}t^m\right)dt + F(z) \\
&= \sum_{m=0}^\infty \frac{(-1)^m}{m!}\int_0^1 t^{z+m-1}\,dt + F(z) \\
&= \sum_{m=0}^\infty \frac{(-1)^m}{m!}\cdot\frac{1}{z+m} + F(z) \\
&= F(z) + \sum_{m=0}^\infty \frac{(-1)^m}{m!(z+m)}
\end{aligned}$$

Now $F(z) = \int_1^\infty t^{z-1}e^{-t}\,dt$ is an entire function of z. This is because it is uniformly integrable at ∞ for all z. Then the same arguments apply. The other term $\sum_{m=0}^\infty \frac{(-1)^m}{m!(z+m)}$ is uniformly convergent on compact subsets of $\mathbb{C}\backslash\{m \in \mathbb{Z} | m \leq 0\}$. In fact, $\sum_{m=0, m\neq k}^\infty \frac{(-1)^m}{m!(z+m)}$ is uniformly convergent on compact subsets of $\mathbb{C}\backslash\{m \in \mathbb{Z} | m \leq 0, m \neq k\}$. This implies that $\Gamma \in \mathcal{M}(\mathbb{C})$ with simple poles at $0, -1, -2, -3, \cdots$ with known residues.

35 The Gamma Function II (04/17)

The Gamma function is defined as

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\,dt$$

for $\mathrm{Re}(z) > 0$. Γ extends to a meromorphic function on $\mathbb{C}$ with simple poles at $0, -1, -2, \cdots$ and no other singularities.

I want to evaluate

$$I_n(z) = \int_0^1 t^{z-1}(1-t)^n\,dt$$

where $n \in \{0, 1, 2, \cdots\}$. Note that

$$\begin{aligned}
I_0(z) &= \int_0^1 t^{z-1}\,dt = \frac{1}{z},\\
I_1(z) &= \int_0^1 t^{z-1}(1-t)\,dt = \frac{1}{z} - \frac{1}{z+1} = \frac{1}{z(z+1)},\\
I_n(z) &= \int_0^1 t^{z-1}(1-t)^n\,dt\\
&= \int_0^1 t^{z-1}(1-t)^{n-1}(1-t)\,dt\\
&= \int_0^1 t^{z-1}(1-t)^{n-1}\,dt - \int_0^1 t^z(1-t)^{n-1}\,dt\\
&= I_{n-1}(z) - I_{n-1}(z+1).
\end{aligned}$$

We get $I_n(z) = I_{n-1}(z) - I_{n-1}(z+1)$ for all $n \geq 1$.

$$I_2(z) = I_1(z) - I_1(z+1) = \frac{1}{z(z+1)} - \frac{1}{(z+1)(z+2)} = \frac{2}{z(z+1)(z+2)}.$$

In special functions, we denote

$$(z)_n = z(z+1)(z+2)\cdots(z+n-1).$$

Guess

$$I_n(z) = \frac{n!}{(z)_{n+1}}$$

and then verify it by induction.

Now go back to

$$\int_0^1 t^{z-1}(1-t)^n\,dt = \frac{n!}{(z)_{n+1}}.$$

I want to change variable, replacing t by $\frac{t}{n}$, I get

$$\begin{aligned}
&\int_0^1 \left(\frac{t}{n}\right)^{z-1} \cdot \left(1 - \frac{t}{n}\right)^n \cdot \frac{1}{n}\,dt = \frac{n!}{(z)_{n+1}}\\
\Rightarrow &\frac{1}{n^z}\int_0^1 t^{z-1}\left(1 - \frac{t}{n}\right)^n\,dt = \frac{n!}{(z)_{n+1}}\\
\Rightarrow &\int_0^1 t^{z-1}\left(1 - \frac{t}{n}\right)^n\,dt = \frac{n^z n!}{(z)_{n+1}}.
\end{aligned}$$

Naively, we try $n \to \infty$, we get

$$\int_0^1 t^{z-1}e^{-t}\,dt = \lim_{n\to\infty} \frac{n^z n!}{(z)_{n+1}}$$

because $\lim_{n\to\infty}(1-\frac{t}{n})^n = e^{-t}$. Note that

$$\lim_{n\to\infty} \frac{n^z n!}{(z)_{n+1}} = \lim_{n\to\infty} \frac{n^{z-1}n!}{(z)_n} \cdot \frac{z}{z+n} = \lim_{n\to\infty} \frac{n^{z-1}n!}{(z)_n}.$$

Can we justify

$$\lim_{n\to\infty} \int_0^n t^{z-1}(1-\frac{t}{n})^n\, dt = \int_0^\infty t^{z-1}e^{-t}\, dt\ ?$$

Yes. How? We could use real analysis. We still need some information about how $(1-\frac{t}{n})^n \to e^{-t}$.

36 The Gamma Function III (04/20)

Last time we arrived at

$$\lim_{n\to\infty} \int_0^n t^{z-1}(1-\frac{t}{n})^n\, dt.$$

Here $\mathrm{Re}(z) > 0$.

Lemma 36.1. *Fix $t \geq 0$ and consider the function $g : (t,\infty) \to \mathbb{R}$ by*

$$g(x) = (1-\frac{t}{x})^x.$$

Then g is an increasing function of x.

Proof. Let $h(x) = \ln(g(x)) = x\ln(1-\frac{t}{x})$. It suffices to show that h is increasing. Well,

$$\begin{aligned} h'(x) &= \ln(1-\frac{t}{x}) + x \cdot \frac{1}{1-\frac{t}{x}} \cdot \frac{t}{x^2} \\ &= \ln(1-\frac{t}{x}) + \frac{t/x}{1-t/x} \\ &= -\sum_{m=1}^\infty \frac{1}{m}(\frac{t}{x})^m + \sum_{m=1}^\infty (\frac{t}{x})^m \\ &= \sum_{m=1}^\infty (1-\frac{1}{m})(\frac{t}{m})^m \\ &\geq 0. \end{aligned}$$

□

Corollary 36.2. *Fix $N > 0, N \in \mathbb{N}$. Then the sequence $\big((1-\frac{t}{n})^n\big)_{n\geq N}$ converges uniformly to e^{-t} on $[0,N]$. Moreover, $(1-\frac{t}{n})^n \leq e^{-t}$ for all $n \geq t$.*

Proof. We know that $(1-\frac{t}{n})^n \to e^{-t}$ pointwise for all $t \geq 0$. Moreover, Lemma 36.1 shows that $(1-\frac{t}{n})^n$ is an increasing sequence once $n > t$. It follows that

$$(1-\frac{t}{n})^n \leq e^{-t}$$

for $n > t$. Fix N as in the statement. The conclusion follows from Dini's Theorem.

Here is the statement of Dini's Theorem. Let X be a compact metric space, (g_n) a sequence of continuous real-valued functions on X such that $(g_n(x))$ is an increasing sequence for each $x \in X$. Also suppose that g is a continuous real-valued function on X and $g(x) = \lim_{n\to\infty} g_n(x)$ for all $x \in X$. Then $g_n \to g$ uniformly.

Now we want to show

$$\lim_{n\to\infty} \int_0^n t^{z-1}(1-\frac{t}{n})^n \, dt = \int_0^\infty t^{z-1}e^{-t}\, dt, \ \mathrm{Re}(z) > 0.$$

Let $\epsilon > 0$. Choose $N > 0, N \in \mathbb{N}$ such that

$$\int_N^\infty t^{\mathrm{Re}(z)-1}e^{-t}\, dt < \frac{\epsilon}{3}.$$

Let $n \geq N$.

$$\left|\int_N^n t^{z-1}(1-\frac{t}{n})^n \, dt\right| \leq \int_N^\infty t^{\mathrm{Re}(z)-1}e^{-t}\, dt < \frac{\epsilon}{3}.$$

Now, for $n \geq N$,

$$\begin{aligned}\left|\int_0^n t^{z-1}(1-\frac{t}{n})^n dt - \int_0^\infty t^{z-1}e^{-t}\, dt\right| &\leq \left|\int_o^N t^{z-1}\left[(1-\frac{t}{n})^n - e^{-t}\right] dt\right| \\ &\quad + \int_N^n t^{\mathrm{Re}(z)-1}(1-\frac{t}{n})^n dt + \int_N^\infty t^{\mathrm{Re}(z)-1}e^{-t} dt \\ &\leq \left|\int_0^N t^{z-1}\left[(1-\frac{t}{n})^n - e^{-t}\right] dt\right| + \frac{2\varepsilon}{3}.\end{aligned}$$

The fact that $(1-\frac{t}{n})^n \to e^{-t}$ uniformly on $[0, N]$ now shows that the first term is $< \frac{\varepsilon}{3}$ for sufficiently large n. The required limit follows. □

Why did we care?

We showed by induction and change of variable that

$$\int_0^n t^{z-1}(1-\frac{t}{n})^n \, dt = \frac{n^z \cdot n!}{(z)_{n+1}} = \frac{n^z \cdot n!}{z(z+1)(z+2)\cdots(z+n)}.$$

Also,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\, dt$$

by induction for $\mathrm{Re}(z) > 0$. We conclude that

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z \cdot n!}{z(z+1)\cdots(z+n)}$$

for all $\mathrm{Re}(z) > 0$. Now we mess around.

$$\begin{aligned}
\frac{n^z \cdot n!}{z(z+1)\cdots(z+n)} &= \frac{n^z}{z(\frac{z+1}{1}\cdot\frac{z+2}{2}\cdots\frac{z+n}{n})} \\
&= \frac{n^z}{z(1+z)(1+\frac{z}{2})\cdots(1+\frac{z}{n})} \\
&= \frac{n^z e^{-z-z/2-\cdots-z/n}}{z(1+z)e^{-z}(1+\frac{z}{2})e^{-z/2}\cdots(1+\frac{z}{n})e^{-z/n}} \\
&= \frac{e^{z(\ln(n)-1-1/2-\cdots-1/n)}}{z\prod_{j=1}^{n}(1+\frac{z}{j})e^{-z/j}}.
\end{aligned}$$

Thus we get for $\mathrm{Re}(z) > 0$,

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} \frac{e^z[\ln(z) - H_n]}{z\prod_{j=1}^{n}(1+\frac{z}{j})e^{-z/j}} \\
&= \frac{e^{-\gamma z}}{z\prod_{j=1}^{\infty}(1+\frac{z}{j})e^{-z/j}}
\end{aligned}$$

where

$$\gamma = \lim_{n\to\infty}(H_n - \ln(n)) = \lim_{n\to\infty}(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n)).$$

A better formula is

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{j=1}^{\infty}(1+\frac{z}{j})e^{-z/j}.$$

The right hand side defines an entire function. Thus $\frac{1}{\Gamma} \in H(\mathbb{C}$. It follows that Γ never takes the value 0.

37 The Gamma Function IV (04/22)

Last time we arrived that

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{m=1}^{\infty}(1+\frac{z}{m})e^{-z/m}$$

valid for all $z \in \mathbb{C}$ with simple poles $0, -1, -2, \cdots$. So

$$\begin{aligned}\frac{1}{\Gamma(z)\Gamma(-z)} &= ze^{\gamma z} \prod_{m=1}^{\infty} \left(1 + \frac{z}{m}\right)^{-z/m} \cdot (-z)e^{-\gamma z} \prod_{m=1}^{\infty} \left(1 - \frac{z}{m}\right)^{z/m} \\ &\quad \text{(both products are absolutely convergent, uniformly on compact sets)} \\ &= -z^2 \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right) \\ &= -\frac{z}{\pi} \cdot \pi z \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2}\right) \\ &= -\frac{z}{\pi} \sin(\pi z). \\ &\quad \text{(from an earlier product evaluation)}\end{aligned}$$

So

$$\frac{1}{\Gamma(z)\Gamma(-z)} = -\frac{z \sin(\pi z)}{\pi}$$

for all $z \neq 0$.

$$\frac{1}{\Gamma(z)(-z)\Gamma(-z)} = \frac{\sin(\pi z)}{\pi}$$

for all $z \neq 0$. So

$$\Gamma(z)\Gamma(1-z) = \frac{\sin(\pi z)}{\pi}$$

for all $z \in \mathbb{C} \backslash \mathbb{Z}$ (recall $w\Gamma(w) = \Gamma(w+1)$). This is called the *Reflection Formula* for the Gamma Function.

Taking $z = \frac{1}{2}$ in this formula we get

$$\left(\Gamma(\frac{1}{2})\right)^2 = \frac{\pi}{\sin(\frac{\pi}{2})} = \pi.$$

From the integral definition we know that $\Gamma(x) > 0$ for all $x > 0$. Thus $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Note that this allows us to evaluate $\Gamma(z)$ whenever z is half an integer because $w\Gamma(w) = \Gamma(w+1)$. For example,

$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma(\frac{1}{2}) = \frac{\sqrt{\pi}}{2}.$$

Also, $\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$. This allows us to evaluate $\Gamma(z)$ for all integral z.

Another perspective on $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ is that we also know that

$$\Gamma(z) = \lim_{n\to\infty} \frac{n^z \cdot n!}{(z)_{n+1}} = \lim_{n\to\infty} \frac{n^{z-1} n!}{(z)_n}.$$

Thus

$$\Gamma\left(\frac{1}{2}\right) = \lim_{n\to\infty} \frac{n^{-\frac{1}{2}} \cdot n!}{(\frac{1}{2})_n}.$$

Note

$$\begin{aligned}
\left(\frac{1}{2}\right)_n &= \left(\frac{1}{2}\right) \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{5}{2}\right) \cdots \left(\frac{1}{2} + n - 1\right) \\
&= \left(\frac{1}{2}\right) \cdot \left(\frac{3}{2}\right) \cdot \left(\frac{5}{2}\right) \cdots \left(\frac{2n-1}{2}\right) \\
&= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \\
&= \frac{1}{2^n} \cdot \frac{1 \cdot 2 \cdot 3 \cdots (2n-1) \cdot 2n}{2 \cdot 4 \cdot 6 \cdots (2n)} \\
&= \frac{1}{2^n} \cdot \frac{(2n)!}{2^n \cdot 1 \cdot 2 \cdot 3 \cdots n} \\
&= \frac{(2n)!}{2^{2n} \cdot n!}.
\end{aligned}$$

So

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right) &= \lim_{n\to\infty} \frac{2^{2n} \cdot n^{-1/2} \cdot (n!)^2}{(2n)!} \\
&= \lim_{n\to\infty} \frac{2^{2n}}{n^{1/2} \cdot \binom{2n}{n}} \\
&= \sqrt{\pi}.
\end{aligned}$$

Recall

$$\sum_{j=0}^{2n} \binom{2n}{j} = 2^{2n}.$$

In addition,

$$\begin{aligned}
\sqrt{\pi} &= \lim_{n\to\infty} \frac{n^{-1/2} \cdot 2^n \cdot n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \\
&= \lim_{n\to\infty} \left(\frac{1}{\sqrt{n}} \cdot \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \right).
\end{aligned}$$

(This is basically the square root of Wallis' formula.)

Back to

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{m=1}^{\infty} (1 + \frac{z}{m}) e^{-z/m}.$$

We can take the logarithmic derivative to get

$$\begin{aligned} -\frac{\Gamma'(z)}{\Gamma(z)} &= \frac{1}{z} + \gamma + \sum_{m=1}^{\infty} \left(\frac{\frac{1}{m}}{1+\frac{z}{m}} - \frac{1}{m} \right) \\ &= \frac{1}{z} + \gamma + \sum_{m=1}^{\infty} \left(\frac{1}{z+m} - \frac{1}{m} \right) \end{aligned}$$

for all $z \in \mathbb{C} \backslash \{0, -1, -2, \cdots\}$. We can set $z = 1$ in this formula to get

$$\begin{aligned} -\frac{\Gamma'(1)}{\Gamma(1)} &= \frac{1}{1} + \gamma + \sum_{m=1}^{\infty} \left(\frac{1}{1+m} - \frac{1}{m} \right) \\ &= 1 + \gamma - 1 \quad \text{(telescoping series)} \\ &= \gamma. \end{aligned}$$

We know that $\Gamma(1) = 1$. We get

$$\Gamma'(1) = -\gamma.$$

We also get

$$-\frac{d}{dt} \left(\frac{\Gamma'(z)}{\Gamma(z)} \right) = -\frac{1}{z^2} + \sum_{m=1}^{\infty} \frac{-1}{(z+m)^2}.$$

So

$$\frac{d}{dt} \left(\frac{\Gamma'(z)}{\Gamma(z)} \right) = \frac{1}{z^2} + \sum_{m=1}^{\infty} \frac{1}{(z+m)^2}.$$

For positive real x, we get

$$\frac{d}{dx} \left(\frac{d}{dx} \ln(\Gamma(x)) \right) = \sum_{m=0}^{\infty} \frac{1}{(x+n)^2} > 0.$$

This tells us that $\ln(\Gamma)$ is convex on $(0, \infty)$ (concave up).

$$\begin{aligned} \frac{d^2}{dx^2} (\ln(f)) &= \frac{d}{dx} \left(\frac{f'}{f} \right) \\ &= \frac{f f'' - (f')^2}{f^2} \\ &= \frac{f''}{f} - \left(\frac{f'}{f} \right)^2. \end{aligned}$$

If $\ln(f)$ is concave up, then

$$\frac{f''}{f} - \left(\frac{f'}{f} \right)^2 \geq 0$$

and so $f'' \geq 0$.

38 The Gamma Function V (04/24)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

is called the Eulerian integral of the 2nd kind. The Eulerian integral of the 1st kind is

$$\int_0^1 t^{z-1}(1-t)^{w-1}\, dt.$$

It converges provided that $\mathrm{Re}(z) > 0, \mathrm{Re}(w) > 0$. Euler wrote

$$B(z,w) = \int_0^1 t^{z-1}(1-t)^{w-1}\, dt$$

and $B(\cdot,\cdot)$ is called the *Beta function*.

Assume that $\mathrm{Re}(z), \mathrm{Re}(w)$ are large (≥ 3 would be fine). One recurrence relation for B is

$$\begin{aligned}
B(z,w) &= \int_0^1 t^{z-1}(1-t)^{w-1}\, dt \\
&= \int_0^1 t^{z-1}(1-t)^{w-2}(1-t)\, dt \\
&= \int_0^1 t^{z-1}(1-t)^{w-2}\, dt - \int_0^1 t^{z}(1-t)^{w-2}\, dt \\
&= B(z, w-1) - B(z+1, w-1).
\end{aligned}$$

Another is

$$\begin{aligned}
B(z,w) &= \int_0^1 t^{z-1}(1-t)^{w-1}\, dt \\
&= \int_0^1 d\left(\frac{1}{z}t^z\right)\cdot(1-t)^{w-1} \\
&= \frac{1}{z}t^z\cdot(1-t)^{w-1}\Big|_0^1 - \int_0^1 \frac{1}{z}t^z\cdot(w-1)(1-t)^{w-2}(-1)dt \\
&= \frac{w-1}{z}\int_0^1 t^z(1-t)^{w-2}\, dt \\
&= \frac{w-1}{z}B(z+1, w-1).
\end{aligned}$$

Thus,

$$\begin{aligned}
B(z,w) &= B(z,w-1) - B(z+1,w-1) \\
&= B(z,w-1) - \frac{z}{w-1}B(z,w)
\end{aligned}$$

and so

$$\begin{aligned}
(1+\frac{z}{w-1})B(z,w) &= B(z,w-1),\\
\frac{z+w-1}{w-1}B(z,w) &= B(z,w-1).
\end{aligned}$$

We can write this as

$$B(z,w)=\frac{z+w}{w}B(z,w+1).$$

We can iterate this formula to get

$$\begin{aligned}
B(z,w) &= \frac{z+w}{w}B(z,w+1)\\
&= \frac{z+w}{w}\cdot\frac{z+w+1}{w+1}B(z,w+2)\\
&= \frac{z+w}{w}\cdot\frac{z+w+1}{w+1}\cdot\frac{z+w+2}{w+2}B(z,w+3)\\
&= \cdots\\
&= \frac{(z+w)_n}{(w)_n}B(z,w+n) \quad \text{for } n\geq 1.
\end{aligned}$$

More explicitly,

$$\begin{aligned}
B(z,w) &= \frac{(z+w)_n}{(w)_n}\int_0^1 t^{z-1}(1-t)^n(1-t)^{w-1}\,dt\\
&= \frac{(z+w)_n}{(w)_n}\int_0^n \left(\frac{t}{n}\right)^{z-1}\left(1-\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^{w-1}\frac{dt}{n}\\
&= \frac{(z+w)_n}{(w)_n}\cdot\frac{1}{n^z}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^{w-1}\,dt\\
&= \frac{(z+w)_n}{n^{z+w-1}\cdot n!}\cdot\frac{n^{w-1}n!}{(w)_n}\int_0^n t^{z-1}\left(1-\frac{t}{n}\right)^n\left(1-\frac{t}{n}\right)^{w-1}\,dt\\
&\quad (\text{Note } \left(1-\frac{t}{n}\right)^{w-1} \to 1 \text{ uniformly on } [0,N]).
\end{aligned}$$

We may take the limit as $n\to\infty$ in this expression to obtain

$$\begin{aligned}
B(z,w) &= \frac{1}{\Gamma(z+w)}\cdot\Gamma(w)\int_0^\infty t^{z-1}e^{-t}dt\\
&= \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.
\end{aligned}$$

Theorem 38.1 (Euler-Beta function evaluation)**.** *For* $Re(z), Re(w)>0$*, we have*

$$\int_0^1 t^{z-1}(1-t)^{w-1}\,dt=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

Proof. We verified this formula for $\mathrm{Re}(z), \mathrm{Re}(w) \geq 3$. However, for fixed w where $\mathrm{Re}(w) > 0$, the function $z \mapsto \int_0^1 t^{z-1}(1-t)^{w-1}dt$ is holomorphic in $\mathrm{Re}(z) > 0$ (by theorems of Fubini and Morera). We have

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)} \quad \text{provided } \mathrm{Re}(w) \geq 3, \mathrm{Re}(z) \geq 3$$

and so the Identity Principle extends the formula to $\mathrm{Re}(z) > 0$, $\mathrm{Re}(w) \geq 3$. Repeat to extend the range to $\mathrm{Re}(w) > 0$ as well. $\square$

39 The Gamma Function VI (04/27)

Let $z \in \mathbb{C}$ and consider

$$(z)_n(z+\frac{1}{2})_n = z \cdot (z+1) \cdots (z+n-1) \cdot (z+\frac{1}{2}) \cdot (z+\frac{1}{3}) \cdot (z+\frac{2n-1}{2}).$$

Then

$$\begin{aligned} 2^{2n}(z)_n(z+\frac{1}{2})_n &= (2z) \cdot (2z+2) \cdots (2z+2n-2) \cdot (2z+1) \cdot (2z+3) \cdots (2z+2n-1) \\ &= (2z)(2z+1)(2z+2) \cdots (2z+2n-2)(2z+2n-1) \\ &= (2z)_{2n}. \end{aligned}$$

That is,

$$(2z)^{2n} = 2^{2n} \cdot (z)_n \cdot (z+\frac{1}{2})_n.$$

Based on this, we consider

$$\begin{aligned} \Gamma(2z) &= \lim_{m\to\infty} \frac{m^{2z-1} \cdot m!}{(2z)_m} \\ &= \lim_{n\to\infty} \frac{(2n)^{2z-1} \cdot (2n)!}{(2z)_{2n}} \\ &= \lim_{n\to\infty} \frac{2^{2z-1} \cdot n^{2z-1} \cdot (2n)!}{2^{2n} \cdot (z)_n \cdot (z+\frac{1}{2})_n} \\ &= 2^{2z-1} \lim_{n\to\infty} \frac{n^{z-1} \cdot n!}{(z)_n} \cdot \frac{n^{z-\frac{1}{2}} \cdot n!}{(z+\frac{1}{2})_n} \cdot \frac{n^{\frac{1}{2}} \cdot (2n)!}{2^{2n} \cdot (n!)^2} \\ &= 2^{2z-1} \lim_{n\to\infty} \frac{n^{z-1} \cdot n!}{(z)_n} \cdot \frac{n^{z-\frac{1}{2}} \cdot n!}{(z+\frac{1}{2})_n} \cdot \frac{n^{\frac{1}{2}} \cdot \binom{2n}{n}}{2^{2n}} \\ &= 2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) \cdot \frac{1}{\sqrt{\pi}}. \end{aligned}$$

(The evaluation of the last limit follows from our earlier discussion of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.) So we end up with

$$\Gamma(2z) = \frac{2^{2\pi} - 1}{\sqrt{\pi}} \Gamma(z)\Gamma(z + \frac{1}{2}).$$

This is called (Legendre's) *Duplication Formula.*

Remark 39.1. *There is a similar formula for*

$$\Gamma(z) \cdot \Gamma(z + \frac{1}{l}) \cdot \Gamma(z + \frac{2}{l}) \cdots \Gamma(z + \frac{l-1}{l})$$

for $l \in \mathbb{N} \backslash \{0\}$*. It is called (Gauss') Division Formula.*

Consider $\mathbb{C} \backslash (-\infty, 0]$. This set is open, simply connected, and it doesn't consider 0. This means there is a determination of logarithm of z on the set. We will fix

$$\log(re^{i\theta}) = \ln(r) + i\theta \quad \text{where } r > 0 \text{ and } -\pi < \theta < \pi.$$

Note that this log agrees with ln on $(0, \infty)$. We also know that $\Gamma \in H(\mathbb{C} \backslash (-\infty, 0])$ and Γ is non-vanishing. This means that Γ has a logarithm on $\mathbb{C} \backslash (-\infty, 0]$ also. Any such logarithm might be written as $\log \Gamma$ provided we recall that it need not be the composition of Γ with a determination of logarithm on $\Gamma(\mathbb{C} \backslash (-\infty, 0])$. In particular, you should never write $\log(\Gamma)$. We can pin $\log \Gamma$ by choosing a value at one point. Since $\Gamma(1) = 1$, we can (and do) insist that $(\log \Gamma)(1) = 0$. We know that the function $x \mapsto (\log \Gamma)(x) - \ln(\Gamma(x))$ takes values in $2\pi i\mathbb{Z}$ for all $x \in (0, \infty)$. The function is continuous, and takes the value 0 at 1. Thus it is constantly zero because $(0, \infty)$ is connected. Thus

$$(\log \Gamma)(x) = \ln(\Gamma(x)) \quad \text{for } x > 0.$$

We know that

$$\Gamma(z + 1) = z\Gamma(z)$$

This means that $(\log \Gamma)(z + 1)$ is a logarithm of $\log(z) + (\log \Gamma)(z)$. When x is real and positive, we have

$$\begin{aligned}
(\log \Gamma)(x + 1) &= \ln(\Gamma(x + 1)) \\
&= \ln(x\Gamma(x)) \\
&= \ln(x) + \ln(\Gamma(x)) \\
&= \log(x) + (\log \Gamma)(x)
\end{aligned}$$

and so we have

$$(\log \Gamma)(z + 1) = \log(z) + (\log \Gamma)(z)$$

for $z \in (0, \infty)$. It follows from the Identity Principle that

$$(\log \Gamma)(z + 1) = \log(z) + (\log \Gamma)(z)$$

for all $z \in \mathbb{C} \backslash (-\infty, 0]$.

Lemma 39.2 (Raabe's Formula)**.** *If $z \in \mathbb{C}\backslash(-\infty, 0]$, then*

$$\int_0^1 (\log\Gamma)(z+t)dt = z\log(z) - z + C$$

for some fixed real constant C.

Proof. Let $F : \mathbb{C}\backslash(-\infty, 0] \to \mathbb{C}$ be

$$F(z) = \int_0^1 (\log\Gamma)(z+t)dt.$$

Then F is holomorphic (by Morera's Theorem) and

$$\begin{aligned} F'(z) &= \int_0^1 \frac{\partial}{\partial z}\left[(\log\Gamma)(z+t)\right]\,dt \\ &= \int_0^1 \frac{\partial}{\partial t}\left[(\log\Gamma)(z+t)\right]\,dt \\ &= (\log\Gamma)(z+1) - (\log\Gamma)(z) \\ &= \log z. \end{aligned}$$

We also have

$$(z\log(z) - z)' = \log(z).$$

Thus

$$F(z) = z\log(z) - z + C$$

for some constant C. The constant is real because both sides are real when $z \in (0, \infty)$. $\square$

40 The Gamma Function VII (04/29)

Last time we proved that

$$\int_0^1 (\log\Gamma)(z+t)dt = z\log(z) - z + C$$

for some real constant C and all $z \in \mathbb{C}\backslash(-\infty, 0]$.

Lemma 40.1. *Let $\varphi : [0, 1] \to \mathbb{C}$ be C^2. Then*

$$\left|\int_0^1 \varphi(t)\,dt - \varphi(\frac{1}{2})\right| \leq \frac{1}{12}\max_{[0,1]}|\varphi''|.$$

Proof. It suffices to show that

$$\left|\int_0^1 \varphi(t)\,dt - \varphi(\frac{1}{2})\right| \leq \frac{1}{12}\max_{[0,1]}|\varphi''|$$

when $\varphi : [0,1] \to \mathbb{R}$. To establish this, let

$$M = \max_{[0,1]}|\varphi''|$$

and observe that by Taylor's Formula, we have

$$\varphi(\frac{1}{2}) + \varphi'(\frac{1}{2})(t-\frac{1}{2}) - \frac{M}{2}(t-\frac{1}{2})^2 \leq \varphi(t) \leq \varphi(\frac{1}{2}) + \varphi'(\frac{1}{2})(t-\frac{1}{2}) + \frac{M}{2}(t-\frac{1}{2})^2$$

Now integrate from 0 to 1 throughout the inequality. □

To apply Lemma 40.1 to $\varphi(t) = (\log\Gamma)(z+t)$ to get the size of $(\log\Gamma)(z+\frac{1}{2})$, I need to estimate $(\log\Gamma)''$. Specifically, I need to know a bound for $(\log\Gamma)''(z)$ when $|z|$ is large. This can't be done on $\mathbb{C}\backslash(-\infty, 0]$. We can do this on

$$S_\alpha = \{re^{i\theta} | r > 0, -\alpha < \theta < \alpha\}$$

with $0 < \alpha < \pi$ (see Figure 9). Recall that

$$(\log\Gamma)''(z) = \sum_{m=0}^{\infty}\frac{1}{(z+m)^2}.$$

Let's assume that $m \geq 1$ and $|z| \geq 1$. We claim that there are constants $c_1, c_2 > 0$ such that

$$|z+m| \geq c_1|z|, |z+m| \geq c_2|z|$$

for all $z \in S_\alpha$. (The constants depends on α.) First, what if $z \in \overline{\pi^{\mathrm{R}}}$? So $|z+m| \geq |z|$, $|z+m| \geq m$ for these z (see Figure 10). What if $z \in \overline{\pi^{\mathrm{L}}}$? Assume $\alpha > \frac{\pi}{2}$, w is the closest point to $-m$ in S_α. Then (see Figure 11)

$$\frac{|w+m|}{m} = \sin(\pi-\alpha),$$

$$\frac{|w+m|}{|w|} = \tan(\pi-\alpha).$$

These are fixed non-zero ratios, which are c_1, c_2. For any other point, the inequalities are better.

To estimate $(\log\Gamma)''(z) = \sum_{m=0}^{\infty}\frac{1}{(z+m)^2}$ for $z \in S_\alpha$ we break the sum

$$(\log\Gamma)''(z) = \sum_{m=0}^{k}\frac{1}{(z+m)^2} + \sum_{m=k+1}^{\infty}\frac{1}{(z+m)^2}.$$

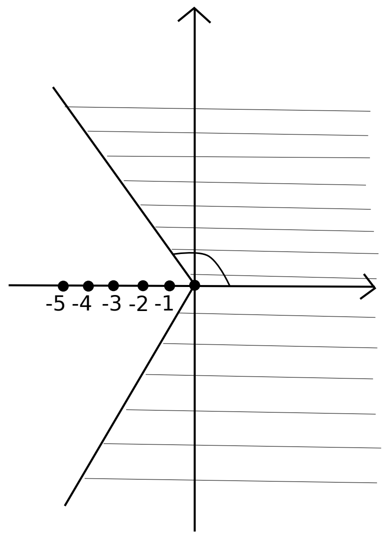


Figure 9: S_α

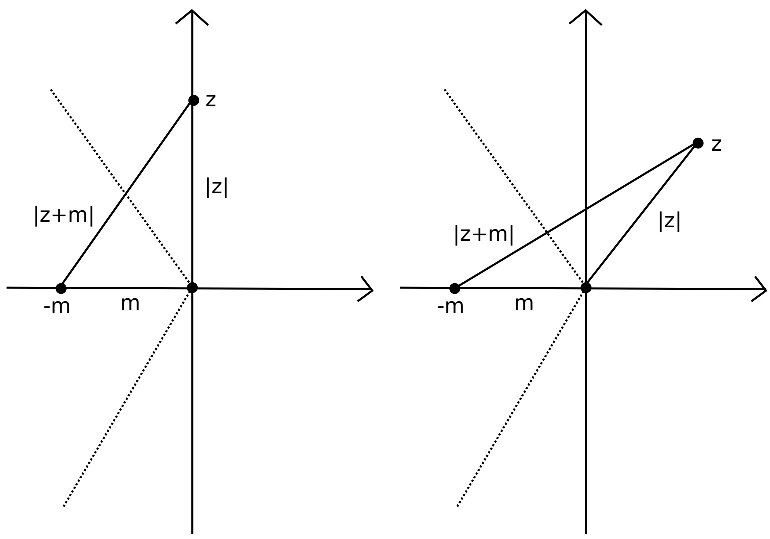


Figure 10: $z \in \overline{\pi^{\mathrm{R}}}$

Thus

$$\begin{aligned}
\left|(\log\Gamma)''(z)\right| &\leq \frac{1}{c_1^2}\sum_{m=0}^{k}\frac{1}{|z|^2} + \frac{1}{c_2^2}\sum_{m=k+1}^{\infty}\frac{1}{m^2} \\
&\leq \frac{k+1}{c_1^2}\cdot\frac{1}{|z|^2} + \frac{1}{c_2^2}\cdot\frac{1}{k} \quad \text{(by integral test)} \\
&= \frac{1}{c_1^2}\cdot\frac{k+1}{|z|^2} + \frac{1}{c_2^2}\cdot\frac{1}{k}.
\end{aligned}$$

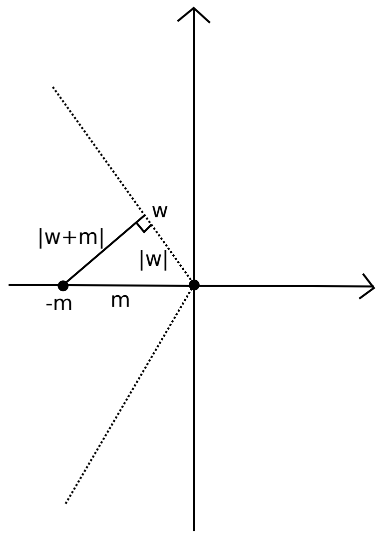


Figure 11: $z \in \overline{\pi^{\mathrm{L}}}$

Now choose k such that

$$|z| \leq k \leq |z| + 1.$$

Then we get

$$\begin{aligned}\left|(\log \Gamma)''(z)\right| &\leq \frac{1}{c_1^2} \cdot \frac{|z|+2}{|z|^2} + \frac{1}{c_2^2} \cdot \frac{1}{|z|} \\ &= \left(\frac{1}{c_1^2} + \frac{1}{c_2^2}\right) \cdot \frac{1}{|z|} + \frac{2}{c_1^2} \cdot \frac{1}{|z|^2} \\ &\leq \frac{3}{c_1^2} + \frac{1}{c_2^2} \quad (\text{since } \frac{1}{|z|^2} < \frac{1}{|z|}).\end{aligned}$$

We can state this as the inequality

$$\left|(\log \Gamma)''(z)\right| \leq \frac{M}{|z|}$$

for all $z \in S_\alpha$ with $|z| \geq 1$. Here M is a constant that depends on α.

Now we assemble everything.

$$\begin{aligned}&\int_0^1 (\log \Gamma)(z+t)\,dt = z\log(z) - z + C, \\ &\left|\int_0^1 (\log \Gamma)(z+t)\,dt - (\log \Gamma)(z - \frac{1}{2})\right| \leq \frac{1}{12} \cdot \max_{t\in[0,1]} \left|(\log \Gamma)''(z+t)\right| \\ &\max_{t\in[0,1]} \left|(\log \Gamma)''(z+t)\right| \leq \frac{k}{|z|}\end{aligned}$$

for $|z| \geq 2$. We arrive at

$$(\log \Gamma)(z + \frac{1}{2}) = z \log z - z + C + E(z)$$

with $|E(z)| \leq \frac{K_\alpha}{|z|}$ for $|z| \geq 2, z \in S_\alpha$. What about C? We find C by $|\Gamma(\frac{1}{2} + it)|$ where $t > 0$. We know $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ (Reflection Formula). Set $z = \frac{1}{2} + it$.

$$\begin{aligned}
\Gamma\left(\frac{1}{2} + it\right)\Gamma\left(\frac{1}{2} - it\right) &= \frac{\pi}{\sin(\frac{\pi}{2} + \pi it})) \\
&= \frac{\pi}{\cos(\pi it)} \\
&= \frac{\pi}{\cosh(\pi t)} \\
&= \frac{2\pi}{e^{\pi t} + e^{-\pi t}}.
\end{aligned}$$

We also know that

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{m=1}^{\infty} (1 + \frac{z}{m}) e^{-z/m}$$

and so

$$\frac{1}{\Gamma(\overline{z})} = \overline{z} e^{\gamma \overline{\overline{z}}} \prod_{m=1}^{\infty} (1 + \frac{\overline{z}}{m}) e^{-\overline{z}/m} = \overline{z e^{\gamma z} \prod_{m=1}^{\infty} (1 + \frac{z}{m}) e^{-z/m}} = \frac{1}{\overline{\Gamma(z)}}.$$

That is, $\overline{\Gamma(z)} = \Gamma(\overline{z})$. Thus

$$\begin{aligned}
\left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 &= \Gamma\left(\frac{1}{2} + it\right)\overline{\Gamma\left(\frac{1}{2} + it\right)} \\
&= \Gamma\left(\frac{1}{2} + it\right)\Gamma\left(\frac{1}{2} - it\right) \\
&= \frac{2\pi}{e^{\pi t} + e^{-\pi t}}.
\end{aligned}$$

So

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}.$$

We can use this to find C exactly.

41 The Gamma Function VIII (05/01)

We know that

$$(\log \Gamma)(z + \frac{1}{2}) = z \log z - z + C + E(z)$$

where C is a real constant, $z \in S_\alpha = \{re^{i\theta} | r > 0, -\alpha < \theta < \alpha\}$, $0 < \alpha < \pi$, $|E_z| \leq \frac{K_\alpha}{|z|}$ for $|z| \geq 2$. We also know from Reflection Formula that

$$\left|\Gamma\left(\frac{1}{2} + it\right)\right|^2 = \frac{2\pi}{e^{\pi t} + e^{-\pi t}}.$$

So

$$\begin{aligned}
\mathrm{Re}\left((\log\Gamma)(z + \frac{1}{2})\right) &= \ln\left|\Gamma\left(\frac{1}{2} + it\right)\right| \\
&= \frac{1}{2}\ln\left(\frac{2\pi}{e^{\pi t} + e^{-\pi t}}\right) \\
&= \frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(e^{\pi t} + e^{-\pi t}) \\
&= \frac{1}{2}\ln(2\pi) - \frac{1}{2}\ln(e^{\pi t}(1 + e^{-2\pi t})) \\
&= \frac{1}{2}\ln(2\pi) - \frac{\pi}{2}t - \frac{1}{2}\ln(1 + e^{-2\pi t}).
\end{aligned}$$

Notice $\frac{1}{2}\ln(1 + e^{-2\pi t}) \to 0$ as $t \to \infty$. On the other hand, (suppose $t > 0$), we have

$$\begin{aligned}
\mathrm{Re}\left((\log\Gamma)(z + \frac{1}{2})\right) &= \mathrm{Re}(it\log(it)) + C + \mathrm{Re}(E(it)) \\
&= \mathrm{Re}\left(it \cdot (\ln(t) + i\frac{\pi}{2})\right) + C + \mathrm{Re}(E(it)) \\
&= -\frac{\pi}{2}t + C + \mathrm{Re}(E(it)).
\end{aligned}$$

Notice $\mathrm{Re}(E(it)) \to 0$ as $t \to \infty$. By comparing the two evaluations of $\mathrm{Re}\left((\log\Gamma)(z + \frac{1}{2})\right)$ as $t \to \infty$, we find that

$$C = \frac{1}{2}\ln(2\pi).$$

Lemma 41.1 (Raabe's Formula)**.**

$$\int_0^1 (\log\Gamma)(z + t)\,dt = z\log(z) - z + \frac{1}{2}\ln(2\pi)$$

for $z \in \mathbb{C}\backslash(-\infty, 0]$.

So

$$(\log\Gamma)(z + \frac{1}{2}) = z\log(z) - z + \frac{1}{2}\ln(2\pi) + E(z)$$

where

$$|E(z)| \leq \frac{K_\alpha}{|z|}.$$

To finish, we put $z+\frac{1}{2}$ in place of z, simplify, exponentiate.

$$
\begin{aligned}
(\log\Gamma)(z+1) &= (z+\frac{1}{2})\log(z+\frac{1}{2}) - z - \frac{1}{2} + \frac{1}{2}\ln(2\pi) + E(z) \\
&= (z+\frac{1}{2})\log(z) + (z+\frac{1}{2})[\log(z+\frac{1}{2}) - \log(z)] - z - \frac{1}{2} + \frac{1}{2}\ln(2\pi) + E(z) \\
&= (z+\frac{1}{2})\log(z) + (z+\frac{1}{2})\log(1+\frac{1}{2z}) - z - \frac{1}{2} + \frac{1}{2}\ln(2\pi) + E(z) \\
&\quad \text{(be careful about arguments)} \\
&= (z+\frac{1}{2})\log(z) + \frac{1}{2} - z - \frac{1}{2} + \frac{1}{2}\ln(2\pi) + \tilde{E}(z) \\
&\quad \text{(because } \log(1+\frac{1}{2z}) = \frac{1}{2z} + \tilde{\tilde{E}}(z) \text{ with } |\tilde{\tilde{E}}(z)| \le \frac{M}{|z|}) \\
&= (z+\frac{1}{2})\log(z) - z + \frac{1}{2}\ln(2\pi) + \tilde{E}(z).
\end{aligned}
$$

Thus

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \quad \text{as } z\to\infty \text{ in } S_\alpha.$$

This means

$$\lim_{z\to\infty, z\in S_\alpha} \frac{\Gamma(z+1)}{\sqrt{2\pi z}\left(\frac{z}{e}\right)^z} = 1.$$

Actually,

$$\left|\frac{\Gamma(z+1)}{\sqrt{2\pi z}\left(\frac{z}{e}\right)^z} - 1\right| \le \frac{L_\alpha}{|z|}.$$

Corollary 41.2.

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} = 1.$$

There is an asymptotic series

$$\Gamma(z+1) \sim \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \left[1 + \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots\right]$$

as $z\to\infty$ in S_α. Here $c_1 = \frac{1}{12}$.

It is possible to write

$$\Gamma(z+1) = \sqrt{2\pi z}\left(\frac{z}{e}\right)^z \left[1 + \frac{d_1}{z} + \frac{d_2}{z(z+1)} + \frac{d_3}{z(z+1)(z+2)} + \cdots\right]$$

that converges in π^{R}. It is a consequence of "Binet's second integral".

There are many other variations.

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