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# Properties of commensurability classes of hyperbolic knot complements

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# Properties of commensurability classes of hyperbolic knot complements

**by**

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To my family.

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# Properties of commensurability classes of hyperbolic knot complements

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This thesis investigates the topology and geometry of hyperbolic knot complements that are commensurable with other knot complements. In chapter 3, we provide an infinite family examples of hyperbolic knot complements commensurable with exactly two other knot complements. In chapter 4, we exhibit an obstruction to knot complements admitting exceptional surgeries in conjunction with hidden symmetries. Finally, in chapter 5, we discuss the role of surfaces embedded in 3-orbifolds as it relates to hidden symmetries.

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# Chapter 1

# Introduction

Two hyperbolic 3-orbifolds are said to be commensurable if they share a common finite sheeted cover. Commensurability forms an equivalence relation on the set of hyperbolic 3-manifolds and 3-orbifolds. In general, it is difficult to identify a commensurability class by a general element. Therefore, one may be interested in finding elements of a commensurability class that appear quite rare. Conjecturally, knot complements in $\mathbb{S}^3$ are rare in a commensurability class as articulated in the following conjecture of Reid and Walsh (see [33, Conj 5.2]):

**Conjecture.** *Let $\mathbb{S}^3 - K$ be a hyperbolic knot complement. Then, there are at most two other knot complements in its commensurability class.*

It is worth mentioning that not all commensurability classes of hyperbolic orbifolds contain a knot complement. There is only one arithmetic knot complement (see [32] and §2.3). Consequently, $\mathbb{H}^3/PSL(2, O_d)$ where $O_d$ is the ring of integers in $\mathbb{Q}(\sqrt{-d})$ (d is square-free), and $d \neq 3$ will not be commensurable with any knot complement.

However the conjecture is non-trivial, in fact, for special classes of knot complements, eg two bridge knots (see [33]) the conjecture is known to hold.

More generally, Boileau, Boyer, Cebanu, and Walsh have shown that the conjecture holds in the case where any knot complement in a given commensurability class does not admit hidden symmetries (see [8] and §2.5). Furthermore, the conjecture can be shown to be sharp in the sense that there are knot complements commensurable with exactly two other knot complements. For example, Fintushel and Stern's $(-2, 3, 7)$ pretzel knot complement famously admits two non-trivial cyclic fillings (see [14]). Each cyclic filling can be exploited in order to construct a knot complement that covers the $(-2, 3, 7)$ knot complement (see [15]). Hence, the $(-2, 3, 7)$ knot complement is commensurable with at least 2 other knot complements. Neumann and Reid showed that the invariant trace field of of the $(-2, 3, 7)$ knot complement is a degree 3 extension of $\mathbb{Q}$ (see [28, §10.2]). This is an obstruction to having hidden symmetries (see [28, Prop 2.7]). Hence, by [33, Cor 5.4], these are the only knot complements in its commensurability class. The ideas of the argument above have been exploited to show that certain knot complements do not admit hidden symmetries (see [21]). The theorem below is proven using a new obstruction to having hidden symmetries using the geometric and algebraic convergence coming from a sequence of Dehn surgeries.

**Theorem 1.** *Let* $n \geq 1$ *and* $(n, 7) = 1$*. For all but at most finitely many pairs of integers* $(n, m)$*, the result of* $(n, m)$ *Dehn surgery on the unknotted cusp of the Berge manifold is a hyperbolic orbifold with exactly three knot complements in its commensurability classes.*

In light of the above theorem and [8], it seems natural to investigate knot complements admitting hidden symmetries. A knot complement admits hidden symmetries if it covers an orbifold with a rigid cusp (see §2.0.2). Also, Boileau, Boyer, Cebanu, and Walsh outlined a program to find commensurable knot complements provided there is an orbifold with two or three finite cyclic fillings ([8, Prop 4.11]). Consequently, one might ask if both finite fillings and hidden symmetries can occur in conjunction. For certain cases, the answer to that question is no, as expressed by the following theorem, which is a less precise version of the actual Theorem 1:

**Theorem 1.** *Let $\mathbb{S}^3 - K$ be a strongly invertible, non-arithmetic, hyperbolic knot complement that admits integral traces and an invariant trace field of class number 1. Then, $\mathbb{S}^3 - K$ does not admit hidden symmetries and a non-trivial exceptional surgery.*

It should be pointed out the theorem can be applied to many knot complements. We remark that all but one hyperbolic knot complement is non-arithmetic (see §2.3 and Thm 2). Additionally, Alan Reid pointed out to the author that all of the hyperbolic knot complements up to 8 crossings have the property that their invariant trace fields are class number 1. If a knot complement admits non-integral traces, then it contains a closed, embedded, essential surface (see [5]). Thus, all small knot complements admit integral traces. Finally, the properties of admitting a non-trivial exceptional surgery and being strongly invertible often occur in conjunction. For instance, all known knot complements exhibiting non-trivial cyclic fillings or special types of toroidal fillings are known to be strongly invertible (see [13] and [17]).

Finally, orbifolds with base space $D^3$ are of particular interest in this work (see Prop 2). If $O$ is an orbifold with a rigid cusp, the branch set of any orbifold is a trivalent graph, $G$ (see §2.0.3 and §2.0.2). The following theorem shows that orbifolds with sufficiently complicated branch set contain a non-trivial two component link in $G$ or they contain an incompressible surface. In particular, if $p\colon \mathbb{S}^3 - K \to O$ is small and $G$ has 8 or more vertices, then $G$ contains at least one two component link.

**Theorem 1.** *Let $O$ be an orbifold with a single rigid cusp and base space $D^3$ that has an isotropy graph with 8 or more vertices. Then either every pair of edge cycles in the isotropy graph of $O$ is linked or $O$ contains a closed, embedded, incompressible 2-orbifold.*

This theorem is also of interest because of the hypotheses of Theorem 1. As mentioned above, if if $O = \mathbb{H}^3/\Gamma$ and $\Gamma$ admits non-integral traces, then $O$ contains a closed embedded essential surface. Therefore, this theorem gives us a window into understanding the complexity of orbifolds with rigid cusps covered by knot complements.

This thesis is organized as follows. The introduction and statements of the main theorems are in chapter 1. Relevant background material can be found in chapter 2. The third chapter contains the proof of Theorem 1. The fourth chapter contains a proof of Theorem 1 and the fifth chapter contains a proof of Theorem 1.

# Chapter 2

# Background

An orientable *hyperbolic 3-manifold* $M$ is a manifold homeomorphic to $\mathbb{H}^3/\Gamma$ where $\Gamma$ is a torsion-free, discrete subgroup $Isom^+(\mathbb{H}^3)$. We will also require that $\mathbb{H}^3/\Gamma$ can be associated to a fundamental domain that is finite volume with respect to the hyperbolic metric. If we identify hyperbolic space with the upper half space model, then $Isom^+(\mathbb{H}^3) \cong PSL(2, \mathbb{C})$.

A hyperbolic 3-manifold can be viewed more generally as a special type of *3-orbifold*, ie a metrizable space equipped with an atlas of compatible local models that are quotients $\mathbb{R}^3$ by finite subgroups of $SO(3, \mathbb{R})$. Since we will be interested in 3-orbifolds that admit a geometric structure, we now recall the definition of a *geometric 3-orbifold* $O$ is a space homeomorphic to $X/\Gamma$, where $X$ is a simply connected Riemannian 3-manifold and $\Gamma$ is a discrete subgroup of $Isom^+(X)$. In particular, since we will be concerned with 3-orbifolds that admit finite volume hyperbolic structures, we recall the following definition for completeness. An orientable *hyperbolic 3-orbifold* is a space homeomorphic to $\mathbb{H}^3/\Gamma$ where $\Gamma$ is discrete subgroup of $Isom^+(\mathbb{H}^3)$ such that $\mathbb{H}^3/\Gamma$ admits a fundamental domain of finite volume under the hyperbolic metric.

Similarly, we also define an orientable *geometric 2-orbifold* $O$ as a space that is homeomorphic to $X/\Gamma$, where $X$ is a simply connected Riemannian 2-manifold and $\Gamma$ is a discrete subgroup of $Isom^+(X)$. In the case where $O = X/\Gamma$, we note that $O$ inherits the Riemannian metric from $X$ and neighborhoods of points in $O$ are isometric to $\mathbb{R}^2$ or $\mathbb{R}^2/\langle\gamma|\gamma^n\rangle$.

Both of the above definitions are due to Thurston and further background can be found in [38, Chapter 13].

### 2.0.1 Finite subgroups of $SO(2, \mathbb{R})$ and $SO(3, \mathbb{R})$

The only finite subgroups of $SO(2, \mathbb{R})$ are cyclic. A finite subgroup $G$ of $SO(2, \mathbb{R})$ preserves the unit disk $D \in \mathbb{R}^2$. If $G$ fixes a point on $\partial D$, then either all of $D$ is fixed or $G$ reverses the orientation on $D$, which is a contradiction. It follows $p : \partial D \to \partial D/G$ is a covering map where $G$ acts freely on $\partial D \cong \mathbb{S}^1$. Hence, $G$ is finite cyclic.

A *disk quotient* is the quotient of a disk by a finite cyclic rotation group of order $n$. We denote this by $D^2(n)$. We also use the notation $\Sigma(n_1, n_2, ..., n_m)$ for a 2-orbifold such that the base space of $\Sigma(n_1, n_2, ..., n_m)$ is a surface $\Sigma$ and for each $n_i$ we remove a closed neighborhood of a point and glue in $D^2(n_i)$ by a homeomorphism of the boundary circles.

We define the *base space* of an orbifold $O$ to be the underlying topological space. For convenience, we use $|O|$ to denote the base space of an orbifold $O$. In dimensions 2 and 3, $|\mathbb{R}^2/\langle\gamma|\gamma^n\rangle| = \mathbb{R}^2$ and $|\mathbb{R}^3/G| = \mathbb{R}^3$. Hence, the base space of a 2-orbifold is a surface and the base space of a 3-orbifold is a 3-manifold. Also, if all neighborhoods of $x \in O$ map to $\mathbb{R}^2/\langle\gamma|\gamma^n\rangle$ or $\mathbb{R}^3/G$, we call x a *cone point* of $O$.

For 2-orbifolds, the Riemann−Hurwitz formula extends the notion of *Euler characteristic* or $\chi(O)$ using the formula:

$$\chi(O) = \chi(|O|) - \sum_i (1 - \frac{1}{r_i})$$

where $r_i$ is the order of the cyclic group fixing the a cone point $x$.

The above definition preserves the property that if $p : S_1 \to S_2$ is a covering map of degree n, then $\chi(S_1) = n \cdot \chi(S_2)$ (see [38, Chapter 13]).

An *elliptic* 2-orbifold is an orientable 2-orbifold that can be covered by $\mathbb{S}^2$. The complete list of orientable 2-orbifolds covered by $\mathbb{S}^2$ is $\mathbb{S}^2$, $\mathbb{S}^2(n,n)$, $\mathbb{S}^2(2,2,n)$, $\mathbb{S}^2(2,3,3)$, $\mathbb{S}^2(2,3,4)$ and $\mathbb{S}^2(2,3,5)$.

Taking the cone over each of these orbifolds produces all of the possibilities for $\mathbb{R}^3/G$ (see Fig 2.1). In particular, $G$ is either trivial, finite cyclic, $D_n$ (a dihedral group of order $2n$), $A_4$, $S_4$, or $A_5$. Finally, if $x \in \mathbb{H}^3/\Gamma$ the *isotropy group* of $x$ is $G \subset \Gamma$ such $g \in G$ if and only if $g(x) = x$.

Using the spherical cosine law (see [35, Lem 3] for application in this context), we can compute the angles between the axes for each type of isotropy group in Figure 2.1. If we consider the three axes fixed by elements of torsion of orders $a$, $b$, and $c$ where the angles between these axes are $\alpha$, $\beta$, $\gamma$ (see in Fig 2.2), then

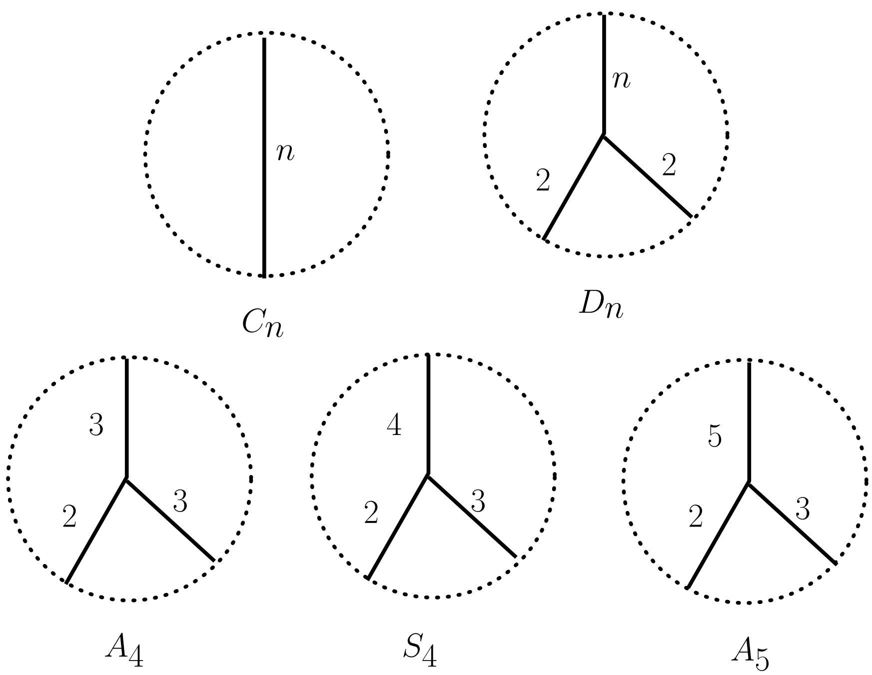



Figure 2.1: The five types of trivalent points that correspond to finite subgroups of $SO(3, \mathbb{R})$

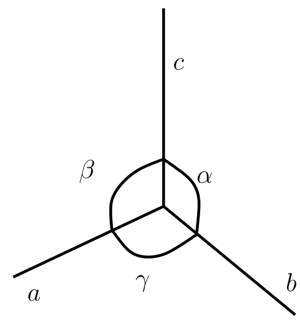



Figure 2.2: The angles between axes of fixed points in finite subgroups of $SO(3,\mathbb{R})$

$$\cos\alpha = \frac{\cos\frac{\pi}{a}+\cos\frac{\pi}{b}\cos\frac{\pi}{c}}{\sin\frac{\pi}{b}\sin\frac{\pi}{c}}$$

$$\cos\beta = \frac{\cos\frac{\pi}{b}+\cos\frac{\pi}{a}\cos\frac{\pi}{c}}{\sin\frac{\pi}{a}\sin\frac{\pi}{c}}$$

$$\cos\gamma = \frac{\cos\frac{\pi}{c}+\cos\frac{\pi}{a}\cos\frac{\pi}{b}}{\sin\frac{\pi}{a}\sin\frac{\pi}{b}}.$$

In particular, if $(a,b,c)$ is $(2,2,n)$, then $(\alpha,\beta,\gamma)$ is $(\frac{\pi}{2},\frac{\pi}{2},\frac{\pi}{n})$. If $(a,b,c) = (2,3,3)$, then $(\alpha,\beta,\gamma)$ is $(\cos^{-1}(\frac{1}{3}),\cos^{-1}(\frac{1}{\sqrt{3}}),\cos^{-1}(\frac{1}{\sqrt{3}}))$. If $(a,b,c) = (2,3,4)$, then $(\alpha,\beta,\gamma) = (\cos^{-1}(\frac{1}{\sqrt{3}}),\frac{\pi}{4},\cos^{-1}(\frac{\sqrt{2}}{\sqrt{3}}))$. If $(a,b,c) = (2,3,5)$, then $(\alpha,\beta,\gamma) = (\cos^{-1}(\frac{\cos(\frac{\pi}{5})}{\sqrt{3}\sin(\frac{\pi}{5})}),\cos^{-1}(\frac{1}{2\sin(\frac{\pi}{5})}),\cos^{-1}(\frac{2\cos(\frac{\pi}{5})}{\sqrt{3}}))$. We note that only the dihedral subgroups have the property that there is an axis perpendicular to all other fixed point axes.

### 2.0.2 Classification of cusps

A finite volume hyperbolic 3-manifold can only have torus cusps. By Selberg's Lemma, any finite volume hyperbolic 3-orbifold $O$ is covered by a 3-manifold, so we see that the cusps of $O$ must be quotients of the torus. In fact, we associate a orientable quotient of the torus to each of the 5 orientation preserving Euclidean wallpaper groups. The five quotients are $T^2$, $\mathbb{S}^2(2,2,2,2)$, $\mathbb{S}^2(2,4,4)$, $\mathbb{S}^2(3,3,3)$ and $\mathbb{S}^2(2,3,6)$. If a cusp $C$ is of the form $S\times[0,\infty)$ where $S$ is one of the last three orbifolds above, then $C$ is said to be a *rigid cusp*. We include presentations for fundamental groups of these orbifolds in the following paragraphs.

The torus admits an involution that fixes four points. The quotient of $T^2$ under this involution is the often called the pillowcase, which we denote by $S^2(2,2,2,2)$. By adding an element of order 2 to $\pi_1(T^2)$, we obtain the following presentation:

$$\pi_1^{orb}(S^2(2,2,2,2)) = \langle a,b,c|aba^{-1}b^{-1}=1, c^2=1, cac^{-1}=a^{-1}, cbc^{-1}=b^{-1}\rangle.$$

In particular, $\pi_1^{orb}(S^2(2,2,2,2)) \cong (\mathbb{Z}\times\mathbb{Z}) \rtimes \mathbb{Z}/2\mathbb{Z}$. The following paragraphs will discuss quotients of tori admitting particular Euclidean structures.

If $T^2$ admits a Euclidean structure such that the $a$ and $b$ are translations of the same length and the angle between the two vectors is $\frac{\pi}{2}$, then $T^2$ will admit a symmetry of order 4. The quotient of $T^2$ by this symmetry is an orbifold with two cone points of order 4 and one cone point of order 2. We use the notation $S^2(2,4,4)$ to denote such a quotient. As the symmetry of order 4 sends $a$ to $b$ and $b$ to $a^{-1}$, we obtain the following presentation:

$$\pi_1^{orb}(S^2(2,4,4)) = \langle a,b,c|aba^{-1}b^{-1}=1, c^4=1, cac^{-1}=b, cbc^{-1}=a^{-1}\rangle.$$

Hence, $\pi_1^{orb}(S^2(2,4,4)) = (\mathbb{Z}\times\mathbb{Z}) \rtimes \mathbb{Z}/4\mathbb{Z}$.

If $T^2$ admits a Euclidean structure such that the $a$ and $b$ are translations of the same length and the angle between the two vectors is $\frac{\pi}{3}$, then the torus will admit a symmetry of order 3 and a symmetry of order 6. If we consider the quotient of $T^2$ under the symmetry of order 3, we will use $S^2(3,3,3)$ to denote the quotient. This symmetry of order 3, sends $a$ to $ba^{-1}$ and $b$ to $a^{-1}$, we obtain the following presentation

$$\pi_1^{orb}(S^2(3,3,3)) = \langle a,b,c | aba^{-1}b^{-1} = 1, c^3 = 1, cac^{-1} = ba^{-1}, cbc^{-1} = a^{-1}\rangle.$$

In particular, this group is isomorphic to $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}/3\mathbb{Z}$.

If we take the quotient of $T^2$ by the symmetry of order 6, we will obtain the orbifold $S^2(2,3,6)$. The symmetry of order 6 sends $a$ to $b$ and $b$ to $ba^{-1}$, therefore we record the following presentation for

$$\pi_1^{orb}(S^2(2,3,6)) = \langle a,b,c | aba^{-1}b^{-1} = 1, c^6 = 1, cac^{-1} = b, cbc^{-1} = ba^{-1}\rangle.$$

Finally, this group is isomorphic to $(\mathbb{Z} \times \mathbb{Z}) \rtimes \mathbb{Z}/6\mathbb{Z}$.

### 2.0.3 Wirtinger presentation

Let $O = X/\Gamma$ be a geometric 3-orbifold where $X$ is the universal cover of $O$. Denote by $F = \{x \in X | \gamma(x) = x, \gamma \in \Gamma - \{1\}\}$. The *isotropy graph*, IG, of an orbifold $O$ is the set of points in $O$ that are in the image $p(F)$, where $p\colon X \to O$ is a covering map. For any closed $O$, the set $IG$ forms a trivalent graph. In the cusped case, it will prove convenient to consider the cusp as a vertex of the graph. Thus, we still have that $IG$ is trivalent except $O$ where

$O$ has $S^2(2,2,2,2)$ cusps. At such points, we see a valence 4 vertex in $IG$ corresponding to each $S^2(2,2,2,2)$ cusp. Finally, if $O$ has a torus cusp $C$ then $IG$ will be disjoint from C.

For a 3-orbifold $O$, we defined $\Gamma$ above as the group of deck transformations on its universal cover $X$. Alternatively, we can construct a presentation for $\Gamma$ using the isotopy graph of $O$ using the Wirtinger presentation. A more general proceedure exists; however, we will restrict to the case where $|O|$ is simply connected. The Wirtinger presentation is derived from a well known algorithm for producing a fundamental group of a knot complement in $\mathbb{S}^3$ from a knot projection. A similar formulation exists for exhibiting the fundamental group of a trivalent graph complement, and a third generalization allows us to construct a presentation for the fundamental group of an orbifold in $\mathbb{S}^3$. We will describe this process below.

Consider an orbifold $O$ with base space $\mathbb{S}^3$ and isotropy graph $G$. We will assume that the projection of $G$ into the plane is in general position, ie there are finitely many edges, crossings, and vertices in the projection and each crossing and vertex is in a neighborhood that does not contain other vertices or crossings. We add a generator for each over strand of an edge in $G$. At each crossing, we introduce the same relation that occurs in the Wirtinger presentation for a knot complement (see Fig 2.3). At each vertex, we introduce the relation $abc = 1$ (see Fig 2.4). Finally, each generator bounds a disk that intersects the graph along an edge in one point. The element of order $n$ associated to this edge induces the relation $a^n = 1$ (see Fig 2.5).

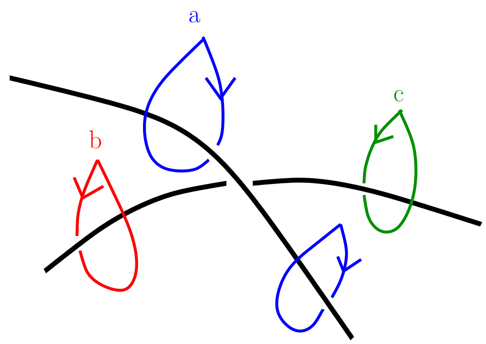



Figure 2.3: The diagram for the relation $aba^{-1} = c$

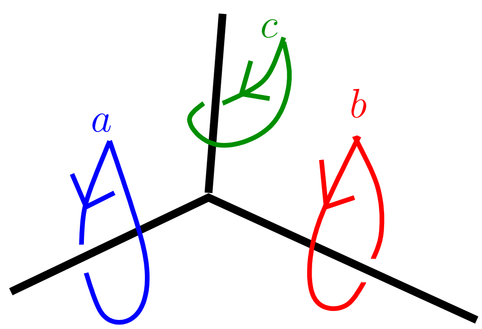



Figure 2.4: The relation $abc = 1$

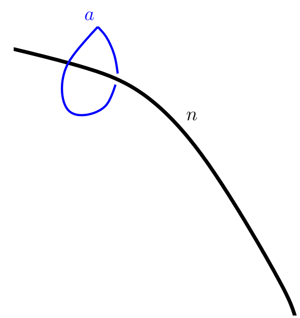



Figure 2.5: Each generator has finite order

If a vertex corresponds to a cusp, then the same algorithm applies. In the case where the base space of $O$ is not simply connected, we must consider how the graph is embedded with respect to $|O|$. However, we will only need to use the Wirtinger presentation for orbifolds with base space $\mathbb{S}^3$ or $D^3$ in the arguments that follow.

### 2.0.4 Incompressible surfaces

The study of incompressible surfaces in 3-manifold topology is of great interest. Recall that $M$ is an *irreducible* 3-manifold if every properly embedded $S^2$ bounds a 3-ball. A compact, orientable surface $\Sigma$ properly embedded in an orientable manifold $M$ is said to be *incompressible* if $\Sigma \ncong S^2$ and every curve $\gamma \in \Sigma$ that bounds a disk in $M$ also bounds a disk in $\Sigma$.

Thurston extended this definition to consider incompressible 2-orbifolds in a closed 3-orbifold (see [38], Chapter 13). To extend this notion to orbifolds, we first introduce a *good* orbifold. An orbifold is said to be good if it has a manifold cover. Not all orbifolds are good, for example a 2-sphere with 1 cone point is not covered by any manifolds. If an orbifold is not good, it is *bad*. We define an *elliptic* 2-orbifold to be an orbifold finitely covered by the 2-sphere. We define an *irreducible* 3-orbifold to be an orbifold that does not contain any bad 2-suborbifolds and where every elliptic 2-orbifold bounds the quotient of a 3-ball.

Let $O$ be a 3-orbifold with orientable base space. If $O'$ is a compact, orientable 2-orbifold and properly embedded in $O$ with $\chi(O') \leq 0$, then $O'$ is an *incompressible suborbifold* of O if for every $O'' \cong S^1$, which bounds a disk quotient in $O - O'$, $O''$ bounds a disk quotient in $O'$. O is *Haken* if it is irreducible and contains an incompressible suborbifold. Finally, we define an *atoroidal* 3-orbifold to be an orbifold that does not contain any incompressible 2-orbifolds that are quotients of Euclidean tori.

The following theorem originally announced by Thurston and published by Boileau, Leeb, and Porti is known as the Orbifold Theorem (see [38, Chap 13], [9, Cor 1.2]).

**Theorem 2.1** (Boileau, Leeb, and Porti, 2005)**.** *Let $Q$ be a compact, connected, orientable, irreducible 3-orbifold with non-empty isotropy graph. If $Q$ is atoroidal, then $Q$ is geometric.*

## 2.1 The trace field and invariant trace field

Mostow-Prasad Rigidity shows that if two finite volume hyperbolic orbifolds, $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$, are homeomorphic then the two groups $\Gamma_1$ and $\Gamma_2$ are conjugate in $Isom(\mathbb{H}^3)$ (see [27], [30], [38]).

If $O = \mathbb{H}^3/\Gamma$ is a finite volume hyperbolic 3-orbifold, a direct consequence of the Mostow-Prasad Rigidity is that the field, $\mathbb{Q}(tr\Gamma) = \mathbb{Q}(tr(\gamma)|\gamma \in \Gamma)$ is a homeomorphism invariant. This field is known to be a finite extension of $\mathbb{Q}$ (see [38, Prop 6.7.4]). Here, we are slightly abusing notation, an element $\gamma \in \Gamma_i$ is of the form

$$\gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \middle| \, ad - bc = 1, a, b, c, d \in \mathbb{C} \right\}.$$

Hence, $tr(\gamma)$ will refer to the trace of one of the elements in the coset that defines $\gamma$.

Let $O = \mathbb{H}^3/\Gamma$, we say $O$ or $\Gamma$ admits an integral representation if for all $\gamma \in \Gamma$, $a, b, c$, and $d$ as defined above are algebraic integers. Also if for all $\gamma$ in $\Gamma$, $tr(\gamma)$ is an algebraic integer, we say $\Gamma$ has *integral traces*. In fact, $\Gamma$ admits an integral representation if and only if it admits integral traces (see [22, Lem 5.2.4]).

As noted in the introduction, two hyperbolic 3–orbifolds, $\mathbb{H}^3/\Gamma_1$ and $\mathbb{H}^3/\Gamma_2$, are said to be *commensurable* if they share a common finite sheeted cover. In terms of their groups, this means that $\exists g \in PSL(2, \mathbb{C})$ such that $\Gamma_1$ and $g\Gamma_2 g^{-1}$ share a subgroup of finite index. In this case, we say that that $\Gamma_1$ and $\Gamma_2$ are *commensurable*.

As noted above, the trace field of a finite volume hyperbolic orbifold $O$ is a homeomorphism invariant for $O$. However, in general, it is not an invariant of the commensurability class. The *invariant trace field* of a hyperbolic 3-orbifold $O = \mathbb{H}^3/\Gamma$ is $k\Gamma = \mathbb{Q}(tr(\gamma^2)|\gamma \in \Gamma)$. This field is known to be both a finite extension of $\mathbb{Q}$ and an invariant of the commensurability class (see [31]). For knot groups, the invariant trace field is equal to the trace field (see [28, Cor 2.3]).

In addition, the property of having integral traces is a commensurability class invariant (see [22, Cor 3.1.4]), so the property of admitting integral representations is a commensurability class invariant.

## 2.2 Class number

Let $k$ be number field and $\mathcal{O}_k$ the ring of integers in $k$. A *fractional ideal* $J$ of $\mathcal{O}_k$ is an $\mathcal{O}_k$-submodule of $k$ such that there exists $\alpha \in \mathcal{O}_k$ with the property that $\alpha J \subset \mathcal{O}_k$.

In the setting of number fields (or more generally Dedekind domains), the set of fractional ideals is well known to form an abelian group under multiplication of ideals. We will call this group the *ideal group of* $k$ and denote it by $I_k$.

We say a fractional ideal $I$ is principal if $I = \langle d \rangle$ for some $d$ in $\mathcal{O}_k$, ie every element of I is of the form $d \cdot r$ for some $r \in \mathcal{O}_k$. Clearly, the set of principal fractional ideals forms a subgroup $P_k$ of $I_k$.

Since $I_k$ is abelian, $P_k$ is a normal subgroup. We define the *class group* as $I_k/P_k$ and the *class number* $h_k$ as the order of the class group. We will focus on the case where the class number is 1. Hence, $\mathcal{O}_k$ is a principal ideal domain, ie all ideals are generated by one element.

## 2.3 Arithmetic 3-manifolds and 3-orbifolds

If $O \cong \mathbb{H}^3/\Gamma_O$ is a finite volume, cusped hyperbolic 3-orbifold $O$, then $\Gamma_O$ is *arithmetic* if $k\Gamma_O = \mathbb{Q}(\sqrt{-d})$ (for d a square-free integer) and $\Gamma_O$ admits integral traces. Also, if $\Gamma_O$ is arithmetic, then $O$ is *arithmetic*. Otherwise, the orbifold is non-arithmetic and the corresponding group is non-arithmetic.

Let $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{-d})$. Then an orbifold of the form $\mathbb{H}^3/PSL(2, \mathcal{O}_d)$ is an arithmetic orbifold, since its invariant trace field is quadratic imaginary and $PSL(2, \mathcal{O}_d)$ has integral traces.

Riley showed that if K is the figure 8 knot then $\pi_1(\mathbb{S}^3 - K) \subset PSL(2, \mathcal{O}_3)$ (see [34]). Reid completely classifies arithmetic knot complements by the following theorem:

**Theorem 2.2** (Reid, 1991)**.** *The figure 8 knot complement is the only arithmetic knot complement.*

Since the invariant trace field and the property of having integral traces are commensurability class invariants, arithmeticity is a commensurability class invariant. Therefore, there are no other knot complements commensurable with the figure 8 knot complement.

Being commensurable with $PSL(2, \mathcal{O}_d)$ (where d is a negative square-free integer) is equivalent to be being arithmetic. Furthermore, if $k = \mathbb{Q}(\sqrt{d})$, It follows from work of Hurwitz that the number of cusps of $\mathbb{H}^3/PSL(2, \mathcal{O}_d)$ is $h_k$ (see [22]). We define a maximal (non-cocompact) arithmetic group $\Gamma$ to be an arithmetic group that is not contained in any other discrete arithmetic group. For $\mathcal{O}_3$, we restate the following special case of [11].

**Theorem 2.3** (Chinburg, Long, and Reid, 2008)**.** *All maximal arithmetic subgroups of $PSL(2, \mathbb{C})$ commensurable with $PSL(2, \mathcal{O}_3)$ have one cusp.*

## 2.4 Tetrahedral orbifolds

We will call a hyperbolic 3-orbifold $\mathbb{H}^3/\Gamma$ *tetrahedral* if $\mathbb{H}^3/\Gamma$ is the orientation double cover of a quotient of a group generated by reflections in the faces of a hyperbolic tetrahedron. We will call a group *tetrahedral* if it is the orbifold fundamental group of a tetrahedral orbifold.

The orbifold fundamental groups of tetrahedral orbifolds are computed by determining the angles between the faces of the tetrahedron. Each angle is a submultiple of $2\pi$. Therefore, we will parametrize tetrahedral groups by six integers. If we have the tetrahedron T in Figure 2.6, as described in [22, Chap 4.7], we denote by

$$\Gamma(m, n, p, r, s, t) = \langle x, y, z | x^m = y^n = z^p = (yz^{-1})^r = (zx^{-1})^s = (xy^{-1})^t = 1 \rangle$$

(the tetrahedral group associated to $T$). Since we are considering our tetrahedral groups to be orientable, $\mathbb{H}^3/\Gamma(m, n, p, r, s, t)$ is homeomorphic to two

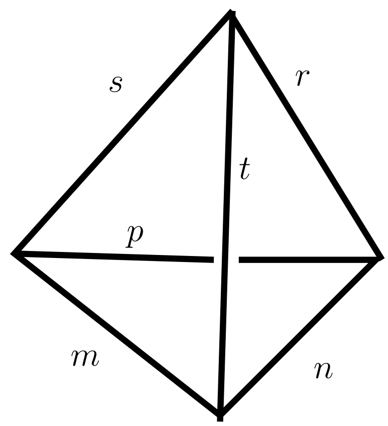



Figure 2.6: The angle between to faces of the tetrahedron above is $\frac{2\pi}{n}$ where n is label on their common edge

copies of the tetrahedron in Figure 2.6 glued together by a set of face pairings.

## 2.5 Hidden symmetries

An important commensurability class invariant of $\Gamma$ or $\mathbb{H}^3/\Gamma$ is the *commensurator*, denoted by

$$Comm^+(\Gamma) = \{g \in PSL(2,\mathbb{C}) | [\Gamma\colon\ \Gamma \cap g\Gamma g^{-1}] < \infty, [g\Gamma g^{-1}\colon\ \Gamma \cap g\Gamma g^{-1}] < \infty\}.$$

Denote by $N^+(\Gamma)$ the normalizer of $\Gamma$ in $PSL(2,\mathbb{C})$. We say that a group $\Gamma$ has *hidden symmetries* if $[Comm^+(\Gamma) : N^+(\Gamma)] > 1$. A hyperbolic orbifold, $Q = \mathbb{H}^3/\Gamma_Q$, has *hidden symmetries* if $\Gamma_Q$ has hidden symmetries.

Margulis showed that if $\Gamma$ is non-arithmetic then $Comm^+(\Gamma)$ is discrete (see [23]). Therefore, if $\Gamma$ is non-arithmetic, we define the *commensurator quotient* to be $\mathbb{H}^3/Comm^+(\Gamma)$. In this case, $\mathbb{H}^3/Comm^+(\Gamma)$ is the minimum volume orbifold in the commensurability class and it is covered by all orbifolds commensurable with $\mathbb{H}^3/\Gamma$.

We defined hidden symmetries in terms of elements of the commensurator above. However, in [28], it has been shown that this algebraic condition is equivalent to an topological condition for knot complements.

**Theorem 2.4** (Neumann and Reid, 1991)**.** *Let* $\mathbb{S}^3 - K \cong \mathbb{H}^3/\Gamma$ *be a non-arithmetic knot complement. Then the following are equivalent:*

1. $\mathbb{S}^3 - K$ *admits hidden symmetries.*
2. $\mathbb{H}^3/Comm^+(\Gamma)$ *has a rigid cusp.*
3. $\mathbb{S}^3 - K$ *non-normally covers some orbifold.*

A key element of the proof of this theorem, which will also be relevant to the following arguments, states that if $p\colon \mathbb{S}^3 - K \to O$ where O does not have a rigid cusp, then $p$ is regular (see [32, Lem 4]). If, as above, we have $\mathbb{S}^3 - K \cong \mathbb{H}^3/\Gamma$, then $N^+(\Gamma)/\Gamma \cong Isom^+(\mathbb{S}^3 - K)$, by Mostow-Prasad Rigidity. Therefore, $N^+(\Gamma)$ is generated by $\Gamma$ and possibly parabolic elements that act on the complement by translation along the longitude and *strong inversions*, ie order 2 elliptic elements that reverse the orientation on the cusp of the knot complement. In $\mathbb{S}^3$, a strong inversion of a knot is realized by an order 2 symmetry of $\mathbb{S}^3$ that sends the knot to itself as an embedded curve in $\mathbb{S}^3$ and fixes two points on the knot. Finally, if a knot complement admits a strong inversion, we say it is *strongly invertible*.

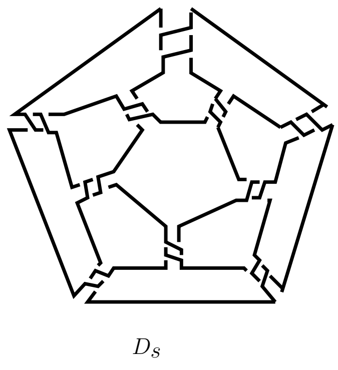



Figure 2.7: The dodecahedral knots

Also, by [28, Prop 2.7], the cusp field of a hyperbolic orbifold is a subfield of the invariant trace field. Thus, if a hyperbolic orbifold has a $S^2(3,3,3)$ or $S^2(2,3,6)$ cusp, $\mathbb{Q}(\sqrt{-3})$ must be a subfield of the orbifold's invariant trace field and if the cusp is $S^2(2,4,4)$, $\mathbb{Q}(i)$ must be a subfield of the orbifold's invariant trace field (see [28, Proof of Thm 5.1(iv)]).

At present, there are only three knot complements known to admit hidden symmetries. The first is the figure 8 knot complement. As noted above, the figure 8 knot complement is arithmetic and so it has a non-discrete commensurator and therefore it has hidden symmetries. In this case, having hidden symmetries can be seen an accident of arithmeticity.

The other examples are the dodecahedral knots of Aitchison and Rubinstein (see [4] and Fig 2.7). These knot complements are constructed by identifying the faces of two regular ideal dodecahedra. Therefore, each knot complement covers a small volume orbifold $\mathbb{H}^3/D$ where $D$ is the group of orientation preserving isometries of T, a tessellation of $\mathbb{H}^3$ by regular ideal dodecahedra. Each dodecahedral knot group is index 120 in $D$. Furthermore, $D$ is tetrahedral, $D = \Gamma(5,2,2,6,2,3)$, and $\mathbb{H}^3/D$ has volume approximately .3430.. (see [28] and [29]). Since $\mathbb{H}^3/D$ has a rigid cusp, the dodecahedral knot complements admit hidden symmetries. Finally, we remark that the dodecahedral knot complements admit integral traces and have an invariant trace field that is class number 1. Therefore, we will be able to apply Theorem 1 to these knot complements.

## 2.6 Adams' classification of orbifolds of small volume

For the remainder of the thesis, we will observe the convention that the volume of a regular ideal tetrahedron in hyperbolic space is denoted by $v_0 \approx 1.01494160$. Also, the volume of an ideal tetrahedron with dihedral angles of $\frac{\pi}{2}$, $\frac{\pi}{4}$, and $\frac{\pi}{4}$ is denoted by $v_1 \approx .9159655941$.

Let $O = \mathbb{H}^3/\Gamma_O$ be a 1-cusped hyperbolic orbifold. We will consider the action of $\Gamma_O$ on upper half space.

Assume that $\Gamma_O$ has an upper triangular parabolic element. Hence, $\infty$ is a parabolic fixed point. Choose another parabolic fixed point $y \in \mathbb{C}$. We will say a Euclidean ball $B_y$ is based at $y$ if the ball is tangent to $\mathbb{C}$ at $y$. We call the set $\{g(B_y) | g \in \Gamma_O\}$ the *horoballs* associated to $\Gamma_O$. If we grow $B_y$ equivariantly with respect to $\Gamma_O$, eventually a ball $B_x$ (based at $x$) will be tangent to the image of $g(B_y)$ where $g \in \Gamma_O$ sends $y$ to $\infty$. In this case, we call the set $\{g(B_y) | g \in \Gamma_O\}$ a set of *maximal horoballs*. For further background on horoballs, see [1]. Given any fundamental domain $D$ for $\mathbb{H}^3/\Gamma_O$, we partition

$D$ into two sets: $D_c$ the set of points in $D$ inside maximal horoballs; and $D_o$ the set of points outside the maximal horoballs. We call the volume $D_c$ the *cusp volume* of $O$.

Meyerhoff used such a sphere packing to classify the hyperbolic 3-orbifold of least volume (see [25]). In particular, Meyerhoff showed that the densest sphere packing implies the that the *cusp density*, ie ratio of cusp volume to total volume, is at most $\frac{2}{v_0\sqrt{3}}$. Using this bound on cusp density and other techniques, Adams improved upon this classification and exhibited the six hyperbolic (orientable) 3-orbifolds of least volume (see [2]). In addition, Adams classified maximal horoball arrangements corresponding to low cusp volume.

The following theorem summarizes Meyerhoff's result and Adams' classification of small cusp volume hyperbolic orbifolds (see [25], [2, Thm 3.2, Cor 4.1, Thm 5.2]).

**Theorem 2.5** (Adams, 1991)**.** *Let $O$ be a 1-cusped hyperbolic 3-orbifold.*

1. *A maximal $S^2(2,3,6)$ cusp in $O$ has volume either $\frac{\sqrt{3}}{24}$, $\frac{\sqrt{3}}{12}$, $\frac{1}{8}$, $\frac{\sqrt{3}(3+\sqrt{5})}{48}$, $\frac{\sqrt{21}}{24}$ or at least $\frac{\sqrt{3}}{8}$.*
2. *A maximal $S^2(3,3,3)$ cusp in $O$ has volume either $\frac{\sqrt{3}}{12}$, $\frac{\sqrt{3}}{6}$, $\frac{1}{4}$, $\frac{\sqrt{3}(3+\sqrt{5})}{24}$, $\frac{\sqrt{21}}{12}$ or at least $\frac{\sqrt{3}}{4}$.*
3. *A maximal $S^2(2,4,4)$ cusp in $O$ has volume either $\frac{1}{8}$, $\frac{\sqrt{2}}{8}$, or at least $\frac{1}{4}$.*

Adams points out that for an orbifold $O$ with a $S^2(2,3,6)$ cusp described in 1) above and each cusp volume in 2), there is a unique orbifold with a $S^2(3,3,3)$ cusp that is the double cover of $O$.

Neumann and Reid provided explicit descriptions of many of the orbifolds corresponding to the cusp volumes in these theorems (see [29]). Many of the orbifolds they describe are arithmetic. More specifically, their notes together with some notes by Adams on the volume of the orbifolds can be summarized in the following proposition (see [2]):

**Proposition 2.6** (Adams 1991, Neumann and Reid 1991)**.** *Let $O$ be a 1-cusped hyperbolic 3-orbifold.*

1. *If $O$ has a maximal $S^2(2,3,6)$ cusp of volume either $\frac{\sqrt{3}}{24}$, $\frac{\sqrt{3}}{12}$, or $\frac{1}{8}$, it is arithmetic. Furthermore, these orbifolds have volumes $\frac{v_0}{12}$, $\frac{v_0}{6}$, and $\frac{5v_0}{24}$, respectively.*

2. *If $O$ has a maximal $S^2(3,3,3)$ cusp of volume either $\frac{\sqrt{3}}{12}$, $\frac{\sqrt{3}}{6}$, or $\frac{1}{4}$, it is arithmetic. Furthermore, these orbifolds have volumes $\frac{v_0}{6}$, $\frac{v_0}{3}$, and $\frac{5v_0}{12}$, respectively.*

3. *If $O$ has a maximal $S^2(2,4,4)$ cusp of volume either $\frac{1}{8}$ or $\frac{\sqrt{2}}{8}$, it is arithmetic. Furthermore, these orbifolds have volumes $\frac{v_1}{6}$ and $\frac{v_1}{4}$, respectively.*

Adams also notes that the orbifold with a $S^2(2,3,6)$ and cusp volume $\frac{\sqrt{3}(3+\sqrt{5})}{48}$ and the orbifold with a $S^2(3,3,3)$ and cusp volume $\frac{\sqrt{3}(3+\sqrt{5})}{24}$ are both tetrahedral. Their volumes are $v \approx .3430..$ and $v' \approx .6860..$, respectively. We remark that the orbifold of volume $v$ is homeomorphic to $\mathbb{H}^3/D$ from the previous section (see [28, §9]).

Following Theorem 3.2 in Adams' paper, there is a conjectural picture of the orbifold with $S^2(2,3,6)$ and cusp volume $\frac{\sqrt{21}}{24}$. Together with the Corollary 4.1 we obtain a conjectural picture of the orbifold with a $S^2(3,3,3)$ cusp and cusp volume $\frac{\sqrt{21}}{12}$. However, a computation of the orbifold fundamental groups shows that these orbifolds turn out to have invariant trace fields that differ from this conjectural picture. The argument is outside of the scope of this background section, so we will delay the computation and discussion of these orbifolds until §4.2.

## 2.7 Dehn filling

Given a 3-orbifold $M$ with a single torus boundary, the result of $\gamma$-*Dehn filling* on $M$ is the space $M \cup_\gamma T = M(\gamma)$ where $T$ is a solid torus, $\partial M$ is a single torus, $\gamma$ is a curve in $\partial M$ and the $\partial T$ and $\partial M$ are identified by a homeomorphism that identifies $\gamma$ with a curve $\mu \in \partial T$ bounding a disk in $T$. If $\gamma$ is a simple curve and $M$ is a manifold then $M(\gamma)$ is a manifold. Here, $\pi_1(M(\gamma)) = \pi_1(M)/\langle\langle\gamma\rangle\rangle_{\pi_1(M)}$ where $\langle\langle\gamma\rangle\rangle_{\pi_1(M)}$ denotes the normal closure of $\gamma$ in $\pi_1(M)$. If $\gamma = \alpha^n$ with $n \geq 2$ ie not a simple curve, then $M(\gamma)$ is an orbifold with base space $M(\alpha)$ and a simple closed curve labeled by $n$ torsion along the core of the torus glued in under the surgery. Here, $\pi_1(M(\gamma)) = \pi_1(M)/\langle\langle\alpha^n\rangle\rangle_{\pi_1(M)}$.

If $M$ has more than one torus boundary, we can define Dehn filling as above by restricting to a particular torus boundary.

For knot exteriors $M$, we can construct canonical coordinates to parametrize the curves we fill along. We define the longitude to be the curve on the boundary torus of $M$ that is homologically trivial. In addition, there is a curve $\mu$ which intersects the longitude in one point and has the property that $M(\mu) = \mathbb{S}^3$. We call such a curve the meridian and the corresponding filling the *trivial filling*. By [18], this filling is unique and so we call any other filling a *non-trivial* filling.

We also define *Dehn surgery* on $M$ if $M$ has a torus cusp. In this case, the cusp is homeomorphic to $T^2 \times [0, \infty)$ and we remove $N = T^2 \times [a, \infty)$ (here $a > 0$) from this cusp. Then we attach a solid torus to the newly created torus boundary via the Dehn filling procedure described above.

### 2.7.1 Exceptional Dehn filling

A Dehn filling is *exceptional* if it does not admit a finite volume hyperbolic structure. It is in general rare for surgery on a hyperbolic 3-manifold or 3-orbifold to be exceptional. The following theorem appears as Theorem 5.8.2 in Thurston's notes (see [38]). A statement is included below for completeness.

**Theorem 2.7** (Thurston's Hyperbolic Dehn Surgery Theorem)**.** *Let $M$ be an $n$-cusped hyperbolic 3-manifold. If we restrict to Dehn filling along a particular cusp, all but at most finitely many Dehn fillings will admit a hyperbolic structure.*

Thus, if $M$ is hyperbolic with many torus boundary components, then Dehn filling along just one boundary component will result in a hyperbolic manifold for all but at most finitely many curves. Dehn fillings.

### 2.7.2 The Six Theorem

We can define length for a parabolic element $\gamma$ for a one-cusp hyperbolic 3-manifold $M$ in the following manner. Considering the upper half space model, conjugate $\gamma$ so it is a parabolic fixing $\infty$. Let $B_\infty$ be the maximal horoball that is based at $\infty$. The length of the translation by $\gamma$ in $g(B_\infty)$ will be the distance between $x$ and $\gamma(x)$ as measured in the boundary of $B_\infty$. We will denote this length by $len(\gamma)$.

Agol and Lackenby (see [3], [19]) both independently proved the "Six Theorem" namely:

**Theorem 2.8** (Agol 2000, Lackenby 2000)**.** *If $M$ is a hyperbolic manifold and $M(\gamma)$ is not hyperbolic, then $len(\gamma) \leq 6$.*

This result improved the bound of the $2\pi$-Theorem attributed to Gromov and Thurston which bounded the length of $\gamma$ by $2\pi$.

As noted in §2.0.2, a rigid cusped orbifold $O$ admit one geometric structure up to Euclidean similarity. If we fix a structure, such that the shortest translation in $\pi_1^{orb}(O)$ is of length 1, the maximal abelian subgroup of $\pi_1^{orb}(O)$ can be identified with a fixed $\mathbb{Z}$-lattice in the complex plane. Specifically, if $O$ is either $S^2(2,3,6)$ or $S^2(3,3,3)$, this lattice $L_\omega$ has 1 and $\omega$ as its basis vectors, where $\omega^2+\omega+1=0$. Note that vectors in $L_\omega$ are of the form $n+m\omega$, $n,m \in \mathbb{Z}$. Vectors of length less than 6 will be of particular interest and so are noted in Figure 2.8(a). Specifically, if a sublattice is generated by 1 and $n+m\omega$ and the lattice has a non-trivial vector $v$ of length less than 6 with $v \neq \pm 1$, then $|m| \leq 6$. If $O$ is $S^2(2,4,4)$, the $\mathbb{Z}$-lattice $L_i$ that corresponds translations in $\pi_1^{orb}(O)$ is of the form $n+m\cdot i$. In this case, we can make the same observation. Namely, if a sublattice is generated by 1 and $n+m\cdot i$ and the lattice has a non-trivial vector $v$ of length less than 6 with $v \neq \pm 1$, then $|m| \leq 6$ (see Fig 2.8(b)).

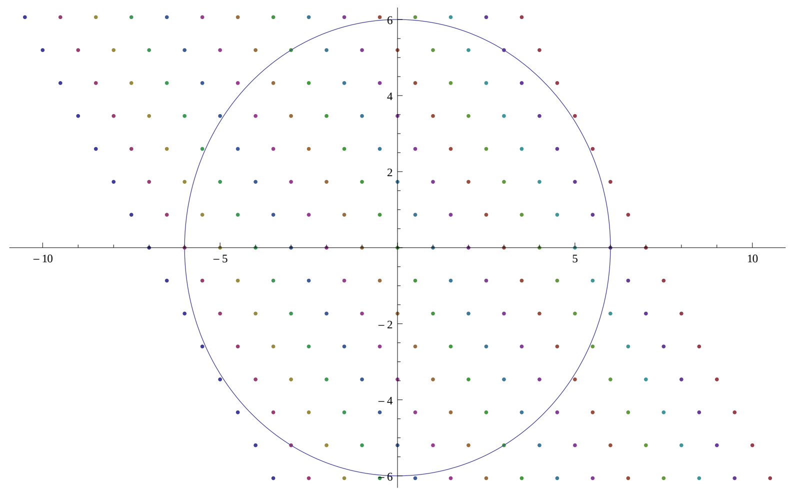
### 2.7.3 Finite cyclic fillings and coverings by knot complements

Given a hyperbolic knot complement $\mathbb{S}^3 - K$, we can bound the number of possible finite cyclic fillings it can admit.

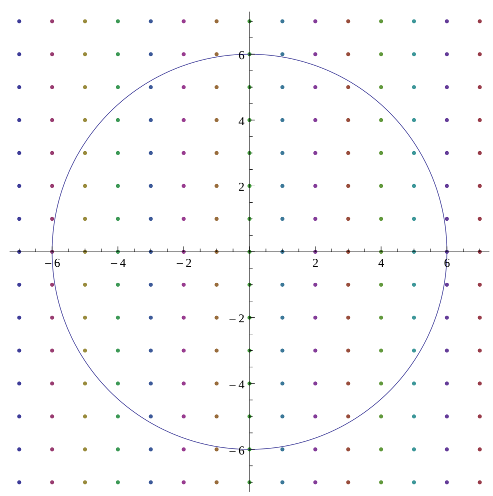
**Theorem 2.9** (Culler, Gordon, Luecke, and Shalen, 1987). *Let $M$ be a hyperbolic 3-manifold with one boundary component. If $\alpha$ is slope in $\partial M$, then $\pi_1(M(\alpha))$ is finite cyclic group for at most three $\alpha$.*

In fact, finite cyclic fillings of hyperbolic knot exteriors correspond to finite coverings by other knot exteriors. More precisely,

**Theorem 2.10** (González-Acuña and Witten, 1992). *If $M$ is a hyperbolic knot exterior, we can construct a covering of $M$ by a knot exterior for each finite cyclic filling of $M$.*




(a) Points in $L_\omega$ that are less than 6 from the origin




(b) Points in $L_i$ less than 6 from the origin

Figure 2.8: Lattice points that are length less than 6 from the origin.

Combining these two results we see that a knot complement can be covered by at most two other knot complements.

The result of Boileau, Boyer, Cebanu and Walsh (see [8]) mentioned earlier generalizes this construction to orbifolds that admit cyclic fillings. They show that a hyperbolic orbifold admits at most three cyclic fillings and that each time a knot complement $\mathbb{S}^3 - K$ covers a torus-cusped orbifold $O$, $O$ admits a cyclic filling. Furthermore they show that if $\mathbb{S}^3 - K$ does not admit hidden symmetries, this completely characterizes the knot complements in a commensurability class. For future reference, we state their results in the following theorem:

**Theorem 2.11** (Boileau, Boyer, Cebanu and Walsh, 2010)**.** *If $\mathbb{S}^3 - K$ is hyperbolic and does not admit hidden symmetries, it is commensurable with at most two other knot complements.*

An important element in the proof the above theorem is a classification of *orbi-lens spaces*, ie orbifolds with finite cyclic fundamental group. This classification is listed below and originally appears as Lemma 3.1 in [8]. Furthermore, this argument generalizes the classification of *lens spaces*, which are manifolds with finite cyclic fundamental group. These manifolds all admit a genus one Heegaard splitting, ie they are the union of two solid tori glued together via a boundary identification. As a result of the Orbifold Theorem (see Thm 1) and the positive solution to the Geometrization conjecture (see [26]), we know that all orbi-lens spaces are quotients of $\mathbb{S}^3$.

**Theorem 2.12** (Boileau, Boyer, Cebanu, Walsh, 2010)**.** *Let $Z$ be a finite cyclic subgroup of $SO(4)$ of order $n$ and fix a generator $\psi$ of $Z$. There are a genus one Heegaard splitting $\mathbb{S}^3 = V_1 \cup V_2$, cores $C_1$,$C_2$ of $V_1$, $V_2$, and integers $a_1$, $a_2 \geq 1$ such that*

1. *both $V_1$ and $V_2$ are $Z$-invariant.*

2. *$\psi$ acts by rotation of order $a_1$ on $C_1$ and order $a_2$ on $C_2$. Moreover, the $Z$-isotropy subgroup of a point in*

   (a) *$\mathbb{S}^3$ $(C_1 \cup C_2)$ is trivial.*

   (b) *$C_1$ is generated by $\psi^{a_1}$ and has order $\bar{a}_2 = n/a_1$,*

   (c) *$C_2$ is generated by $\psi^{a_2}$ and has order $\bar{a}_1 = n/a_2$.*

   *Thus, $n = lcm(a_1, a_2)$, $\bar{a}_1 = a_1/gcd(a_1, a_2)$, $\bar{a}_2 = a_2/gcd(a_1, a_2)$, so $gcd(\bar{a}_1, \bar{a}_2) = 1$.*

3. *$|\mathbb{S}^3/Z|$ is the lens space with fundamental group $Z/gcd(a_1, a_2)$ and genus one Heegaard splitting $(V_1/Z) \cup (V_2/Z)$.*

## 2.8 Notation

We now establish the notation that will be used throughout the thesis.

First,

$$\Gamma_K = \pi_1(\mathbb{S}^3 - K)$$

where $K$ is an embedded knot in $\mathbb{S}^3$.

If $O \cong \mathbb{H}^3/\Gamma$ or $O \cong \mathbb{S}^3/\Gamma$, we say that

$$\Gamma = \Gamma_O.$$

Also, we use $P$ to denote the peripheral subgroup of a 1-cusped orbifold $O$. In arguments where multiple 1-cusped orbifolds are being discussed, we denote the peripheral subgroup of $O$ by $P_O$ for clarity. Similarly, if $P$ is the peripheral subgroup of a knot complement, we will simply use $P_K$.

# Chapter 3

# Commensurability classes containing exactly three hyperbolic knot complements

In this chapter, we exhibit infinitely many commensurability classes containing exactly three hyperbolic knot complements. Each commensurability class can be identified by an orbifold that is the result of surgery on the *Berge manifold*, which is the complement of the two-component link shown in Figure 3.1. In particular, we prove the following theorem:

**Theorem 3.1.** *Let $n \geq 1$ and $(n, 7) = 1$. For all but at most finitely many pairs of integers $(n, m)$, the result of $(n, m)$ Dehn surgery on the unknotted cusp of the Berge manifold is a hyperbolic orbifold with exactly three knot complements in its commensurability classes.*

An important ingredient of the proof is an orbifold coming from $(n, m)$ surgery on the unknotted cusp of the Berge manifold, which we denote by $\beta_{n,m}$. Each of these orbifolds admits three finite cyclic fillings (see §3.1.1). For each one of these fillings we can construct a covering by a knot complement (see Lem 5). In general, classifying manifolds three cyclic fillings remains an open problem, however in the case where $\beta_{n,m}$ has non-empty branch set more is known. We finish the chapter with a complete classification of orbifolds that

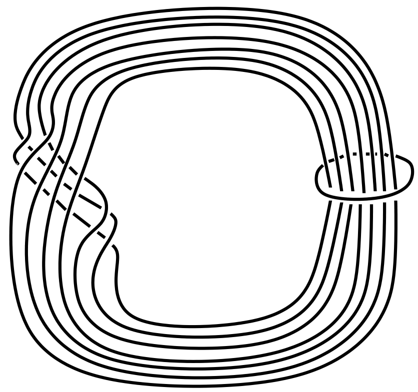


Figure 3.1: The Berge manifold is the complement of this link.

admit three cyclic fillings, are covered by three hyperbolic knot complements, and have non-empty branch set (see Thm 6).

## 3.1 Preliminaries

If a knot complement admits hidden symmetries, it covers an orbifold with a rigid cusp (see Thm 4). In the arguments that follow, we will show that certain manifolds and orbifolds do not cover rigid cusped orbifolds as an obstruction to having hidden symmetries.

First, we investigate some self-covers of Euclidean 2-orbifolds. Like the torus, the orbifolds $S^2(3,3,3)$, $S^2(2,4,4)$, and $S^2(2,3,6)$, can admit self-covers. As noted in the previous chapter, the groups of deck transformations for the above orbifolds are of the form $(\mathbb{Z} \times \mathbb{Z}) \rtimes_f \mathbb{Z}/n\mathbb{Z}$ (where n is 3, 4, or 6 respectively), there exists a degree $m$ self-cover of one of these orbifolds for each index $m$ subgroup of $\mathbb{Z} \times \mathbb{Z}$ preserved by $f$, an outer automorphism of

$\mathbb{Z} \times \mathbb{Z}$. The following proposition shows that there are certain degrees such that $S^2(3,3,3)$ does not admit a self-cover.

**Proposition 3.2.** *Let $p\colon O' \to O$ be a covering map. If $O \cong S^2(3,3,3)$, then degree of $p$ is not 2 or 5.*

*Proof.* Denote by $\Gamma = \pi_1^{orb}(O)$ and note that the abelianization of $\Gamma$ is $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ (see §2.0.2). In particular, $\Gamma$ has no index 2 subgroups.

Assume that $\Gamma' \subset \Gamma$ has index 5. The abelianization of $\Gamma$ excludes the case that $\Gamma' \lhd \Gamma$.

Notice, $\Gamma$ has a torsion-free subgroup $T \cong \mathbb{Z} \times \mathbb{Z}$ of index 3. Also, $[\Gamma : \Gamma' \cdot T][\Gamma' \cdot T : T] = 3$ and since $\Gamma'$ has torsion elements, $[\Gamma : \Gamma' \cdot T] = 1$. Thus, we get the following lattice of subgroups:

$$
\begin{tikzcd}[arrows=dash]
 & \Gamma \arrow[dl, "3"'] \arrow[dr, "5"] & \\
T \arrow[dr, "5"'] & & \Gamma' \arrow[dl, "3"] \\
 & T \bigcap \Gamma' &
\end{tikzcd}
$$

Note that $T \lhd \Gamma$. Hence, we have that $T \bigcap \Gamma' \lhd \Gamma'$ by the Second Isomorphism Theorem. Also, $T \bigcap \Gamma' \lhd T$ since T is abelian. Using $\Gamma = \Gamma' \cdot T$, we obtain that $T \bigcap \Gamma' \lhd \Gamma$. Thus, $\Gamma / T \bigcap \Gamma'$ is isomorphic to a cyclic group of order 15, which is a contradiction to abelianization of $\Gamma$ being $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. This completes the proof. $\square$

### 3.1.1 Special fillings of the Berge manifold

In the explanation below, we assume that the $(n, m)$ surgery on the Berge manifold comes from a standard framing on the cusps of this link complement as in §2.7.

Each $\beta_{n,m}$ admits several surgery slopes of interest. These surgeries can be explained by understanding surgeries on the Berge manifold. First, if we perform Dehn surgery along the $(1, 0)$ slope of the unknotted cusp of the Berge manifold, we will obtain the $(-2, 3, 7)$ pretzel knot (see [14]). Also, if we drill out a solid torus along the unknotted cusp of the manifold we would obtain the one knot in the solid torus (defined up to homeomorphism of the solid torus) that admits three $D^2 \times S^1$ fillings (see [6, Cor 2.9]). Furthermore, if we perform Dehn surgery along the $(1, r)$ slope and then drill along the core of the surgered torus, we would also obtain a knot complement in $D^2 \times S^1$ that admits three $D^2 \times S^1$ surgeries. In fact, by the above mentioned corollary, these are the only knots in solid tori with this property.

The above construction shows that Dehn surgery along a $(1, r)$ slope of the unknotted cusp of the Berge manifold produces knot complements that produce three lens space surgeries. In fact, it is well known that the $(1, 0)$, $(18, 1)$ and $(19, 1)$ surgery slopes on the $(-2, 3, 7)$ pretzel knot admit lens space surgeries (see [14]). By drilling out the unknotted cusp of the Berge manifold, these are also the surgery slopes that produce a solid torus filling. Since the linking number of the knotted cusp and the unknotted cusp is 7, the longitude gets sent to the curve $(49r, 1)$ after $(1, r)$ Dehn surgery on the unknotted cusp while the meridian $(1,0)$ remains fixed (for further background see [36, Sect 9.H]). So the $(1,0)$, $(18,1)$, and $(19,1)$ surgery parameters get sent to $(1,0)$, $(49r+18,1)$, and $(49r+19,1)$ respectively after $(1,r)$ Dehn surgery on the unknotted cusp. Therefore, these fillings produce solid tori in the new coordinates. Furthermore, we can use the surgery parameters to compute the homology of the manifolds resulting from lens space surgeries on the knot complements. In fact, we see that for these knots we obtain $S^3$ and two lens spaces - one with fundamental group of order $|49r+18|$ and another of order $|49r+19|$.

More generally, if we allow Dehn surgery along any $(p,q)$ slope of the unknotted cusp of the Berge manifold where (p,q)=1, and either $(1,0)$, $(18,1)$, or $(19,1)$ Dehn surgery on the knotted cusp, we will also get lens spaces. Again, we see that the $(1,0)$ surgery slope corresponds to a lens space of order $|p|$, $(18,1)$ surgery slope corresponds to a lens space of order $|49q+18p|$, and $(19,1)$ surgery slope corresponds to a lens space of order $|49q+19p|$.

### 3.1.2 The commensurability class of the Berge manifold

As defined in §2.3, $v_0 \approx 1.01494146$ as the volume of the regular ideal tetrahedron. The Berge manifold is comprised of four such tetrahedra and therefore its volume is $4v_0$. Denote by $\Omega_L$ the fundamental group of the Berge manifold.

We saw that the figure 8 knot complement is the only arithmetic knot complement (see Thm 2). However, there are two component link complements that are arithmetic. In particular the Berge manifold is the complement of the link in Figure 3.1 and is arithmetic.

**Proposition 3.3.** *The Berge manifold is arithmetic. Furthermore, the invariant trace field of the Berge manifold is* $\mathbb{Q}(\sqrt{-3})$.

*Proof.* As stated above the Berge manifold is a link complement made up of four regular ideal tetrahedra. Let $\Gamma$ be the group generated by the order 24 group of symmetries of the ideal tetrahedron. Then $\mathbb{H}^3/\Gamma$ is the smallest volume non-orientable hyperbolic 3-orbifold (see [25]). Let $\Gamma^+$ be the orientation preserving subgroup of $\Gamma$. Then, $\Gamma^+ = PGL(2, \mathcal{O}_3)$ which can be identified with the group of isometries that preserve the tessellation of $\mathbb{H}^3$ by regular ideal tetrahedra (see [29]). Since the Berge manifold is comprised of regular ideal tetrahedra, $[\Gamma^+ \colon \Omega_L] = 48$. We make the observation that arithmeticity is preserved by passing to finite index subgroups to complete the first part of the proof. We finish the proof by noting that the invariant trace field of $PGL(2, \mathcal{O}_3)$ is $\mathbb{Q}(\sqrt{-3})$. $\square$

The proof of the following lemma takes advantage of the fact that the Berge manifold has relatively low volume in order to show that it cannot cover an orbifold with a torus cusp and a rigid cusp. It is worth mentioning that $PGL(2, \mathcal{O}_3)$ is the orbifold fundamental group of hyperbolic orbifold with a single $S^2(2, 3, 6)$ cusp. Additionally, the following proof will consider $PSL(2, \mathcal{O}_3)$, which is an index 2 subgroup of $PGL(2, \mathcal{O}_3)$, and so $\mathbb{H}^3/PSL(2, \mathcal{O}_3)$ is a two-fold cover of the orbifold $\mathbb{H}^3/PSL(2, \mathcal{O}_3)$ (see §2.6). Hence for this paper, we consider $PGL(2, \mathcal{O}_3)$ under the image of its representation into $PSL(2, \mathbb{C})$. Also, we will consider all other groups as subgroups of $PSL(2, \mathbb{C})$ where necessary.

**Lemma 3.4.** *The Berge manifold does not cover an orbifold with a torus cusp and a rigid cusp.*

*Proof of 4.* Assume $Q_T$ is an orbifold with a torus cusp and a rigid cusp covered by the Berge manifold. Since the invariant trace field of the Berge manifold is $\mathbb{Q}(\sqrt{-3})$, the rigid cusp of $Q_T$ must be either $S^2(3, 3, 3)$ or $S^2(2, 3, 6)$. In either case, consideration of the unknotted torus cusp of the Berge manifold covering the rigid cusp shows the degree of such a cover is $3k$ for some integer $k \geq 1$. Also, since the Berge manifold is arithmetic and the class number of $\mathbb{Q}(\sqrt{-3})$ is 1 (see Prop 3 and §2.2), it follows from Theorem 3 that any maximal group commensurable with the Berge manifold has exactly one cusp. Thus, there exists a one-cusped orbifold $Q_M$ covered by $Q_T$.

Denote the Berge manifold by B. By consideration of a torus cusp of B covering the rigid cusp of $Q_T$, we see that $p_1 \colon B \to Q_T$ is a covering map of degree $3k$ ($k \geq 1$). Also, by consideration of the torus cusp of $Q_T$ covering the rigid cusp of $Q_M$, $p_2 \colon Q_T \to Q_M$ is a covering map of degree at least 3. If $Q_M$ has a $S^2(3, 3, 3)$ cusp, we use the fact that $vol(Q_M) \leq \frac{4v_0}{9}$ to show that it must be one of the orbifolds classified by Adams (see Thm 5 and Prop 6). However, as pointed out in the comment after the theorem, each of the orbifolds is a double cover of an orbifold with a $S^2(2, 3, 6)$ cusp. Since we may assume $Q_M$ is corresponds to a maximal subgroup of $PSL(2,\mathbb{C})$, and is therefore not a cover of any smaller volume orbifold, we only have to consider to the case that $Q_M$ has a $S^2(2,3,6)$ cusp. In this case, $Q_M$ has a $S^2(2,3,6)$ cusp and the degree of $p_2$ is at least 7. Since the torus cusp of $Q_T$ is at least a six fold-cover of the rigid cusp of $Q_M$ and the rigid cusp of $Q_T$ is at least a one-fold cover of the rigid cusp of $Q_M$. Hence, $vol(Q_M) \leq \frac{2v_0}{9}$ and $Q_M$ is described by Adams (see §2.6). Furthermore, since $vol(B) = 4v_0$, $vol(Q_M)$ is either $\frac{v_0}{6}$ or $\frac{v_0}{12}$. Thus, the covering of $Q_M$ by the Berge manifold is of order 24 or 48, respectively. We will consider these two cases separately by further analyzing the possible covering maps $p_1$ and $p_2$.

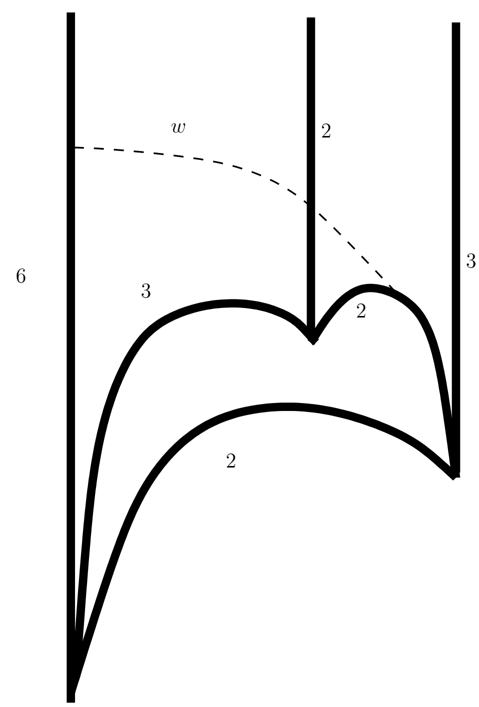



Figure 3.2: The fundamental domain for Γ together with the involution $w$

**Case 1: $Q_M$ has volume $v_0/6$ and the degree of the cover $p_1\colon B \to Q_M$ is 24.**

By noting that $\Gamma_{Q_M}$ has an index 2 subgroup:

$\Gamma = \langle x, y, z | x^2, y^2, z^3, (yz^{-1})^2, (zx^{-1})^6, (xy^{-1})^3 \rangle$ and $\Gamma_{Q_M} = \langle \Gamma, w \rangle$ where $w$ is the order 2 rotation on the fundamental domain of $\Gamma$, we obtain a presentation for $\Gamma_{Q_M}$ (see §2.4 and Fig 3.2).

Thus, we obtain the following presentation:

$$\Gamma_{Q_M} = \langle w, x, y, z | x^2, y^2, z^3, w^2, (yz^{-1})^2, (zx^{-1})^6, (xy^{-1})^3, (wx)^2, wywyz^{-1} \rangle.$$

By the constraints mentioned above, the degree $p_1\colon B \to Q_T$ must be 3 and the degree of $p_2\colon Q_T \to Q_M$ must be 8. However, using GAP, the above group $\Gamma_{Q_M}$ does not have any index 8 subgroups. Thus, there can be no orbifold $Q_T$.

**Case 2: $Q_M$ has volume $v_0/12$ and the degree of the cover $p_2\colon B \to Q_M$ is 48.**

In this case, $Q_M \cong \mathbb{H}^3/PGL(2, \mathcal{O}_3)$ and so we will consider the group picture. Here, $[PGL(2, \mathcal{O}_3) : \Gamma_{Q_T}] = 8$ or 16, since degree of $p_2 \geq 7$ and the degree of $p_1 = 3k$ $(k \geq 1)$.

First, assume $[PGL(2, \mathcal{O}_3) : \Gamma_{Q_T}] = 8$. If $\Gamma_{Q_T} \subset PSL(2, \mathcal{O}_3)$, $[PSL(2, \mathcal{O}_3) : \Gamma_{Q_T}] = 4$. Using GAP, there is a unique index 4 subgroup $G$ of $PSL(2, \mathcal{O}_3)$. However, $G$ has finite abelianization, and therefore cannot be the orbifold group of $Q_T$.

Thus, we may assume that $\Gamma_{Q_T} \not\subset PSL(2, \mathcal{O}_3)$ and deduce that there is a unique subgroup $\Lambda$ of index 2 in $\Gamma_{Q_T}$ such that $\Lambda \subset PSL(2, \mathcal{O}_3)$. By covolume considerations $\Lambda$ has index 8 in $PSL(2, \mathcal{O}_3)$. Also, $\mathbb{H}^3/\Lambda$ has a torus cusp and a $S^2(3, 3, 3)$ cusp. Since $\mathbb{H}^3/PSL(2, \mathcal{O}_3)$ has a $S^2(3, 3, 3)$ cusp, the degree of the covering $p\colon \mathbb{H}^3/\Lambda \to \mathbb{H}^3/PSL(2, \mathcal{O}_3)$ has to be $3l + m$. However, $m \neq 2, 5$ (see Prop 2), a contradiction.

$$\begin{tikzcd}
PGL(2, \mathcal{O}_3) \arrow[d, dash, "2"] \arrow[ddr, dash, "8"] & \\
PSL(2, \mathcal{O}_3) \arrow[ddr, dash, "8"'] & \\
& \Gamma_{Q_T} \arrow[d, dash, "2"] \\
& \Lambda
\end{tikzcd}$$

Now, assume that $[PGL(2, \mathcal{O}_3) : \Gamma_{Q_T}] = 16$. We know that $p_1\colon B \to Q_T$ is of degree 3 and therefore, $Q_T$ has a $S^2(3, 3, 3)$ cusp and a torus cusp. Thus, $\Gamma_{Q_T} \subset PSL(2, \mathcal{O}_3)$ and $[PSL(2, \mathcal{O}_3) : \Gamma_{Q_T}] = 8$, giving us the same contradiction as in above paragraph.

This completes the proof. $\square$

## 3.2 Knot complements covering $\beta_{n,m}$

In this section, we show that for fixed n and m, $\beta_{n,m}$ admits three finite cyclic surgeries. We also show directly it is covered by three knot complements if $n \neq 7$.

**Lemma 3.5.** *The orbifolds $\beta_{n,m}$ are covered by three knot complements. Furthermore, the degrees of the corresponding covering maps are distinct.*

*Proof.* For a fixed $\beta_{n,m}$, let $r = (n, m)$ and consider $\beta_{n,m}$ as the union of the complement of a knot in a solid torus, $T_1$ and a solid torus with core a singular locus of order r, $T_2$ (see Fig 3.3).

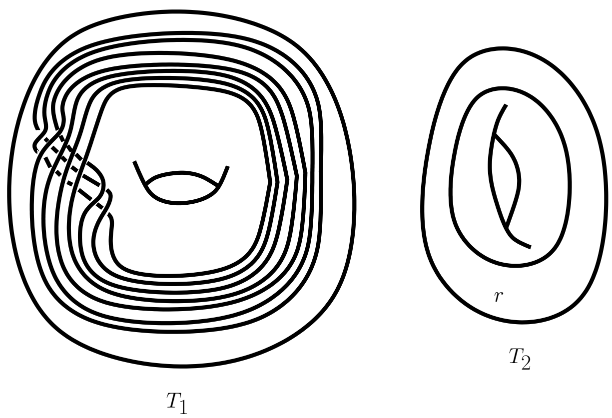



Figure 3.3: The decomposition of a surgered $\beta_{n,m}$ along a torus

By [6, Cor 2.9], $T_1$ admits three Dehn surgeries that result in a solid torus. Thus, $\beta_{n,m}$ admits three Dehn surgeries that are homeomorphic to $T_2$ and a solid torus glued together along their boundaries. Each orbifold $O_j$ ($j \in \{1, 2, 3\}$) resulting from one of these Dehn surgeries has underlying space a lens space with $\Gamma_{O_j}$ finite cyclic.

In fact, $|\Gamma_{O_j}|$ is distinct for each choice of $j$. To see this, we observe, as noted above, that $O_j$ is an orbifold with underlying space a lens space.

Moreover, this underlying space is a lens space with fundamental group of order either $\frac{n}{r}$, $|49\frac{m}{r} + 18\frac{n}{r}|$, or $|49\frac{m}{r} + 19\frac{n}{r}|$ depending on the choice of surgery on $T_1$ (see §3.1). Splitting $O_j$ into a solid torus coming from the Dehn surgery on $T_1$ and $T_2$ the solid torus core a singular curve, we can compute $\Gamma_{O_j}$ using van Kampen's theorem. Thus, the orders of the each fundamental group increase by a factor of r and $|\Gamma_{O_j}|$ is either $n$, $r \cdot |49\frac{m}{r} + 18\frac{n}{r}|$, or $r \cdot |49\frac{m}{r} + 19\frac{n}{r}|$ which take on three distinct values for fixed $n$, $m$, and $r$.

In addition, by the Orbifold Theorem (see [10, Thm 2]) and the above argument that $\Gamma_{O_j}$ is finite cyclic, each $O_j$ has $S^3$ as its universal cover. Denote this covering map by $\phi_j \colon S^3 \to O_j$. We may view $O_j$ as the union of the solid torus coming from the cusp Dehn filling of $\beta_{n,m}$ and the complement of this solid torus, which we denote by $B$. Hence, $\phi_j^{-1}(B)$ is a knot or link exterior in $\mathbb{S}^3$. Since $(n, 7) = 1$ and the singular set of $T_2$ has linking number 7 with the knotted cusp of $\beta_{n,m}$, the boundary of $\phi_j^{-1}(B)$ is connected. Hence, if $(n, 7) = 1$, $\beta_{n,m}$ will be covered by three knot complements in $\mathbb{S}^3$. Also, since the orders of $|\Gamma_{O_j}|$ are distinct, the covering degree of $\phi_j$ will take on a distinct value for each $j$. $\square$

*Remark* 3.2.1. When $n = 1$, the classification of exceptional Dehn surgeries in [24, Table A.1, Rem A.3] shows that $\beta_{n,m}$ is hyperbolic. Hence, $\beta_{1,m}$ is a hyperbolic knot complement that admits three cyclic surgeries. In addition, Eudave-Muñoz realized these these knot complements from tangle filling in [13]. Therefore, the $\beta_{1,m}$ are strongly invertible.

## 3.3 Proof of Theorem 1

In this section, we prove Theorem 1. Also for this section, we consider $\Omega_{n,m}$, $\Delta_{n,m}$, and $\Omega_L$ as subgroups of $PSL(2,\mathbb{C})$.

*Proof of Theorem 1.* Using Lemma 5, each $\beta_{n,m}$ is covered by three knot complements such that the covers are of distinct degrees. Also, Thurston's Hyperbolic Dehn Surgery Theorem (see Thm 7) shows that all but at most finitely many of the $\beta_{n,m}$ are hyperbolic. For the rest of the proof we only consider those $\beta_{n,m}$ that are hyperbolic. Given this condition, each $\beta_{n,m}$ we consider is covered by three distinct knot complements. By Theorem 11, to prove Theorem 1 it suffices to show that the knot complements covering $\beta_{n,m}$ do not have hidden symmetries.

Suppose an infinite number of the hyperbolic knot complements that cover $\beta_{n,m}$ admit hidden symmetries. By the discussion in §2.5, every such knot complement will non-normally cover an orbifold $Q_{n,m}$ with a rigid cusp. Furthermore, on passage to a subset of the $\beta_{n,m}$, we can assume that the orbifolds $Q_{n,m}$ have the same type of rigid cusp, $C$. Let $\Omega_{n,m} = \Gamma_{\beta_{n,m}}$, $\Delta_{n,m} = \Gamma_{Q_{n,m}}$ and let $P \subset PSL(2,\mathbb{C})$ be the peripheral subgroup of $\Delta_{n,m}$. We may assume that each $\Omega_{n,m}$ is conjugated so that $P$ has a fixed representation in $PSL(2,\mathbb{C})$. Since $\beta_{n,m}$ has one cusp, notice that $\Delta_{n,m} = P \cdot \Omega_{n,m}$.

By Thurston's Hyperbolic Dehn Surgery Theorem (see Thm 7), the volumes of the $\beta_{n,m}$ are bounded from above by the volume of the Berge manifold. In addition, the minimum volume of a non-compact oriented hyperbolic

3–orbifold is $\frac{v_0}{12}$ (see §2.6). Hence, $vol(Q_{n,m}) \geq \frac{v_0}{12}$. Thus, we can further subsequence to arrange that $\beta_{n,m}$ covers $Q_{n,m}$, that the $Q_{n,m}$'s have the same type of rigid cusp, and that the covering degree is fixed, say d.

Since $\beta_{n,m}$ is obtained by Dehn surgery on the Berge manifold, the $\Omega_{n,m}$ will converge geometrically to $\Omega_L$, the fundamental group of the Berge manifold (see [38, Thm 5.8.2]). As P is a fixed group in our construction, $\Delta_{n,r}$ also converges algebraically and geometrically to $P \cdot \Omega_L$.

We have the following diagram:

$$\begin{tikzcd}
\Delta_{n,m} \arrow[r, "(n,m)\to\infty"] & P \cdot \Omega_L \\
\Omega_{n,m} \arrow[u, hook, "d"'] \arrow[r, "(n,m)\to\infty"] & \Omega_L \arrow[u, hook, "d"']
\end{tikzcd}$$

Note, $[P \cdot \Omega_L : \Omega_L] = d < \infty$. Let $Q_T = \mathbb{H}^3 / P \cdot \Omega_L$. $Q_T$ has two cusps: a torus cusp, corresponding to the cusp created by geometric convergence from Dehn surgery, and a rigid cusp, corresponding to the cusp with peripheral group $P$.

However by Lemma 4, such a limiting $Q_L$ cannot exist. Hence, at most finitely many of the $\beta_{n,m}$ have hidden symmetries. $\square$

*Remark* 3.3.1. To find explicit examples of hyperbolic knot complements with three knot complements in the commensurability class, we can use the computer program snap (see [16]) to show directly that there are no hidden symmetries. Specifically, for m=0 and n=1,2,3,4,5,6,7, $\beta_{n,m}$ is hyperbolic and snap shows us that $\beta_{n,m}$ has an invariant trace field with real embeddings. These fields cannot contain $\mathbb{Q}(i)$ or $\mathbb{Q}(\sqrt{-3})$ as subfields. Thus, the knot complements covering such $\beta_{n,m}$ do not have hidden symmetries (recall §2.5) and there are exactly three knot complements in each of these commensurability classes.

## 3.4 Remarks

The following theorem provides a partial classification of hyperbolic orbifolds covered by three knot complements. It can be seen as a direct corollary to a result of [8], however we provide a proof for completeness.

**Theorem 3.6.** *Let $O$ be a closed 3–orbifold and let $K$ be a knot in $O$ that is disjoint from the singular locus of $O$. If $O - K$ is:*

1. *hyperbolic,*
2. *covered by 3 knot complements,*
3. *does not admit hidden symmetries, and*
4. *$O$ has non-empty singular locus,*

*then $O - K \cong \beta_{n,m}$ for some pair $(n, m)$.*

*Proof.* Let $\gamma$ be the singular locus of $O$. Denote $|O|$ the underlying space of $O$. By [8, Thm 1.2] and the assumptions above, we know that $|O|$ is a lens space, $\gamma$ is a non-empty subset of the cores of a genus 1 Heegaard splitting of

$|O|$, and if $S^3 - K$ covers $O - K$, then it does so cyclically and corresponds to a finite cyclic filling of $O - K$. Finally, denote $M = O - \gamma - K$.

First assume $\gamma$ has one component. Each of the three knot complements covering $O - K$ corresponds to M admitting a $S^1 \times D^2$ filling along its knotted cusp. Again, we appeal to the fact that there is a unique family of knots in solid tori that admits 3 non-trivial $S^1 \times D^2$ fillings (see [6, Cor 9.1]). Hence, M is obtained by performing $(1, m)$ surgery on the unknotted cusp of the Berge manifold then drilling out the core of the surgered torus. Gluing back in the neighborhood of the fixed point set of $\langle\gamma\rangle$ gives us $\beta_{n,m}$ for some $n, m$.

Now, assume that $\gamma$ has two components $\gamma_1$ and $\gamma_2$. $M = T^2 \times I - K'$, where $K'$ is a knot. Each cyclic filling on $O - K$ corresponds to $M$ admitting a $T^2 \times I$ filling. Hence, Dehn filling along the cusp corresponding to $\gamma_1$ will produce a knot complement in $D^2 \times S^1$ with three $D^2 \times S^1$ fillings.

Denote $l_1$ to be the linking number of $\gamma_1$ and $K'$ and $l_2$ to be the linking number of $\gamma_2$ and $K'$. If $l_1$ is zero, $K'$ would be a knot in a solid torus that is not a 1-braid after $(1, 0)$ on $\gamma_2$ but has two non-trivial $S^1 \times D^2$ fillings. This contradicts [6, Cor 9.1]. Hence, we may assume $l_1 \neq 0$ and $l_2 \neq 0$.

Also, $(1, n)$ surgery on $\gamma_2$ will produce a knot $K''$ in a solid torus that has linking number $l_2 + n \cdot l_1$ with $\gamma_2$. In particular for large enough $n$, $l_2 + n \cdot l_1 \neq 7$. Hence, it cannot be in the family of knots that admit two non-trivial $S^1 \times D^2$ fillings. $\square$

One might hope to relax condition (4) above. However, Brandy Guntel pointed out to the author that the $k(2,2,0,2)$ knot complement (see Fig 3.4) is hyperbolic and admits two non-trivial cyclic surgeries. In fact, John Berge first showed that this knot complement produced two lens space surgeries in unpublished work. Additionally, Mario Eudave-Muñoz gave a construction of the two non-trivial lens space surgeries of this knot complement (see [13]). Let $M(r)$ denote Dehn filling the torus cusp with respect to the slope $r$ on the cusp torus. Furthermore, we will observe the convention that $\frac{1}{0}$ is the meridian and $\frac{0}{1}$ the homologically determined longitude. From the discussion following [13, Prop 5.4], we obtain that k(2,2,0,2)($\frac{32}{1}$) and k(2,2,0,2)($\frac{31}{1}$) are lens space surgeries where the fundamental groups of these lens spaces are of orders 32 and 31 respectively (see [13, Prop 5.3]). By our original discussion in §3.1.1, knot complements obtained by Dehn surgery on the unknotted cusp of the Berge manifold have lens spaces of order $|49r - 18|$ and $|49r - 19|$, none of which can be 32. Hence, the $k(2,2,0,2)$ complement is not one of the $\beta_{n,m}$. However, since the invariant trace field of the $k(2,2,0,2)$ is an odd degree extension of $\mathbb{Q}$, we see that this knot complement does not admit hidden symmetries and the $k(2,2,0,2)$ has exactly three knot complements in its commensurability class (see [33, Cor 5.4]).

Ken Baker pointed out to the author that the $(-2,3,7)$ pretzel knot and the $K(7,5,2,-1)$ knot complement both come from $(p'',1)$ surgery on the unknotted cusp of the Whitehead sister link. Although perhaps known to others (see [24], [7]), this leads to the following conjecture (which is a subcase of the generalized Berge conjecture):

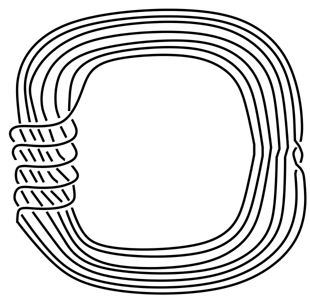


Figure 3.4: The k(2,2,0,2) knot

**Conjecture.** *Let $L(p,q) - K$ be a knot in a lens space that admits 3 cyclic fillings. If $L(p,q) - K$ is covered by a knot complement, then $L(p,q) - K$ is either the result of $(p',q')$ surgery on the Berge manifold $(p',7) = 1$ or the result of $(p'',1)$ surgery on the Whitehead sister link $(p'',5) = 1$.*

The restrictions that $(p',7) = 1$ and $(p'',5) = 1$ are imposed in order to ensure that $L(p,q) - K$ is covered by a knot complement.

As mentioned above $(1,m)$ surgery on the unknotted cusp of the Berge manifold produces Berge knots. It seems natural to ask if any hyperbolic Berge knots can have hidden symmetries. More generally, we might ask if any hyperbolic knot complements can have hidden symmetries and admit non-trivial lens space surgeries. As discussed in §2.5, there are three hyperbolic knot complements known to have hidden symmetries: the complements of the two dodecahedral knots of Aitchison and Rubinstein, and the figure 8 knot complement. Using SnapPea [39], one can see that both dodecahedral knots are amphichiral. Thus, by [12, Cor 4] they cannot admit a lens space surgery.

Additionally, it is well known that the figure 8 knot complement does not admit a lens space surgery (see [37] for example).

# Chapter 4

# Hidden symmetries and exceptional surgeries

In this chapter, we prove Theorem 1, which was stated in the introduction. As seen in the previous chapter, a knot complement $\mathbb{S}^3 - K$ can be commensurable with another knot complement if it admits a finite cyclic surgery. However, the dodecahedral knot complements are commensurable because they cover a common orbifold with a rigid cusp. Motivated by the desire to understand hidden symmetries, a natural question to ask is "Can a knot complement admit a non-trivial exceptional surgery (eg a finite cyclic surgery) and hidden symmetries?" The theorem below provides a negative answer for certain knot complements.

**Theorem 4.1.** *Let $\mathbb{S}^3 - K$ be a non-arithmetic hyperbolic knot complement that covers an orbifold $O$ with a rigid cusp such that $\Gamma_K$ admits integral traces and the invariant trace field of $\mathbb{S}^3 - K$ has class number 1.*

1. *If $\mathbb{S}^3 - K$ covers an orbifold with a $S^2(3,3,3)$ cusp or a $S^2(2,4,4)$ cusp, then $\mathbb{S}^3 - K$ does not admit a non-trivial exceptional filling.*

2. *If $\mathbb{S}^3 - K$ is strongly invertible and covers an orbifold with a $S^2(2,3,6)$ cusp, then $\mathbb{S}^3 - K$ does not admit a non-trivial exceptional filling.*

Note that this theorem is slightly stronger than the theorem stated in the introduction.

This chapter is broken up into five sections. The first section establishes some preliminaries. The second section uses the assumptions of Theorem 1 to construct an integral representation of a knot group for which the meridian is a particular parabolic element. The third section establishes lower bounds for the minimum degree of a covering $p : \mathbb{S}^3 - K \to Q$, where $Q$ is an orbifold with a rigid cusp. The proof of Theorem 1 is contained in section 4. Finally, in section 5, we discuss the hypotheses of Theorem 1 in greater detail and some consequences of the lemmas and propositions of this chapter.

## 4.1 Preliminaries

Let $Q = \mathbb{H}^3/\Gamma_Q$ be a 1-cusped hyperbolic 3-orbifold. Denote by $|\gamma|$ the order of an element in $\Gamma_Q$ and denote by $R = \{\gamma | \gamma \in P_Q, |\gamma| < \infty\}$. We define the *cusp killing homomorphism* to be

$$f\colon\ \Gamma_Q \to \Gamma_Q/\langle\langle R\rangle\rangle_{\Gamma_Q}.$$

We will make use of the following proposition, which was also noticed by M. Kapovich.

**Proposition 4.2.** *Let $\mathbb{S}^3 - K$ be a hyperbolic knot complement. Suppose $\mathbb{S}^3 - K$ covers an orientable orbifold $Q$ with a non-torus cusp. Denote the cusp killing homomorphism by $f$. Then, $f(\Gamma_Q)$ is trivial. Furthermore, $|Q| \cong D^3$ and each component of the isotropy graph of $Q$ is connected to the cusp.*

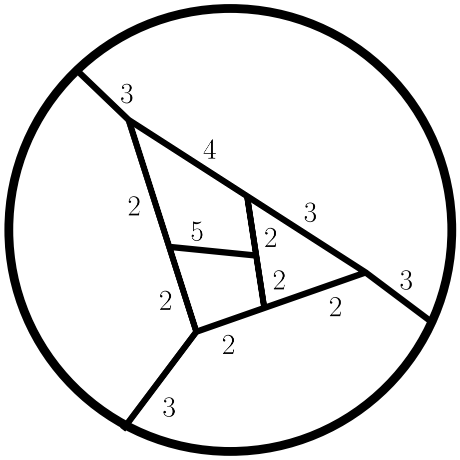
*Proof.* First note that $\Gamma_Q = P_Q \cdot \Gamma_K$.

Since a meridian $\mu$ of $\Gamma_K$ is contained in $P_Q$ and $P_Q$ is generated by torsion elements on the cusp (we recall §2.0.2) killing these torsion elements kills $\langle\langle\mu\rangle\rangle_{\Gamma_Q}$ as well as killing $P_Q$. However, $\Gamma_K = \langle\langle\mu\rangle\rangle_{\Gamma_K}$ and $\langle\langle\mu\rangle\rangle_{\Gamma_K} \subset \langle\langle\mu\rangle\rangle_{\Gamma_Q}$. Hence, the cusp killing homomorphism kills the whole group $\Gamma_Q$.

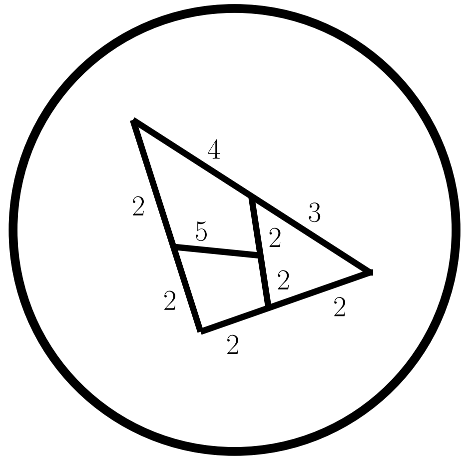
Thus, $|Q|$ is a simply connected space with $S^2$ boundary. Therefore, $|Q| \cong D^3$ by the solution to the Geometrization Conjecture (see [26]).

If there were any pieces of the isotropy graph not connected to the cusp, then there would be elements of finite order that are non-trivial under the cusp killing homomorphism. Hence, the isotropy graph is connected. □

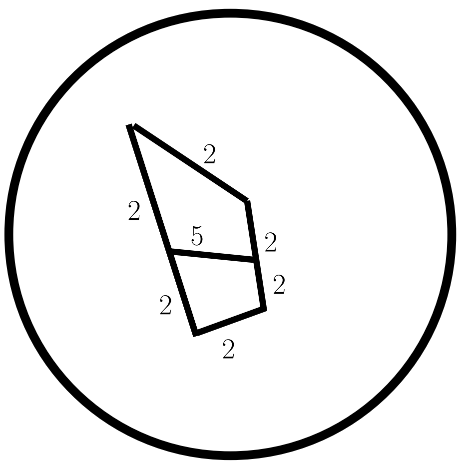
*Remark* 4.1.1. We can also interpret the effects of the cusp killing homomorphism on the isotropy graph of $Q$ when $|Q|$ is simply connected. Viewing the isotopy graph as a weighted graph that generates the fundamental group of $Q$ via the Wirtinger presentation (see §2.0.3), killing elements of torsion on the cusp corresponds to erasing edges of the graph. For each endpoint $x$ of an erased edge corresponding an elliptic element $\gamma$, we introduce the relation $\gamma = 1$ in the local isotropy group at $x$. If $x$ corresponds to a $S^2(2,2,2,2)$ cusp, then the new isotropy group at $x$ is a quotient of the Klein 4 group. If not, then $x$ corresponds to a trivalent vertex of the isotropy graph, say each edge corresponds to torsion elements $\gamma$, $a$, and $b$. Introducing the relation that $\gamma = 1$, to $ab\gamma = 1$ (see Fig 2.4) yields $a = b^{-1}$. In particular $a$ and $b$ have the same order. Therefore, in the image $f(\Gamma_Q)$, $f(a) = f(b^{-1})$ and graphically we



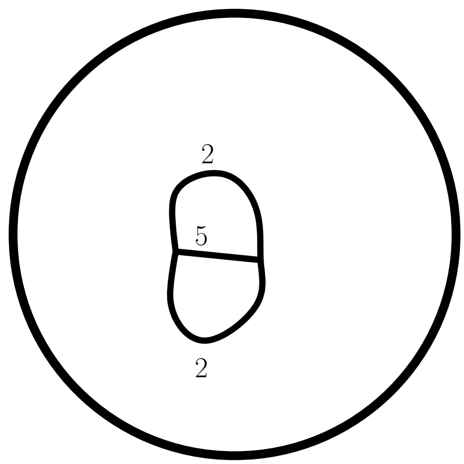

(a) The isotropy graph before reduction




(b) The graph after removing edges corresponding to torsion elements fixing points on the cusp




(c) The graph after resolving the degree 2 vertices




(d) The result of cusp killing is a graph corresponding to a dihedral group

Figure 4.1: A step by step graphical interpretation of the cusp killing homomorphism

can relabel the weights corresponding to $a$ and $b$ with $gcd(|a|,|b|)$ (see §2.0.3 and Fig 4.1). Relabeling the edges could introduce further reductions to the graph, however since the isotropy graph has a finite number of vertices and edges and each edge is weighted by a finite integer, this process will terminate in a finite number of steps.

## 4.2 Representations of knot groups

In this section, we begin by constructing a particularly useful representation of a knot group using the assumptions of Theorem 1. We then show that no orbifold with cusp volume $\frac{\sqrt{21}}{12}$ or $\frac{\sqrt{21}}{24}$ can be covered by a knot complement. We conclude this section by showing that no 1-cusped orbifold admitting integral traces with cusp volume $\frac{\sqrt{3}}{4}$ can be covered by a knot complement and no 1-cusped orbifold admitting integral traces with cusp volume $\frac{\sqrt{3}}{8}$ can be covered by a knot complement.

For the rest of the chapter, we will use the notation that if $k$ is a number field, then $\mathcal{O}_k$ is the ring of integers in $k$. Also, we will use $\mathbb{A}$ to denote the ring of algebraic integers in $\mathbb{C}$.

A subgroup of $PSL(2,\mathbb{C})$ is *elementary* if there is a finite orbit in its action on $\mathbb{H}^3 \cup \partial\mathbb{H}^3$. Otherwise, it is *non-elementary*.

Recall that a quaternion algebra over a field $L$ is a four dimensional $L-$*space* and with basis vectors 1,i,j,k such that 1 is the multiplicative identity, $i^2 = a1$, $j^2 = b1$, $ij = -ji = k$ for some $a, b \in L$, and multiplication is extended linearly so A is an associative algebra over $L$ (for further background see [22, Chapter 2]).

The proof of the following proposition exploits the fact that if $\langle g, h\rangle$ is a non-elementary subgroup of $\Gamma_Q$ for some finite volume hyperbolic 3-orbifold $Q$, then $\mathbb{Q}(\mathrm{tr}\Gamma_Q)[I, g, h, gh]$ is a quaternion algebra over $\mathbb{Q}(\mathrm{tr}\Gamma_Q)$ (see [22, Cor 3.2.3]). Here, we again abuse notation and consider $g, h$ as elements of $SL(2, \mathbb{C})$. Also, denote by

$$A_0\Gamma_Q = \{\sum_{finite} a_i\gamma_i | a_i \in \mathbb{Q}(\mathrm{tr}\Gamma_Q), \gamma_i \in \Gamma_Q\}.$$

The key facts for us is

$$A_0\Gamma_Q = \mathbb{Q}(\mathrm{tr}\Gamma_Q)[I, g, h, gh]$$

(see [22, § 3.2]). Also, by construction $\Gamma_Q \subset A_0\Gamma_Q$, so $\Gamma_Q \subset \mathbb{Q}(\mathrm{tr}\Gamma_Q)[I, g, h, gh]$.

**Proposition 4.3.** *Let $Q_1$, $Q_2$ be 1-cusped hyperbolic 3-orbifolds so that $p\colon\ Q_1 \rightarrow Q_2$. Denote by $k_1 = \mathbb{Q}(tr\Gamma_{Q_1})$ and $k_2 = \mathbb{Q}(tr\Gamma_{Q_2})$. Then, we may conjugate $\Gamma_{Q_1}$ and $\Gamma_{Q_2}$ so that*

1. *$\Gamma_2 \subset PSL(2, k_2)$ and*
2. *$\Gamma_1 \subset PSL(2, k_1) \cap \Gamma_2$.*

*Proof.* Let $g, h \in \Gamma_{Q_1}$ be non-commuting parabolic elements. Then $\langle g, h\rangle$ is non-elementary subgroup of $PSL(2, \mathbb{C})$ and $A_0\Gamma_{Q_1} = \mathbb{Q}(\mathrm{tr}\Gamma_{Q_1})[I, g, h, gh]$ and $A_0\Gamma_{Q_2} = \mathbb{Q}(\mathrm{tr}\Gamma_{Q_2})[I, g, h, gh]$ from above.

Since $g, h$ are non-commuting parabolic elements, we can conjugate $\Gamma_{Q_1}$ by $\gamma \in PSL(2, \mathbb{C})$ such that

$$\gamma g \gamma^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \gamma h \gamma^{-1} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \in \gamma \Gamma_{Q_1} \gamma^{-1}.$$

Also, $\gamma \Gamma_{Q_1} \gamma^{-1} \subset \gamma \Gamma_{Q_2} \gamma^{-1}$. For the remainder of the proof, we will suppress this conjugation for ease of notation.

Note that $\mathbb{Q}(\mathrm{tr}\Gamma_{Q_1})[I, g, h, gh] \subset M_2(k_1)$ and $\mathbb{Q}(\mathrm{tr}\Gamma_{Q_2})[I, g, h, gh] \subset M_2(k_2)$. Furthermore, we have equality for both because both algebras are 4-dimensional over their respective fields. Hence, $A_0\Gamma_{Q_1} = M_2(k_1)$ and $A_0\Gamma_{Q_2} = M_2(k_2)$. Also, $\Gamma_{Q_2} \subset PSL(2, k_2)$ and by this construction $\Gamma_{Q_1} \subset PSL(2, k_1) \cap \Gamma_{Q_2}$. $\square$

We now refine the above proposition to show that if $Q$ admits integral traces and a trace field $k$ that is class number 1, then we can get an integral representation of $\Gamma_Q$ into $PSL(2, \mathcal{O}_k)$.

**Proposition 4.4.** *Let $Q = \mathbb{H}^3/\Gamma_Q$ be a cusped hyperbolic 3-orbifold with integral traces. If $k = \mathbb{Q}(tr\Gamma_Q)$ is class number 1, then $\Gamma_Q$ is conjugate into $PSL(2, \mathcal{O}_k)$.*

*Proof.* We begin by choosing two non-commuting parabolic elements $g, h$ as in Proposition 3. As above, we can conjugate $\Gamma_Q$ so that

$$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } h = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

Thereby identifying $A_0\Gamma_Q = M_2(k)$ and so $\Gamma_Q \subset M_2(k)$.

Define $\mathcal{O}\Gamma_Q = \{\sum \alpha_i\gamma_i | \alpha_i \in \mathcal{O}_k, \gamma_i \in \Gamma_Q\}$ where each sum is finite. Then, $\mathcal{O}\Gamma_Q$ is an order of $M_2(k)$ [22, Lem 2.2.7]. Let $\mathcal{O}$ be a maximal order of $M_2(k)$ containing $\mathcal{O}\Gamma_Q$.

Now $M_2(\mathcal{O}_k)$ is a maximal order (see [22, Example 2.2.6.3]) and since $k$ is class number 1, all maximal orders of $M_2(k)$ are conjugate to $M_2(\mathcal{O}_k)$ (see [22, Cor 2.2.10]). Hence, $\mathcal{O}$ is conjugate to $M_2(\mathcal{O}_k)$, and so we have $\Gamma_Q$ is conjugate to a subgroup of $PSL(2, \mathcal{O}_k)$. $\square$

Finally, we are ready to prove the key lemma of this section.

**Lemma 4.5.** *Let $S^3 - K$ be a hyperbolic knot complement that covers an orbifold $Q$ with a rigid cusp. Let $k$ be the trace field of $\Gamma_K$ and let $L$ be the trace field field of $\Gamma_Q$. If $k$ is class number 1 and $\Gamma_K$ admits integral traces, then*

1. *$\Gamma_Q$ is conjugate into $PSL(2, \mathcal{O}_L)$,*
2. *$\Gamma_K$ is conjugate into $\Gamma_Q \cap PSL(2, \mathcal{O}_k)$,*
3. *A meridian of the above representation of $\Gamma_K$ is of the form $\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.*

*Proof.* First, we may assume that $\Gamma_K$ and $\Gamma_Q$ are conjugate in $PSL(2, \mathbb{C})$ to subgroups of $PSL(2, k)$ and $PSL(2, L)$, respectively (see Prop 3). Furthermore, by the same argument, we may assume $\Gamma_K \subset PSL(2, k) \cap \Gamma_Q$.

Since $\Gamma_K$ admits integral traces, $\Gamma_Q$ admits integral traces as well (see §2.1). Hence, $\mathcal{O}\Gamma_K$ and $\mathcal{O}\Gamma_Q$ are orders in $M_2(k)$ and $M_2(L)$ respectively. As in Proposition 4, $k$ is class number 1, so $\mathcal{O}\Gamma_K$ is conjugate into $M_2(\mathcal{O}_k)$ by $\gamma \in PSL(2,k)$. Conjugating $\mathcal{O}\Gamma_Q$ by $\gamma$, defines a maximal order $\mathcal{O}$ of $A_0\Gamma_Q$ with $\Gamma_Q \subset \mathcal{O}$ and $M_2(\mathcal{O}_k) \subset \mathcal{O}$.

Here, we have shown that $\Gamma_K \subset PSL(2,\mathcal{O}_k)$. to show that $M_2(\mathcal{O}_L) \subset \mathcal{O}$ for some maximal order. Therefore, $\mathcal{O} = M_2(\mathcal{O}_L)$ and $\Gamma_Q \subset PSL(2,\mathcal{O}_L)$.

We observe that $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is an $\mathcal{O}_k$-basis for $M_2(\mathcal{O}_k)$ and an $\mathcal{O}_L$-basis for $M_2(\mathcal{O}_L)$.

Since $\mathcal{O}$ is an order $\mathcal{O} = \sum_{i=1}^{n} a_i e_i$ for $a_i \in \mathcal{O}_L$, $M_2(\mathcal{O}_L) \subset \mathcal{O}$. However, $M_2(\mathcal{O}_L)$ is a maximal order. Therefore $M_2(\mathcal{O}_L) = \mathcal{O}$.

We claim that conjugating $\Gamma_K$ by some element of $PSL(2,\mathcal{O}_k)$ sends a meridian $\mu$ to an upper triangular parabolic element, while preserving the property $\Gamma_Q \subset \mathcal{O}$. This follows from work of Hurwitz, however, we provide an explicit computation. Let $\lambda$ be a parabolic element $\lambda = \begin{pmatrix} 1+\beta\alpha & -\beta^2 \\ \alpha^2 & 1-\beta\alpha \end{pmatrix}$. Then $\lambda$ fixes $\frac{\beta}{\alpha} \in k$. Since $\mathcal{O}_k$ is a principal ideal domain, $\frac{\beta}{\alpha} = \frac{r}{s}$ such that $r, s \in \mathcal{O}_k$ and $pr+qs = 1$ for some $p, q \in \mathcal{O}_k$. Thus, $h_0 = \begin{pmatrix} p & q \\ -s & r \end{pmatrix} \in PSL(2,\mathcal{O}_k)$ and $h_0(\frac{\beta}{\alpha}) = \infty$. Therefore $h_0 \lambda h_0^{-1}$ is an upper triangular parabolic, as desired. $\square$

We further refine the representation above such that the upper right entry of $\mu$ is a unit.

**Lemma 4.6.** *Assuming the representations* $\Gamma_K \subset PSL(2,\mathcal{O}_k)$ *and* $\Gamma_Q \subset$

*$PSL(2, \mathcal{O}_L)$ as in Lemma 5 where $k$ is class number 1, then $x$ is a unit in $\mathcal{O}_k$. Furthermore, by conjugation in $PSL(2, \mathbb{A})$, we may assume that $\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.*

*Proof.* Given these assumptions, we have $\Gamma_K \subset PSL(2, \mathcal{O}_k)$, and $\Gamma_K$ has an upper triangular meridian, $\mu = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

We claim that $x$ is a unit in $\mathcal{O}_k$. For if not, then $x \in I$ for some maximal ideal $I \subset \mathcal{O}_k$. Since $\Gamma_K$ is normally generated by $\mu$, $\Gamma_K$ is trivial under the homomorphism $f\colon \Gamma_K \to PSL(2, \mathcal{O}_k/I)$ induced from reduction mod $I$. Hence, for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$ $a = i + 1$, $d = j + 1$ where $i, j \in I$ and $b, c \in I$.

If we assume that $I = \langle \delta \rangle \subset \mathcal{O}_k$, we conjugate $\Gamma_K$ by $h_1 = \begin{pmatrix} \sqrt{\delta} & 0 \\ 0 & \frac{1}{\sqrt{\delta}} \end{pmatrix}$, then $h_1 \cdot g \cdot h_1^{-1} = \begin{pmatrix} a & \delta \cdot b \\ \frac{c}{\delta} & d \end{pmatrix}$.

Under this conjugation, $h_1 \Gamma_K h_1^{-1}$ remains a subgroup of $PSL(2, \mathcal{O}_k)$ since $c \in I$ and therefore $\frac{c}{\delta} \in \mathcal{O}_k$. Also, the upper right entry of $h_1 \mu h_1^{-1}$ is $x \cdot \delta$ and so $f(h_1 \Gamma_K h_1^{-1})$ is trivial.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_K$. We know from above that $c \in I$. In fact, for some $m$, $c = r \cdot \delta^m$ with $r \notin I$. We may assume that $\gamma$ is an element of $\Gamma_K$ with lowest such $m$. Hence, conjugation of $\gamma$ by $h_1^m$ is a matrix where the lower right entry is not in $I$. However, under this conjugation $\Gamma_K$ remains a subgroup of $PSL(2, \mathcal{O}_k)$. Thus, the reduction homomorphism $f$ is still well defined and $h_1^m \gamma h_1^{-m}$ would be non-trivial under $f$. However, $h_1^m \mu h_1^{-m}$ remains trivial, which is a contradiction. Hence, $x$ is a unit.

Finally, since $x$ is a unit, $\sqrt{x}$ and $\frac{1}{\sqrt{x}}$ are also units (possibly in $\mathcal{O}_{k'}$ where $[k' \colon k] = 2$). Therefore, if we conjugate $\Gamma_Q$ by $h_2 = \begin{pmatrix} \frac{1}{\sqrt{x}} & 0 \\ 0 & \sqrt{x} \end{pmatrix}$, we still have integral representations for $\Gamma_K$ and $\Gamma_Q$ where $\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, as desired. $\square$

Note that this final conjugation may move $\Gamma_K$ out of $PSL(2, \mathcal{O}_k)$ (and $\Gamma_Q$ out of $PSL(2, \mathcal{O}_L)$), since $k(\sqrt{x})$ can be a quadratic extension of $k$. In this case, $k(\sqrt{x})$ could have class number strictly bigger than 1. However, as noted above, we still have that $\Gamma_K$ and $\Gamma_Q$ are an integral representations with the above meridian $\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Also, we call attention to the fact that because of the form of $\mu$, we can identify the possible meridians of knot complements in $P_Q$. If the cusp of $\mathbb{H}^3/\Gamma_Q$ is $S^2(3,3,3) \times [0,\infty)$, the integral representation for $\Gamma_Q$ defined above is of the form where $P_Q = \left\langle \begin{pmatrix} \omega & r \\ 0 & \omega^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ where $\omega^2 + \omega + 1 = 0$ and $r$ an algebraic integer. We may conjugate $\Gamma_Q$ by $\begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$. Thus, we may assume that $\Gamma_Q \subset PSL(2, \mathbb{A})$ and $P_Q = \left\langle \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$

Therefore, all parabolic elements in $P_Q$ have $y = n + m\omega$ in their upper right entries. Hence, if $y$ is a unit, $y = (-\omega)^i$ where $i \in \{0, .., 5\}$. A nearly identical argument shows that if the cusp of $\mathbb{H}^3/\Gamma_Q$ is $S^2(2,3,6) \times [0,\infty)$, $y$ is of the same form. However, if the cusp of $\mathbb{H}^3/\Gamma_Q$ is $S^2(2,4,4) \times [0,\infty)$, then $P_Q = \left\langle \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right\rangle$ where $\ell = e^{\frac{i\pi}{4}}$. Therefore, all parabolic elements in $P_Q$ have $y = n + m \cdot i$ in their upper right entries. And if $y$ is a unit, then $y = \pm 1$ or $y = \pm i$.

Lemmas 8 and 10 use similar ideas to those above in order to show that certain orbifolds described in §2.6 cannot be covered by a knot complement. We begin with a proposition that also exhibits a correspondence between units in $\mathcal{O}_k$ and meridians in $\Gamma_K$.

**Proposition 4.7.** *Let $k$ be a number field and $\mathcal{O}_k$ the ring of integers in $k$. If $\Gamma_K$ be a knot group such that $\Gamma_K \subset PSL(2, \mathcal{O}_k)$ and*

$$\Gamma_K = \left\langle \mu_1 = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \mu_2 = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}, \mu_3, \ldots \mu_n \right\rangle$$

*where $\mu_i$ are meridians, then $x$ and $y$ are units in $\mathcal{O}_k$.*

*Proof.* Assume that $x$ is not a unit in $\mathcal{O}_k$. Then there is a prime ideal $I \subset \mathcal{O}_k$ such that $x \in I$. Just as in the proof of Lemma 6, we may define a homomorphism $f : PSL(2, \mathcal{O}_k) \to PSL(2, \mathcal{O}_k/I)$. Under $f$, $\mu_1$ is trivial and since $\Gamma_K$ is normally generated by $\mu_1$, $f(\Gamma_K)$ is trivial. But there is an element $g \in \Gamma$ such that $g\mu_1 g^{-1} = \mu_2$. Such an element has a 0 in the 1,1 entry and hence, $\Gamma_K$ cannot be trivial under $f$. This is a contradiction and so $x$ is a unit.

The same argument carries through if we assume $y$ is not a unit. $\square$

We use this observation to prove the following lemma. Also, in the following proofs, we will identify $\mathbb{H}^3$ with $\{z + tj \in \mathbb{H} | z \in \mathbb{C}, t > 0, j^2 = -1\}$ (upper-half space) and $\partial\mathbb{H}^3$ with $\mathbb{C} \cup \{\infty\}$. We also denote by $B_x$ the horoball that is tangent to $\partial\mathbb{H}^3$ at $x$.

**Lemma 4.8.** 1. *Any orbifold with a $S^2(3,3,3)$ cusp and cusp volume $\frac{\sqrt{21}}{12}$ cannot be covered by a knot complement.*

*2. Any orbifold with a $S^2(2,3,6)$ cusp and cusp volume $\frac{\sqrt{21}}{24}$ cannot be covered by a knot complement.*

*Proof.* First, we appeal to Adams' characterization of orbifolds of small cusp volume (see §2.6) to reduce to case (2) as any orbifold with a $S^2(3,3,3)$ cusp and cusp volume $\frac{\sqrt{21}}{12}$ covers an orbifold with $S^2(2,3,6)$ cusp and cusp volume $\frac{\sqrt{21}}{24}$.

Hence, let $Q$ be a 3-orbifold with a $S^2(2,3,6)$ cusp and cusp volume $\frac{\sqrt{21}}{24}$. A diagram of the horoballs associated to $Q$ first appeared in Adams' paper (see [2, Fig 5]), however we include it here as Figure 4.2 for the sake of completeness. Furthermore, following the discussion of this horoball diagram in [2], we use the following notation: $O = 0$, $D = \sqrt[4]{7}$, $X = \frac{5+i\sqrt{3}}{2\sqrt{7}}$, and $Y = \frac{\sqrt[4]{7}}{2} + i\frac{\sqrt[4]{7}}{2\sqrt{3}}$.

In this figure, there are four horoballs pictured. Following the description of this diagram from Adams' work, the horoballs $B_O$ and $B_D$ are of Euclidean diameter 1 and maximal in the sense that they are tangent to the horoball based at $\infty$. The horoball $B_X$ has Euclidean diameter $\frac{1}{\sqrt{7}}$ and the horoball $B_Y$ has Euclidean diameter $\frac{3}{7}$. The line segment $\overline{OY}$ is length $w = \frac{\sqrt[4]{7}}{\sqrt{3}}$ while the line segment $\overline{OX}$ is length $\frac{1}{\sqrt[4]{7}}$. In particular, $\frac{\sqrt[4]{7}}{\sqrt{3}} \approx 0.939104416 < 1$.

Under Adams' description of $Q$, we see that $\Gamma_Q$ contains a parabolic element $T$ such that $T(\infty) = \infty$ and $T(0) = \sqrt[4]{7}$. In addition, $\Gamma_Q$ contains an order 6 rotation $R$ that fixes 0 and $\infty$. Finally, as Adams notes, all horoballs of Euclidean diameter 1 are equivalent under the action of $P_Q$. Therefore, there

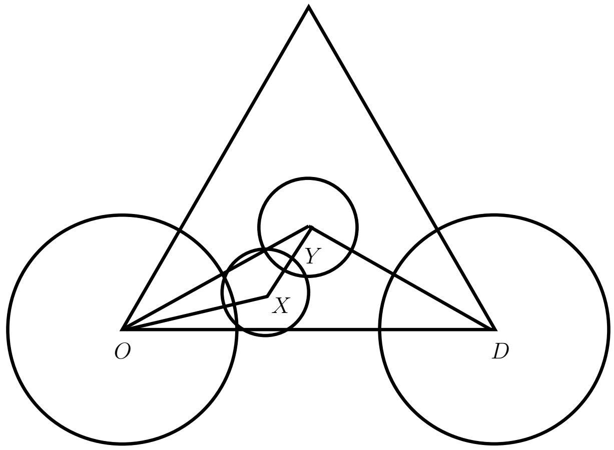



Figure 4.2: A horoball diagram for $Q$

is an element $\gamma$ that exchanges 0 and $\infty$ while sending $\sqrt[4]{7}$ to $\frac{5+\sqrt{-3}}{2\cdot\sqrt[4]{7^3}}$ (see [2, Lem 1.2]). Hence,

$$T = \begin{pmatrix} 1 & \sqrt[4]{7} \\ 0 & 1 \end{pmatrix},\ R = \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix},\ \text{and } \gamma = \begin{pmatrix} 0 & i\cdot b \\ \frac{i}{b} & 0 \end{pmatrix}$$

where $\ell = \frac{\sqrt{3}+i}{2}$ and $b = \frac{\sqrt{5+i\sqrt{3}}}{\sqrt{2\sqrt{7}}}$.

The isometric sphere of $\gamma$ is of radius 1 and centered at 0 (see Fig 4.3). Hence, the isometric sphere for $T\cdot\gamma\cdot T^{-1}$ is radius 1 and centered at $\sqrt[4]{7}$. Let $\Gamma = \langle T, R, \gamma\rangle$. Since these two isometric spheres bound a fundamental domain for $\Gamma$ away from $\mathbb{C}$, $\Gamma$ has finite co-volume. Also, since the cusp co-volume of $\Gamma$ is $\frac{\sqrt{21}}{24}$, $[\Gamma_Q : \Gamma] = 1$.

Let $\lambda = \frac{1}{\sqrt{b}}$ and

$$c = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

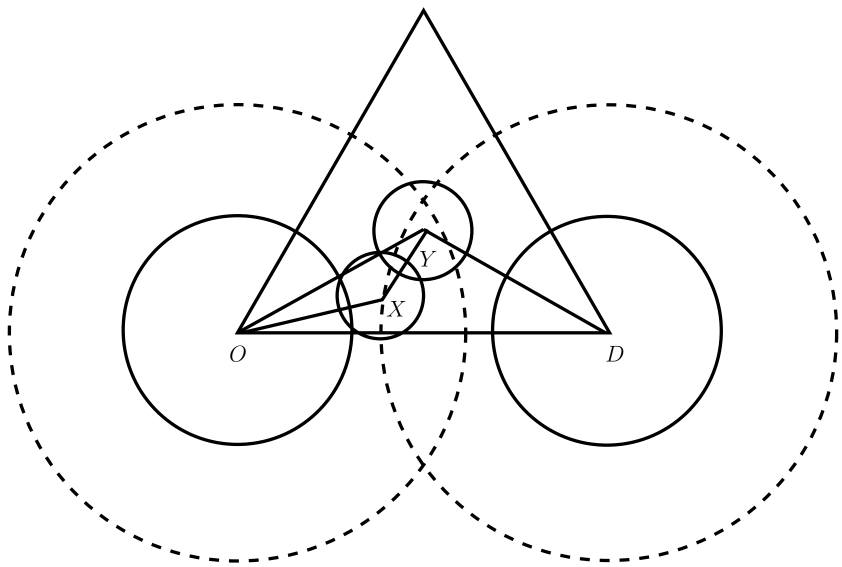



Figure 4.3: A horoball diagram for $Q$ with the isometric spheres represented by dotted curves around 0 and $D$

Then,

$$c \cdot \Gamma \cdot c^{-1} = \left\langle T' = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}, \gamma' = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \right\rangle$$

where $\alpha = \sqrt{\frac{14}{5+i\sqrt{3}}}$. Note the minimal polynomial for $\alpha$ is $q^4 - 5q^2 + 7 = 0$. Thus, $\alpha$ is an algebraic integer, but not a unit since the constant term of this polynomial is not 1.

Under this integral representation of $\Gamma_Q$, there are upper and lower triangular parabolic elements ($T$ and $\gamma' \cdot T \cdot \gamma'^{-1}$, respectively). Hence, a knot complement covering $Q$ would contradict Proposition 7. $\square$

The following proposition discusses which finite groups can act on a point of tangency between two horoballs.

**Proposition 4.9.** *Consider a maximal horoball packing corresponding to an orbifold with a rigid cusp. Denote by $B_x$ the horoball centered $x$ and denote by $B_\infty$ the horoball at $\infty$. If $y$ is the point of tangency of $B_x$ and $B_\infty$ and $y$ fixed by an element $\gamma \in Stab_\infty$, then the isotropy group of $y$ is $C_n$ or $D_n$ where $n = 2, 3, 4, 6$.*

*Proof.* First $\gamma$ is order 2,3,4, or 6 because it fixes $\infty$. If $Stab_y$ is cyclic or dihedral, we are done.

Denote by $\gamma'$ be an element of the isotropy group of $y$ such that axis fixed by $\gamma'$ intersects the axis fixed $\gamma$ at the smallest angle possible. Deont this angle by $\alpha$. If $\alpha = \frac{\pi}{2}$, then, $\langle \gamma, \gamma' \rangle$ is dihedral (see §2.0.1). Hence, we may assume that $\alpha < \frac{\pi}{2}$. Therefore, $\gamma'$ fixes points inside of $B_\infty$. However, $\gamma'(B_\infty) \cap B_\infty = \emptyset$ and $\gamma'$ does not fix $\infty$, which is a contradiction. □

We are now ready to prove the following lemma.

**Lemma 4.10.**

1. *Any orbifold $Q$ with a $S^2(3,3,3)$ cusp and cusp volume $\frac{\sqrt{3}}{4}$ such that $\Gamma_Q$ admits integral traces cannot be covered by a knot complement.*
2. *Any orbifold $Q$ with a $S^2(2,3,6)$ cusp and cusp volume $\frac{\sqrt{3}}{8}$ such that $\Gamma_Q$ admits integral traces cannot be covered by a knot complement.*

*Proof.* We begin by assuming that $Q$ has a $S^2(3,3,3)$ cusp and has cusp volume $\frac{\sqrt{3}}{4}$, and $\Gamma_Q$ admits integral traces. Consider a horoball diagram for the fundamental domain of Q viewed from the point at $\infty$ (see Fig 4.4). In this

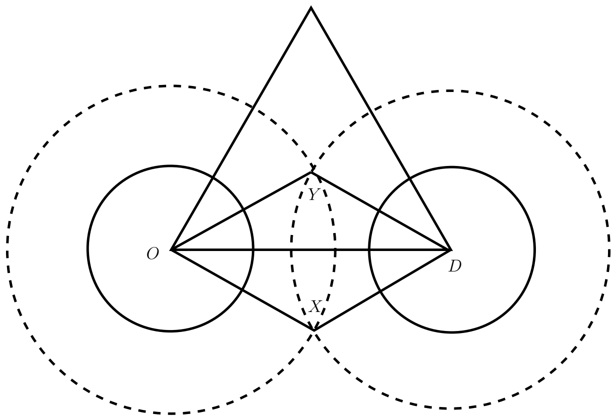



Figure 4.4: A fundamental domain for $Q$ lies above $OXDY$

figure, $O = 0$, $Y = \frac{\sqrt{3}+i}{2}$, $X = \frac{\sqrt{3}-i}{2}$, and $D = \sqrt{3}$ and there are horoballs that are tangent to the horoball at $\infty$, which are of Euclidean diameter 1 tangent and to $\partial\mathbb{H}^3$ at 0 and $D$. Also, there are elliptic elements of order 3 in $\Gamma_Q$ fixing 0 and $\infty$, $X$ and $\infty$, $Y$ and $\infty$, and $D$ and $\infty$.

We know that the point stabilizer of $0 + j$ is $D_3$ or $C_3$ (see Prop 9).

**Case 1:** Assume the point stabilizer of $0 + j$ is $D_3$.

Then, there is an element $\gamma = \begin{pmatrix} 0 & ie^{i\theta} \\ ie^{-i\theta} & 0 \end{pmatrix}$, which fixes $e^{i\theta}$ and $-e^{i\theta}$. Therefore, the isometric sphere corresponding to $\gamma$ has radius 1 and is centered at 0. Also, let $R$ be the element of order 3 fixing $Y$ and $\infty$. Then, $R = \begin{pmatrix} \omega & \frac{\sqrt{3}+i}{2} \\ 0 & \omega^{-1} \end{pmatrix}$ and $R\gamma R^{-1}$ admits an isometric sphere of radius 1 centered at $\sqrt{3}$. The boundaries of these isometric spheres in $\mathbb{C}$ are depicted by dotted lines in Figure 4.4. Finally, let $T = \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}$.

Since $\Gamma' = \langle \gamma, R, T \rangle$ is a subgroup with covolume $\leq v_0$ (see Fig 4.4), it must be finite index in $\Gamma_Q$. Also, by combining the assumption that cusp

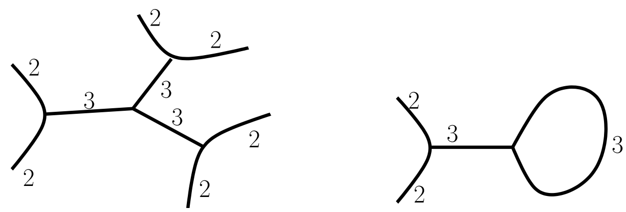



Figure 4.5: The two possible types of isotropy graphs for the orbifold $Q$ described in Case 1 of Lemma 10

volume is $\frac{\sqrt{3}}{4}$ with the upper bound on the cusp density of $\frac{2}{v_0\sqrt{3}}$ (see §2.6), we know that $covolume(\Gamma_Q) \geq \frac{v_0}{2}$. Hence, $[\Gamma_Q : \Gamma'] = 1, 2$.

If $[\Gamma_Q : \Gamma'] = 2$, then $covolume(\Gamma_Q) = \frac{v_0}{2}$ and there are horoballs based at $\frac{\sqrt{3}+i}{2}$ and $\frac{\sqrt{3}-i}{2}$ of Euclidean diameter 1. Thus, by Proposition 9, the point stabilizers above these points are either both $D_3$ or both $C_3$. Hence, the cusp corresponds to a vertex in the isotropy graph that is either 1) connected to three vertices labeled by $D_3$ isotropy groups or 2) there is a loop labeled by 3-torsion and the cusp connects to one vertex labeled by a $D_3$ isotropy group (see Fig 4.5). In either case, $\Gamma_Q$ cannot be trivial under the cusp killing homomorphism.

Therefore, we consider the case that $[\Gamma_Q : \Gamma'] = 1$. Here, $\mathrm{tr}(\gamma \cdot R) = -i(\frac{\sqrt{3}+i}{2})e^{-i\theta}$. Since, $-i(\frac{\sqrt{3}+i}{2})$ is a unit and we are assuming integral traces, $e^{-i\theta}$ is an algebraic integer. Hence,

$$\left\langle \begin{pmatrix} 0 & ie^{i\theta} \\ ie^{-i\theta} & 0 \end{pmatrix}, \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \omega & \frac{\sqrt{3}+i}{2} \\ 0 & \omega^{-1} \end{pmatrix} \right\rangle$$

is a representation of $\Gamma_Q$ with parabolic elements fixing 0 and $\infty$ where all entries of the generators are algebraic integers. If $\Gamma_K \subset \Gamma_Q$, then $\Gamma_K$ admits integral representation. However, the maximal abelian subgroup $A_Q$ of $P_Q$ is of the form:

$$\left\langle \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \omega\sqrt{3} \\ 0 & 1 \end{pmatrix} \right\rangle.$$

In particular, $A_Q$ vanishes under reduction modulo the prime ideal $I$, if $\sqrt{3} \in I$. Also, $\Gamma_Q$ contains the upper triangular parabolic element $T$ and lower triangular parabolic element $\gamma \cdot T \cdot \gamma^{-1}$. Hence, $\Gamma_K$ contains upper and lower triangular parabolics as well. Therefore, no knot group $\Gamma_K$ is a subgroup by Proposition 7.

**Case 2:** The point stabilizer of $0 + j$ is $C_3$.

In this case, there is a group element $\gamma'$ that identifies $0 + j$ with a point above either $\frac{\sqrt{3}+i}{2}$ or $\frac{\sqrt{3}-i}{2}$. We may assume that $0 + j$ is identified with $\frac{\sqrt{3}+i}{2} + tj$. Since $\gamma'$ can be decomposed into reflections in the plane defined by hemisphere of radius 1 centered at 0 and vertical planes, $0 + j$ and $\frac{\sqrt{3}+i}{2} + tj$ are the same Euclidean distance above $\mathbb{C}$ and $t = 1$.

Hence, under $\gamma'$, $\infty \mapsto 0$, $\frac{\sqrt{3}+i}{2} \mapsto \infty$, and $\frac{\sqrt{3}+i}{2} + j \mapsto j$. Here, $\gamma' = \begin{pmatrix} 0 & \frac{\sqrt{3}+i}{2} \\ \frac{-\sqrt{3}+i}{2} & 1 \end{pmatrix}$.

Let $R' = \begin{pmatrix} \omega & \frac{-\sqrt{3}-3i}{2} \\ 0 & \omega^{-1} \end{pmatrix}$. Since $\gamma'$ admits an isometric sphere of radius 1 at $\frac{\sqrt{3}+i}{2}$, $\gamma'^{-1}$ admits an isometric sphere of radius 1 at 0, and $R' \cdot \gamma'^{-1} \cdot R'^{-1}$ admits an isometric sphere of radius 1 at $\sqrt{3}$.

Hence, $\Gamma_3 = \langle \gamma', R', T \rangle$ is a subgroup of finite covolume (here $T$ is defined in Case 1) and $\Gamma_3$ is finite index in $\Gamma_Q$. However, $k\Gamma_3 = \mathbb{Q}(\sqrt{-3})$ and

$\Gamma_3$ has integral traces. Thus, $\Gamma_3$ is arithmetic and therefore, $\Gamma_Q$ is arithmetic. However, the only knot complement that can cover $Q$ is the figure 8 knot complement (see Thm 2). Cusp volume considerations would force the figure 8 knot complement to be a 4-fold cover of $Q$. However, $Q$ has 3 torsion on the cusp. Hence in this case, $Q$ is not covered by a hyperbolic knot complement.

Finally, if $Q$ has a $S^2(2,3,6)$ cusp and cusp volume $\frac{\sqrt{3}}{8}$ and $\Gamma_Q$ admits integral traces. Then, the point stabilizer of $0+j$ is $D_6$. Hence, an identical argument to Case 1 shows $Q$ is not covered by a knot complement. □

## 4.3 Bounding the degree of covering

In this section, we establish bounds on the minimum degree of covering form $p\colon \mathbb{S}^3 - K \to Q$ where $Q$ is a rigid cusped orbifold. We achieve these bounds by analyzing the isotropy graph associated to $Q$. A *loop* in a graph is a edge in a graph that connects a vertex to itself.

We begin by classifying the possible abelian quotients of $\Gamma_Q$. We denote the abelianization of $\Gamma_Q$ by $\Gamma_Q^{ab}$.

**Proposition 4.11.** *Let $\mathbb{S}^3 - K$ be a hyperbolic knot complement that covers an orbifold $Q$.*

1. *If $Q$ has a $S^2(2,3,6)$ cusp, then $\mathbb{Z}/2\mathbb{Z}$ surjects $\Gamma_Q^{ab}$.*
2. *If $Q$ has a $S^2(3,3,3)$ cusp, then $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ surjects $\Gamma_Q^{ab}$.*
3. *If $Q$ has a $S^2(2,4,4)$ cusp, $\Gamma_Q^{ab}$ is trivial, $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, or $\mathbb{Z}/4\mathbb{Z}$.*

*Furthermore, $\Gamma_Q^{ab} \cong \mathbb{Z}/4Z$ if and only if the isotropy graph of $Q$ has a loop.*

*Proof.* We first note that $\Gamma_Q = P_Q \cdot \Gamma_K$. Therefore, we claim that

$$\Gamma_Q = \left\langle \left\langle t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, r = \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix} \right\rangle \right\rangle_{P_Q \cdot \Gamma_K},$$

where $r$ is an elliptic element of order 3, 4, or 6 depending on the cusp type. The claim follows from the fact that $r$ and $t$ are generators for $P_Q$ and $P_Q$ contains a meridian of $\Gamma_K$.

First assume Q has a $S^2(2,3,6)$ cusp, then $\ell = e^{\frac{i\pi}{6}}$. Since $P_Q$ abelianizes to $\mathbb{Z}/6\mathbb{Z}$ (see 2.0.2), we know $\Gamma_Q^{ab}$ is a quotient of $\mathbb{Z}/6\mathbb{Z}$. Also, the torsion element of order 6 is connected to an interior vertex with the isotropy group a $D_6$, (dihedral group of order 12). Under the abelianization of this isotropy group, the element of order 6 maps to an element of order 2. Thus, $\Gamma_Q^{ab}$ is a quotient of $\mathbb{Z}/2\mathbb{Z}$, as desired.

Next assume Q has a $S^2(3,3,3)$ cusp. In this case, $\ell = e^{\frac{2i\pi}{3}}$. Then $\Gamma_Q^{ab}$ is a quotient of $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$, the abelianization of the peripheral subgroup (see 2.0.2).

Finally assume Q has a $S^2(2,4,4)$ cusp, then $\ell = e^{\frac{i\pi}{4}}$. Hence, $\Gamma_Q^{ab}$ is a quotient of the abelianization of $P_Q$, which is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ (see 2.0.2).

Consider an edge $e$ labeled by 4-torsion that connects the cusp $c$ to another vertex $x$. Then $x$ is either the cusp itself, or it corresponds to an isotropy group, $D_4$ or $S_4$ (see §2.0.1).

**Case 1:** $e$ is a loop. In this case, $e$ connects the cusp back to itself. Then, considering a Wirtinger presentation for $\Gamma_Q$ coming from the isotropy graph. $\Gamma_Q^{ab}$ is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. However, the second case cannot occur since this would imply the existence of a cycle labeled by only even numbers that starts and ends at the cusp (by the cusp killing homomorphism) and includes the 2 torsion on the cusp. Such a cycle would kill the 4 torsion on the cusp and the maximal abelian quotient would be $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

**Case 2:** $x$ corresponds to $D_4$ or $S_4$. Under the abelianizations of these groups, elements of order 4 are mapped to elements of order 2. Under the abelianization of the cusp, the peripheral elements of order 4 all have the same order in $\Gamma_Q^{ab}$. Therefore, $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ surjects $\Gamma_Q^{ab}$. In this case, the isotropy graph of $Q$ does not contain a loop. This completes the proof. □

We now establish lower bounds on the degree of the cover

$$p\colon\ \mathbb{S}^3 - K \to Q$$

using the following propostion and lemma. We establish the following notation for the remainder of the chapter.

**Proposition 4.12.** *If $p\colon\ \mathbb{S}^3 - K \to Q$ with a $S^2(2,4,4)$ cusp and the isotropy graph for $Q$ has a loop labeled 4, then deg$(p) \geq 24$.*

*Proof.* Assume that $Q$ is covered by a knot complement and has an edge labeled 4 in its isotropy graph (see Fig 4.6 for a possible example). Then

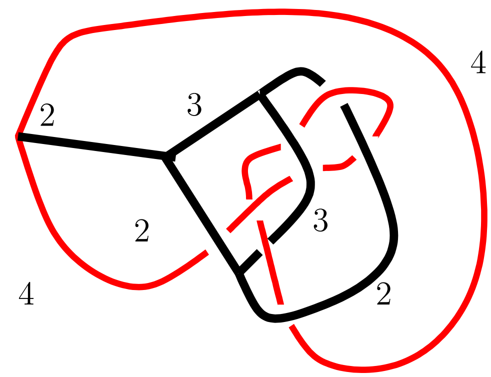


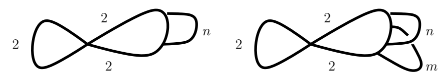

Figure 4.6: A possible isotropy graph of an orbifold with a $S^2(2,4,4)$ cusp

$\Gamma_Q^{ab}$ is $\mathbb{Z}/4\mathbb{Z}$ (see Prop 11). Hence, there is a unique orbifold $Q'$ which has a $S^2(2,2,2,2)$ cusp and is 2-fold cover of $Q$.

Since $P_{Q'}$ and $P_Q$ have the same parabolic subgroup, $P_{Q'}$ contains a meridian $\mu$ of $\Gamma_K$. The abelianization of $\Gamma_Q$ is $\mathbb{Z}/4\mathbb{Z}$ so $\Gamma_{Q'}$ is characteristic in $\Gamma_Q$. In particular, $\langle\langle\mu\rangle\rangle_{\Gamma_K} \subset \Gamma_{Q'}$. Therefore, $\mathbb{S}^3 - K$ covers $Q'$. Also, $Q'$ has $S^2(2,2,2,2)$ cusp and the cover $p' : \mathbb{S}^3 - K \to Q'$ is a regular covering (see §2.5). Furthermore, there is a unique 2-fold cover of $Q'$ that has a torus cusp. We denote this orbifold by $Q_T$. We know that $\mathbb{S}^3 - K$ covers $Q_T$ by an identical argument to that above.

Since $Q_T \cong (\mathbb{S}^3 - K)/Z$ for some cyclic group $Z$ (see §2.5), we see that $Q_T$ is the complement of a knot in an orbi-lens space (see §2.7.3). The isotropy graph for $Q_T$ is either 0, 1, or 2 unknotted circles (see Thm 12). Therefore, the isotropy graph for $Q'$ contains 0, 2, or 4 internal vertices.

The isotropy graph for $Q'$ cannot contain 0 vertices because that would imply that the isotropy graph for $Q$ only had vertices labeled by Klein 4 groups.




(a) 2 vertex graph (b) 4 vertex graph

Figure 4.7: The possible isotropy graphs Q'

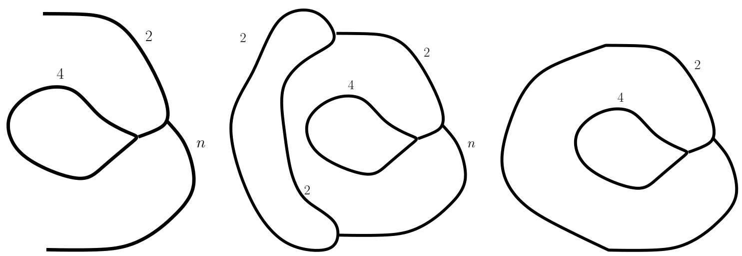
Such a graph would be non-trivial under the cusp killing homomorphism. Since the 2-fold of the loop labeled by 4 is a loop labeled by 2, the possible graphs as defined up to graph isomorphism type can be see Figure 4.7.

We claim that the isotropy graph for $Q'$ cannot contain 2 internal vertices as well. First, notice that the isotropy graph for $Q'$ has an edge $e$ labeled by 2-torsion with both endpoints on the cusp. Thus, the isotropy graph $Q$ takes the form of Figures 4.8(a), 4.8(b), and 4.8(c). In the latter two cases, the orbifold is non-trivial under the cusp killing homomorphism and therefore cannot be covered by a knot complement. In the first case, we cannot close up the isotropy graph. Therefore, no such orbifold $Q$ can be covered by a knot complement.

Finally, if $Q'$ contains 4 internal vertices, then $Q_T$ is the complement of a knot in an orbi-lens space with an isotropy graph consisting of two unknotted circles. Since the circles are labeled by $n$-torsion and $m$-torsion with $n, m \geq 2$ and $(n, m) = 1$, $\mathbb{S}^3 - K$ is at least a 6-fold cover of $Q_T$ and therefore at least a 24-fold cover of $Q$. This completes the proof. $\square$

**Lemma 4.13.** *1. If $Q$ has a $S^2(3,3,3)$ cusp, then deg$(p) = 12n$ $n \geq 1$.*

*2. If $Q$ has a $S^2(2,4,4)$ cusp, then the deg$(p) \geq 24$.*




(a) The first case (b) The second case (c) The third case

Figure 4.8: The three cases for an isotropy graph with 2 vertices

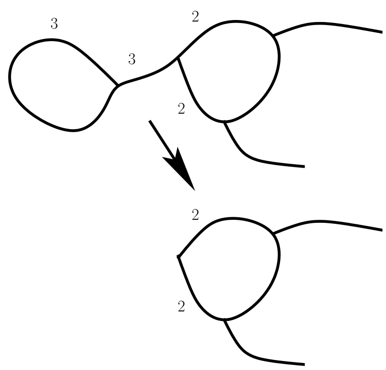



Figure 4.9: Application of the cusp killing homomorphism to a graph with a cycle labeled 3 and a vertex labeled $D_3$

*Proof.* **1) Assume that $Q$ has a $S^2(3,3,3)$ cusp.** First, consider the isotropy graph of $Q$. If there is an edge of the graph with two endpoints on the cusp, then the third edge cannot connect the cusp to a point with isotropy group $D_3$. For this case, $\Gamma_Q$ would be non-trivial under the cusp killing homomorphism (see Fig 4.9).

Therefore, this vertex is fixed by a group $G$ where $G$ is either $A_4$, $S_4$ or $A_5$ and to lift to a torsion-free group $deg(p)$ must be a multiple of the order of $G$. Hence, $deg(p) = 12n$ $(n \in \mathbb{Z})$.

If we assume that there is no edge in the isotropy graph with both endpoints on the cusp, then there must be at least one vertex adjacent to the cusp labeled with a $A_4$, $S_4$ or $A_5$ subgroup. Otherwise, all vertices are labeled with $D_3$ and just as above $\Gamma_Q$ would be non-trivial under the cusp killing homomorphism. Thus, $deg(p) = 12n$.

**2) Assume that $Q$ has a $S^2(2,4,4)$ cusp.** Since $\mathbb{S}^3 - K$ is a manifold, all isotropy subgroups of $\Gamma_Q$ must vanish in the lift. Either, the order 4 elements in the cusp are part of the same fixed axis (see Fig 4.6) or the four torsion on the cusp connects to a pair of distinct vertices in the isotropy graph.

In the first case, $deg(p) \geq 24$ (see Prop 12). In the second case, the vertices are either of type $D_4$ or $S_4$ isotropy subgroup. If there is a vertex of type $S_4$, then $deg(p) \geq 24$. If we have a vertex of type $D_4$. There must be some edges in the isotropy graph labeled with odd integers otherwise the graph would be non-trivial under the cusp killing homomorphism. Thus, $deg(p) = 8(2k+1)n$ for some $n, k \geq 1$ and $deg(p) \geq 24$ . □

## 4.4 Proof of Theorem 1

In this section, we prove Theorem 1. The proof relies on accounting for short curves in the peripheral subgroups of our knot complements. We defined the length of a parabolic element in §2.7.2. We discuss discuss length for peripheral elements fixing $\infty$ below. Let $Q$ be a 1-cusped hyperbolic 3-orbifold and fix a representation for $\Gamma_Q$ in $PSL(2,\mathbb{C})$ such that $P_Q$ is upper triangular and we consider $\Gamma_Q$ acting on upper half space. Denote by $\frac{1}{c}$ the height of a maximal horoball tangent to $\infty$ and denote by $S_c$ the horosphere centered at $\infty$ of Euclidean height $\frac{1}{c}$. If $\gamma \in \Gamma_Q$ is a parabolic element fixing $\infty$, we measure the $len(\gamma)$ by its translation length in $S_c$. If $\gamma = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$, then $len(\gamma) = c \cdot |x|$. Therefore, if $\gamma$ corresponds to an exceptional slope, then by the Six Theorem $c \cdot |x| \leq 6$ (see Thm 8). Finally, by Lemma 6, we will only consider representation of $\Gamma_Q$ such that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_Q$. Since the interiors of maximal horoballs are disjoint, we know that $c \geq 1$. Also, $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix}$ correspond to the same slopes in terms of Dehn surgery parameters, so for convenience we consider them as one curve in our accounting of short parabolic elements.

With $\Gamma_Q$, $P_Q$ as above, recall that if $A$ is the area of the the fundamental domain for $P_Q$ in the horosphere of Euclidean height 1, the cusp volume of $Q$ is

$$\int_{\frac{1}{c}}^{\infty} \frac{A}{z} dz = \frac{c^2 \cdot A}{2}.$$

*Proof of Theorem 1.* Assume $\mathbb{S}^3 - K$ admits a non-trivial exceptional surgery, $\Gamma_K$ has integral traces, and $k\Gamma_K$ is class number 1. Also, assume $p\colon\ \mathbb{S}^3 - K \to Q$, where $Q$ has a rigid cusp (see §2.0.2). We break the proof into three cases, one for each cusp type of $Q$.

**Case 1: $Q$ has a $S^2(2,4,4)$ cusp.** We know $deg(p) \geq 24$ (see Lemma 13). By Lemma 6 and the Six Theorem (see §8), $P_K$ is of the form: $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 6i \\ 0 & 1 \end{pmatrix} \right\rangle$ and $c = 1$ (with $c$ defined above). In this case, the cusp volume of $Q$ is $\frac{3}{24}$. By §2.6, there is one such orbifold, which is arithmetic with invariant trace field $\mathbb{Q}(i)$. However, no knot complement can cover such an orbifold (see Thm 2).

**Case 2: $Q$ has a $S^2(3,3,3)$ cusp.** By Lemma 6, we can find a representation for $\Gamma_K$ where the $P_K$ is of the form: $\left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & n\omega \\ 0 & 1 \end{pmatrix} \right\rangle$, such that $\omega^2+\omega+1=0$. By the Six Theorem (see §2.7), $n \leq 6$. However, $3n$ must be a multiple of 12. Hence, $n=4$ and $deg(p)=12$ (see Lemma 13). Here, the two shortest parabolic elements (excluding inverses) are $\mu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\lambda = \begin{pmatrix} 1 & 2+4\omega \\ 0 & 1 \end{pmatrix}$, ($|2+4\omega| = 2\sqrt{3}$). In order to have two curves $\gamma_1$, $\gamma_2 \in P_K$ with $len(\gamma_i) \leq 6$, the horoballs tangent to $B_\infty$ have Euclidean height greater than $\frac{1}{\sqrt{3}}$ and so $c \leq \sqrt{3}$. (Note if $\lambda$ is a longer element ,say $\lambda = \begin{pmatrix} 1 & 3+4\omega \\ 0 & 1 \end{pmatrix}$, then we must have $c \leq \frac{6}{\sqrt{13}} < \sqrt{3}$). Thus, the cusp volume of $\mathbb{S}^3 - K$ is in the range $[\sqrt{3}, 3\sqrt{3}]$ $(1 \leq |c| \leq \sqrt{3})$ and the cusp volume of $Q$ is in the range $[\frac{\sqrt{3}}{12}, \frac{\sqrt{3}}{4}]$.

Since any orbifold $Q$ with $S^2(3,3,3)$ cusp and cusp volume $\frac{\sqrt{3}}{12}$, $\frac{\sqrt{3}}{6}$, or $\frac{1}{4}$ is arithmetic (see §2.6), the only knot complement that can cover $Q$ is the figure 8 knot (see Thm 2), which is excluded by the hypothesis that $\mathbb{S}^3 - K$ is non-arithmetic. Hence, we only have to consider orbifolds with possible cusp volume, $\frac{3\sqrt{3}+\sqrt{15}}{24}$, $\frac{\sqrt{21}}{12}$, or $\frac{\sqrt{3}}{4}$ (see §2.6). The first case, implies that $Q$ is $\mathbb{H}^3/\Gamma(5,2,2,3,3,3)$ which has an order 60 isotropy group. Thus, it cannot have a 12-fold manifold cover. In the second case, we know no such orbifold can be covered by a knot complement by Lemma 8. Finally, we show the third case cannot occur by appealing to Lemma 10.

**Case 3: $Q$ has $S^2(2,3,6)$ and $\mathbb{S}^3 - K$ admits a strong inversion** Since the element of 6 torsion is part of a dihedral group of order 12, we know $deg(p) = 12n$.

If $deg(p) > 24$, we claim $\mathbb{S}^3 - K$ cannot admit two exceptional surgeries. In this case, $P_K = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2n\omega \\ 0 & 1 \end{pmatrix} \right\rangle$ with $n > 3$ and $\omega^2 + \omega + 1 = 0$. Since $|c| \geq 1$, if $\gamma \in P_K$ with $len(\gamma) \leq 6$, then $\gamma = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}$ (see §2.7.2). However, these curves both correspond to surgery along the meridian.

If $deg(p) = 24$, then $P_K$ is same as in Case 2 above. Therefore, $\mathbb{S}^3 - K$ has cusp volume in $[\sqrt{3}, 3\sqrt{3}]$ and $Q$ has cusp volume in $[\frac{\sqrt{3}}{24}, \frac{\sqrt{3}}{8}]$. For cusp volume in $[\frac{\sqrt{3}}{24}, \frac{\sqrt{3}}{8})$, these orbifolds fit Adams' list and only the figure 8 knot complement can cover $Q$ as shown in Case 2. If the cusp volume is exactly $\frac{\sqrt{3}}{8}$, we appeal to Lemma 10.

If $deg(p) = 12$, we may consider $\Gamma_Q = P_Q \cdot \Gamma_K$. In this case,

$$P_Q = \left\langle r = \begin{pmatrix} \ell & 0 \\ 0 & \ell^{-1} \end{pmatrix}, t = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \right\rangle$$

where $\ell = e^{\frac{i\pi}{6}}$.

Since $\mathbb{S}^3 - K$ admits a strong inversion $N^+(\Gamma_K) = \langle r^3, \Gamma_K \rangle$ or $\langle r^3, t, \Gamma_K \rangle$. In the second case, we may consider $\Gamma_Q$ acting on $\Gamma_K$ by conjugation. Since $r^3, t \in N^+(\Gamma_K)$, the distinct conjugates of $\Gamma_K$ are $\Gamma_K$, $r \cdot \Gamma_K \cdot r^{-1}$ and $r^2 \cdot \Gamma_K \cdot r^{-2}$. Since $\Gamma_Q = P_Q \cdot \Gamma_K$, if $\gamma \in \Gamma_Q$, then $\gamma = p \cdot g$ with $p \in P_Q$ and $g \in \Gamma_K$. Therefore, $\Gamma_Q = \langle r, \mu_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mu_2, ...\mu_n \rangle$, where $r$ defined above and $\mu_i$ is a meridian of $\Gamma_K$. Conjugation of $r \cdot \Gamma_K \cdot r^{-1}$ and $r^2 \cdot \Gamma_K \cdot r^{-2}$ by $\Gamma_Q$ produces a homomorphism onto $S_3$. Here, $\mu_1$ maps trivially and $r$ maps to an element of order 3. Hence, $\Gamma_Q$ admits a $\mathbb{Z}/3\mathbb{Z}$ quotient by the action of $\Gamma_Q$ on $\Gamma_K$. However, since $Q$ has a $S^2(2,3,6)$ cusp, this is not a valid abelian quotient for $\Gamma_Q$ (see Prop 11).

Thus, we may assume $N^+(\Gamma_K) = \langle r^3, \Gamma_K \rangle$. Then the conjugates of $\Gamma_K$ in $P \cdot \Gamma_K$ are $\Gamma_K$, $r \cdot \Gamma_K \cdot r^{-1}$,$r^2 \cdot \Gamma_K \cdot r^{-2}$, $t \cdot \Gamma_K \cdot t^{-1}$, $rt \cdot \Gamma_K \cdot (rt)^{-1}$ and $r^2 t \cdot \Gamma_K \cdot t^{-1} r^{-2}$. In this case, $t$ maps to product of three 2-cycles in $S_6$. Hence, $P_Q \cdot \Gamma_K$ has a $\mathbb{Z}/2\mathbb{Z}$ quotient. Therefore, $Q$ is covered by an orbifold $Q'$ with a $S^2(3,3,3)$ cusp with $[P_Q : P_{Q'}] = 2$ and $\Gamma_{Q'} = P_{Q'} \cdot \Gamma_K$. However, $\mathbb{S}^3 - K$ would be a 6-fold cover of $Q'$, which is a contraction to the minimum degree cover of $p\colon\ \mathbb{S}^3 - K \to Q'$ (see Lem 13).

□

## 4.5 Remarks on Theorem 1

First, we remark that for small volume manifolds we may use the lower bounds on the degree of the covering $p : \mathbb{S}^3 - K \to Q$ to our advantage. Let $\mathbb{S}^3 - K$ be a knot complement arising from $\frac{1}{m}$ surgery on the unknotted cusp of the Berge manifold (see Chapter 3). We observe that $\mathbb{S}^3 - K$ is hyperbolic (see Rem 3.2.1). Notice the volume of $\mathbb{S}^3 - K$ is bounded from above by the volume of the Berge manifold, $4v_0$. If $\mathbb{S}^3 - K$ admits hidden symmetries, $\mathbb{S}^3 - K$ would cover an orbifold with a rigid cusp (see Thm 4). If $Q$ has $S^2(3,3,3)$ cusp, then the $deg(p) \geq 12$ and so the volume of $Q$ would be less than $\frac{v_0}{3} \approx 0.3383138$ (see Lemma 13). In particular, this volume forces $Q$ to be arithmetic (see Prop

6 and §2.6). Now consider $Q$ with a $S^2(2,3,6)$ cusp. Since $\mathbb{S}^3 - K$ is strongly invertible (see Rem 3.2.1), $deg(p) \geq 24$ by Case 3 of the above proof. Here volume of $Q$ would be less than $\frac{v_0}{6} \approx 0.1691569$. Again, this volume forces $Q$ to be arithmetic (see Prop 6 and §2.6). For either cusp, this would imply that $\mathbb{S}^3 - K$ is the figure 8 knot (see Thm 2), but the figure 8 knot does not admit finite cyclic fillings (see [37]). Therefore, it cannot be of the form $\beta_{1,m}$.

If $Q$ has a $S^2(2,4,4)$ cusp, then $deg(p) \geq 24$ and volume of $Q$ is less than $\frac{v_0}{6} \approx 0.1691569344$, no non-arithmetic orbifolds with $S^2(2,4,4)$ cusps exists at such low volume (see Thm 5 and Prop 6). Therefore, $\mathbb{S}^3 - K$ does not admit hidden symmetries. This is a slight improvement from the results of the previous chapter because it exhibits an infinite family of knot complements such that all of the members do not admit hidden symmetries and are commensurable with exactly two other knot complements (see Lem 5).

Also, Long and Reid recently produced knot complements with invariant trace fields with class number $> 1$ (see [20, §7]). However, these knot complements admit a high degree of symmetry (of order $\geq 257$) and the quotient $Q_T$ of these knot complements under this symmetry is homeomorphic to orbifold Dehn surgery on the link in Figure 4.10. This link complement has volume $4v_0$. If any knot complement covering $Q_T$ admits hidden symmetries, then $Q_T$ would have to cover an orbifold $Q_r$ with a rigid cusp. However, the degree of such a cover would put $Q_r$ on Adams' list of small volume orbifolds (see §2.6). In fact, just as above such an orbifold $Q_r$ would be arithmetic, but the figure 8 knot complement is the only arithmetic knot complement and it

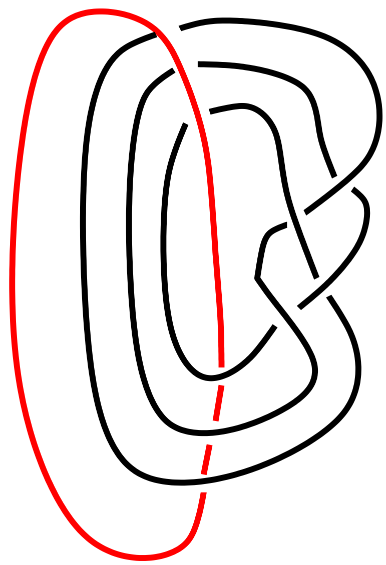


Figure 4.10: An orbifold $Q$ resulting from $(m, 0)$-surgery ($m \geq 257$) along the unknotted cusp of the above link has invariant trace field with class number bigger than 1

does not admit a symmetry of such high order.

# Chapter 5

# Links in the isotropy graph

In this chapter, we establish a relationship between the complexity of the isotropy graph of an orbifold and the existence of an incompressible surface in the orbifold. Specifically, we prove the following theorem.

**Theorem 5.1.** *Let $O$ be an orbifold with a single rigid cusp and base space $D^3$ that has an isotropy graph with 8 or more vertices. Then either every pair of edge cycles in the isotropy graph of $O$ is linked or $O$ contains a closed, embedded, incompressible 2-orbifold.*

For this theorem, we have the immediate corollary regarding knot complements with hidden symmetries.

**Corollary 5.2.** *Let $O$ be an orbifold covered by a small hyperbolic knot complement $S^3 - K$. Then, either the isotropy for $O$ has fewer than 8 vertices or every pair of disjoint cycles of the isotropy graph is an embedded, non-split, two-component link.*

In some sense, the above theorem can be viewed as an analog to the proposition in Thurston's notes that shows that a non-orientable orbifold $N$ with base space $D^3$ is either $\mathbb{H}^3/T$ where $T$ is the group of reflections in some tetrahedron or $N$ contains a closed, incompressible 2-orbifold (see [38, Prop 13.5.2] and §2.4). In the later case, the orientable double cover of $N$ is Haken (see §2.0.4).

## 5.1 Graphs and incompressible 2-orbifolds

In this section, we use properties of the isotropy graph $G$ associated to an orbifold $O$ (see § 4.1) in order to find cases where $O$ contains an incompressible 2-orbifold.

**Proposition 5.3.** *If a trivalent graph $G$ has eight or more vertices, then it contains at least two disjoint cycles.*

*Proof.* Throughout the proof we will assume that $G$ has $n$ vertices with $n \geq 8$. Since $G$ is trivalent, this implies that $G$ has $\frac{3n}{2}$ edges.

First, we claim that $G$ has one cycle. If $G$ has no cycles, then $\frac{3n}{2} \leq n-1$ since a tree has $n-1$ edges and the disjoint union of $m$ trees has $n-m$ edges. However, this implies $n \leq -2$, which is a contradiction.

Thus, $G$ has at least one cycle. Assume that $C$ is a smallest cycle in $G$, in the sense that $C$ has the fewest edges of any cycle. Denote by $k$ the number of edges in $C$ and denote by $G - C$ the graph obtained by removing vertices in $C$ and all edges incident to these vertices.

**Case 1:** Assume $k \leq 4$. Then $G-C$ has at least $\frac{3n}{2} - k - k$ edges since there are $k$ edges in C and at most $k$ edges connecting to the vertices of $C$.

To be a tree or the disjoint union of trees $G - C$ would have at most $n-k-1$ edges. If $\frac{3n}{2} - 2k \leq n - k - 1$. However, this contradicts $n \geq 8$.

**Case 2:** Assume $k > 4$. $C$ has more than 4 vertices. Then, all edges that connect to vertices of C but are not a part of C, connect to vertices outside of $C$. Otherwise, $C$ would not be a smallest cycle. Also, there are no vertices of $G - C$ that have degree 1 because this would also contradict the minimality of $C$.

Hence, $G - C$ cannot be a tree and must contain a cycle. $\square$

Let $O$ be an orbifold such that $|O|$ is simply connected and denote by $G$ the isotropy graph of $O$. Consider $G$ as an embedded graph in $|O|$. We say that $G$ has a pair of *split cycles*, $C_1$ and $C_2$ if there exist two 3-balls $B_1$ and $B_2$ embedded in $|O|$ such that $B_1$ and $B_2$ are disjoint and $C_1 \subset B_1$ and $C_2 \subset B_2$. If $G$ does not contain a pair of split cycles, then any pair of two disjoint cycles in $G$ from a non-split 2-component link.

We now prove Theorem 1.

*Proof of Theorem 1.* Denote by $G$ the isotropy graph of $O$. Assume that $G$ has a least 8 vertices. There exist at least two disjoint cycles in $G$ (see Prop 3). Assume that $G$ contains a pair of split cycles.

Pick two such split cycles $C_1, C_2$. We may isotope $G$ in $O$ such that 1) there exists an embedded ball $B_1$ containing $C_1$, 2) there exists an embedded ball $B_2$ containing $C_2$, 3) $G$ intersects the Seifert surface $F_1$ bounding $C_1$

minimally, and 4) $B_1 \cap G$ is the cycle $C_1$, untwisted, unknotted parts of the edges incident to $C_1$ and a set of untwisted, unknotted arcs that transversely intersect $F_1$. Denote the boundary of $B_1$ by $S_1$ and the boundary of $B_2$ by $S_2$. We say the inside of $B_i$ is the side of $S_i$ that contains the cycle $C_i$. (Hence, the outside $S_1$ contains part of the outside of $S_2$ and all of the inside of $S_2$).

Assume $B_1$ contains an arc $\alpha$ which connects two vertices of $C_1$ (likewise, for $B_2$ and $C_2$) that can be isotoped inside $B_1$ without forcing any other pieces of $G$ into $B_1$. Then, assume $\alpha$ is isotoped inside $B_1$ to reduce intersection number of $G$ and $S_1$. This isotopy prevents the existence of a disk quotient that is boundary compressibling on two sides. Note, such an arc $\alpha$ is part of a cycle and therefore will not affect the incompressibility of $S_1$ the inside of $B_1$.

Notice $S_1$ and $S_2$ are separating and incompressible on the side containing the cycle since any embedded disk is either boundary parallel or intersects $G$ in two places along the cycle contained in the sphere. Assume that that there exists a compressing disk $D$ for $S_1$. Then $D$ is contained in the outside of $S_1$ and intersects $S_1$ circle which divides $S_1$ into two pieces $D'$ and $D''$. Either $D \cup D'$ or $D \cup D''$ is a sphere that encloses $C_1$ and contains the union a set of knotted arcs and vertices of $G$. Otherwise, $D$ would be boundary parallel to $S_1$ and not a compressing disk. Call this sphere $S'$.

Also, $S'$ is incompressible on the inside because a compressing disk would lead a sphere hitting $G$ in 2 points which would contradict the irreducibility of $O$.

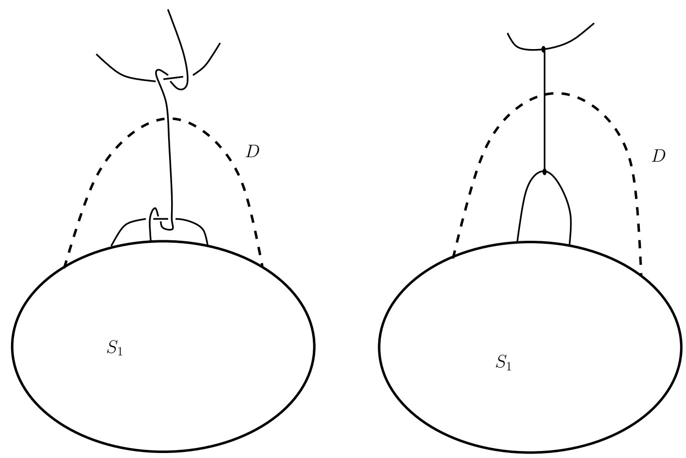


Figure 5.1: Two examples of compressing disks outside of $S_1$

We may repeat this process for each compressing disk. Since this process will put a new vertex or crossing of $G$ inside $S'$, (see Fig 5.1) it must terminate by the fact that $\Gamma_O$ has finite Wirtinger presentation. Furthermore, no disks can intersect the inside of $S_2$ and the surface constructed here is incompressible on both sides. $\square$

It is worth mentioning that we cannot relax the condition on the number of vertices in $G$. Since we are interested in hyperbolic 3-orbifolds with a rigid cusp, the isotropy graphs we are interested in are all trivalent. Therefore, they all have an even number of vertices. The $K(3,3,3)$ has six vertices but does not contain two disjoint cycles. Thus, we cannot improve on the hypothesis of the above theorem.

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