LEXICOGRAPHIC IDEALS

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Macaulay proved in 1927 that every Hilbert function in the polynomial ring $R = k[x_1, \cdots, x_n]$ is attained by a lexicographic ideal. We study the combinatorial and homological properties of lexicographic ideals, and identify other settings in which lexicographic ideals attain every Hilbert function, and other classes of ideals with similar properties. We develop the theory of compression, which makes many of our arguments possible and leads to shorter new proofs of results of Macaulay, Bigatti, Hulett, and Pardue. Using compression, we extend results of Green and Clements-Lindström, and prove a special case of Evans' Lex-Plus-Powers conjecture.

BIOGRAPHICAL SKETCH

Jeff Mermin was born on May 6, 1978, in Chapel Hill, North Carolina. He discovered his interest in mathematics at an early age, and his parents worked hard to foster it throughout his elementary school career.

In middle school, Mermin discovered the MathCounts program, which, together with its high school relatives, tremendously expanded his love for, and skill with, the subject. Mermin remembers the activities surrounding competitive mathematics as among the happiest of his youth; since leaving high school, he has continued to work with gifted middle and high school mathematicians whenever he has had the chance.

After graduating from Chapel Hill High School in 1996, Mermin joined many of his high school rivals at Duke University, where he would earn a bachelor's degree in mathematics and classical languages. After graduating from Duke in 2000, Mermin came to Cornell, where he has studied commutative algebra ever since.

Dedicated to Jonathan Andrews, Jeremy Weiss, and Jonathan Woodward,
my friends, rivals, and teammates through high school, whose examples have constantly inspired me to improve.

But most of all to Bud Stuart,
our teacher and coach; whose passion for the subject remains with me even now, almost twelve years after the last class; who taught us so much more than mathematics; whose profound effects on my life I can never hope to repay.

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Chapter 1

Introduction

1.1 Hilbert Functions

Let k be a field, S be a quotient of the polynomial ring $k[x_1, \cdots, x_n]$ (where each x_1 has degree 1) by a homogeneous ideal, and M a finitely generated graded S-module. (Usually, S will in fact be the polynomial ring $S = k[x_1, \cdots, x_n]$, and $M \subset S$ will be a homogeneous ideal.) Then, by definition, M is a direct sum,

$$M = \bigoplus_{d \in \mathbb{Z}} M_d,$$

where each M_d is the k-vector space of degree-d elements of M, $M_d = \{m \in M : \deg(m) = d\}$. Since M and S are finitely generated, each M_d is finite-dimensional. The *Hilbert Function* of M associates to each degree d the dimension of the vector space M_d:

Definition 1.1.1. Let M be a graded module over the finitely generated graded k-algebra S. The *Hilbert Function* of M is

$$\mathrm{Hilb}_M^S : \mathbb{Z} \to \mathbb{Z}_{\geq 0}$$
$$\mathrm{Hilb}_M^S(d) = \dim_k M_d$$

The Hilbert function measures the size of M, and is one of the most important combinatorial invariants. (In fact, it arises very naturally in combinatorics: if $S = k[x_1, \cdots, x_n]/(x_1^2, \cdots, x_n^2)$ and $M = S/I_\Delta$ is the Stanley-Reisner ring of a simplicial complex Δ, then the Hilbert function of M is the f-vector of Δ).

Because dimension is an additive functor on k-vector spaces, the Hilbert function is additive. That is, if

$$0 \to M' \xrightarrow{\alpha} M \xrightarrow{\beta} M'' \to 0$$

is an exact sequence of graded S-modules, and the maps α and β are of degree 0 (i.e., $\deg(\alpha(m')) = \deg(m')$, $\deg(\beta(m)) = \deg(m)$ for all homogeneous m', m), then we have $\mathrm{Hilb}_M^S = \mathrm{Hilb}_{M'}^S + \mathrm{Hilb}_{M''}^S$.

The Hilbert function is intimately related to two other invariants, the *Hilbert-Poincare Series* and the *Hilbert Polynomial*:

Definition 1.1.2. The *Hilbert-Poincare Series* of M is the formal power series defined by

$$\mathrm{HS}_M^S = \sum_{d \in \mathbb{Z}} \mathrm{Hilb}_M^S(d) t^d.$$

Theorem 1.1.3 (Hilbert)**.** *The Hilbert-Poincare series of M is a rational function of the form $\frac{p(t)}{(1-t)^n}$, for some polynomial $p(t)$.*

Proof. The proof is by induction on the number of variables n. If $n = 0$, then M is a finite-dimensional vector space and so HS_M^S is a polynomial. In general, we have the exact sequence

$$0 \to K(-1) \to M(-1) \xrightarrow{\mu} M \to L \to 0,$$

where μ is the map "multiplication by x_n", K and L are the kernel and cokernel of μ, and $K(-1)$ and $M(-1)$ are the modules K and M shifted by one degree (so $K(-1)_d = K_{d-1}$). K and L are both annihilated by x_n, and so are finitely generated over $k[x_1, \cdots, x_{n-1}]$. By the additivity of the Hilbert series, we have $\mathrm{HS}_{K(-1)}^S - \mathrm{HS}_{M(-1)}^S + \mathrm{HS}_M^S - \mathrm{HS}_L^S = 0$. Rearranging, this becomes $(1-t)\,\mathrm{HS}_M^S = \mathrm{HS}_L^S - t\,\mathrm{HS}_K^S$; the right-hand-side is a rational function of the form $\frac{p(t)}{(1-t)^{n-1}}$ by induction. $\square$

Now the t^d-coefficient of $\frac{1}{(1-t)^n}$ is $\binom{n+d-1}{n-1}$, which is a polynomial in d, with coefficients in $\frac{1}{(n-1)!}\mathbb{Z}$, for positive d. Similarly, the t^d coefficient of $\frac{t^k}{(1-t)^n}$ agrees with a polynomial for $d > k$. Thus, there exists a polynomial agreeing with the Hilbert function of M in sufficiently large degree:

Definition/Theorem 1.1.4 (Hilbert)**.** There exists a polynomial $\mathrm{HP}_M^S(d)$, with coefficients in $\frac{1}{(n-1)!}\mathbb{Z}$, called the *Hilbert polynomial of* M, such that, for all sufficiently large d, we have $\mathrm{HP}_M^S(d) = \mathrm{Hilb}_M^S(d)$.

If S is the polynomial ring $k[x_1, \cdots, x_n]$ and M is the coordinate ring of a projective variety $X \subset \mathbb{P}^{n-1}(k)$, then the degree of HP_M^S is the dimension of X, and $(n-1)!$ times the leading coefficient is its multiplicity. Proofs are found in many texts on commutative algebra (cf. [Ei, Chapter 12]).

It is thus a problem of considerable interest to classify the Hilbert functions of quotients of S by homogeneous ideals (or, equivalently, in view of the exact sequence $0 \to M \to S \to S/M \to 0$, to classify the Hilbert functions of homogeneous ideals $M \subset S$).

The first solution to this problem was given by Macaulay [Ma], who showed that every ideal $M \subset S = k[x_1, \cdots, x_n]$ has the same Hilbert function as some lexicographic ideal.

1.2 Lexicographic Ideals

Definition 1.2.1. Let $u = x_1^{e_1} \cdots x_n^{e_n}$ and $v = x_1^{f_1} \cdots x_n^{f_n}$ be two monomials of the same degree. We say that u is *lex-before* v, $u >_{\mathrm{Lex}} v$, if the following condition is satisfied:

Let i be minimal such that $e_i \neq f_i$. Then $e_i > f_i$.

Definition 1.2.2. Let $M \subset S$ be an ideal generated by monomials. We say that M is *lex* or *lexicographic* if the following condition is satisfied:

(1.2.3) Suppose that u and v are monomials of the same degree, with u lex-before v and $v \in M$. Then $u \in M$ as well.

Example 1.2.4. The ideal $(a^3, a^2b, a^2c, ab^3, ab^2c^2)$ is lexicographic in $S = k[a, b, c]$.

The notion of lex ideal defined above is sensible, in that an initial segment of the lex order in any degree generates a lex ideal:

Proposition 1.2.5. *Let $M \subset S_d$ be a vector space spanned by monomials, satisfying (1.2.3). Then $(x_1, \cdots, x_n)M \subset S_{d+1}$ is a vector space spanned by monomials, and satisfies (1.2.3).*

Proof. Let $v \in (x_1, \cdots, x_n)M$ be a monomial, so we may write $v = x_i v'$, with $v' \in M$, and suppose u is a degree-$(d+1)$ monomial lex-before v. Let x_j be the lex-latest monomial dividing u. Then $u' = \frac{u}{x_j}$ is lex-before v', and so $u' \in M$ and $u \in (x_1, \cdots, x_n)M$ as desired. □

Macaulay [Ma] proved that the lex ideals attain every Hilbert function in the polynomial ring:

Theorem 1.2.6 (Macaulay)**.** *Let $I \subset S = k[x_1, \cdots, x_n]$ be any homogeneous ideal. Then there exists a lex ideal L such that* $\mathrm{Hilb}_I^S = \mathrm{Hilb}_L^S$.

Macaulay's original proof is ten pages long and nearly unreadable. Many other proofs have been produced in the interim, most notably that of Green [Gr]. In theorems 3.4.1, 4.2.13, and 6.3.1, we obtain three more.

Macaulay's result is not limited to the polynomial ring, however. For example, it holds in the quotient $R = k[x_1, \cdots, x_n]/(x_1^2, \cdots, x_n^2)$ (where it is known as

Kruskal-Katona's theorem [Kr,Ka]), and in fact in any quotient by an ascending sequence of powers $R = k[x_1, \cdots, x_n]/(x_1^{a_1}, \cdots, x_n^{a_n})$, $a_1 \leq \cdots \leq a_n \leq \infty$ (where we use the convention that $a_i^\infty = 0$); this is known as Clements-Lindström's theorem [CL]. Chapters 3,4, and 5 are primarily concerned with identifying rings where Macaulay's theorem holds.

A popular characterization of Macaulay's theorem is a combinatorial inequality governing the growth of lex ideals known as the *Macaulay representation*, which we derive in theorem 2.2.8.

One of the most interesting currently open conjectures about Hilbert functions is the Eisenbud-Green-Harris conjecture [EGH], which posits that a generalization of Macaulay's theorem holds in the polynomial ring.

Conjecture 1.2.7 (Eisenbud-Green-Harris)**.** *Let* $f_1, \cdots, f_r \in S = k[x_1, \cdots, x_n]$ *be a regular sequence of homogeneous forms, with* $\deg f_i = e_i$ *and* $e_1 \leq \cdots \leq e_r$*. Put* $P = (x_1^{e_1}, \cdots, x_r^{e_r})$*. Let* I *be any ideal containing* $(f_1, \cdots, f_r)$*. Then there exists a lex ideal* L *such that* $\mathrm{Hilb}_{L+P}^S = \mathrm{Hilb}_I^S$*.*

(A *regular sequence* is a sequence that behaves like powers of the variables: f_i must be a non-zero-divisor in the quotient $S/(f_1, \cdots, f_{i-1})$ and $(f_1, \cdots, f_r)$ must not be the unit ideal. By dimension theory, $r \leq n$.)

The Eisenbud-Green-Harris conjecture is known to hold in only a very few special cases (cf. [FR]). In section 5.2 we prove that it holds when the f_i are all monomials.

1.3 Betti Numbers

Another important combinatorial invariant of a graded ideal is its Betti numbers, defined below.

Definition 1.3.1. Let $I \subset S$ be a homogeneous ideal. A *free resolution* of I is an exact sequence

$$\mathbb{F} : \cdots \to F_2 \to F_1 \to F_0 \to I \to 0,$$

with each F_i a free S-module. $\mathbb{F}$ is *minimal* if every F_i has minimum possible rank, and *graded* if each map is homogeneous of degree zero.

Finite free resolutions exist by the Hilbert syzygy theorem; the minimal free resolution is the (unique up to an isomorphism of complexes) resolution where the image of F_{i+1} in F_i is contained in $(x_1, \cdots, x_n)F_i)$ for all i; any resolution may be graded by inductively assigning a degree to the homogeneous generators of F_i.

Definition 1.3.2. Let $\mathbb{F}$ be the graded minimal free resolution of I, and write $F_i = \oplus S(-j)^{b_{i,j}}$. Then the $b_{i,j}$ are the *graded Betti numbers* of I.

Lex ideals are important to the study of Betti numbers because of the following theorem, due to Bigatti [Bi], Hulett [Hu], and Pardue [Pa], which states that lex ideals have maximal Betti numbers:

Theorem 1.3.3 (Bigatti, Hulett, Pardue)**.** *Let $I \subset S = k[x_1, \cdots, x_n]$ be any homogeneous ideal, and let L be the lexicographic ideal with the same Hilbert function as I. Then for all i, j, we have $b_{i,j}(L) \geq b_{i,j}(I)$.*

Aramova, Herzog, and Hibi [AHH] have shown an analogous result in the case of squarefree ideals:

Theorem 1.3.4 (Aramova, Herzog, Hibi). *Let $I \subset S$ be generated by squarefree monomials, and let $I' \subset R = S/(x_1^2, \cdots, x_n^2)$ be the image of I in the quotient by the squares of the variables. By Kruskal and Katona's theorem, there exists a lex ideal $L'$of R having the same the same Hilbert function as I'. Let $L \subset S$ be generated by the monomials of L' (so L is the squarefree lex ideal with the same Hilbert function as I). Then:*

1. *The Betti numbers of L' over R are greater than or equal to those of I'.*

2. *The Betti numbers of L over S are greater than or equal to those of I.*

(1) is widely conjectured to hold in every setting where Macaulay's theorem is known; there are counterexamples to the natural generalization of (2), arising from situations where L and I do not have the same Hilbert function.

One of the most interesting open problems in commutative algebra is the lex-plus-powers conjecture of Graham Evans, which extends the Eisenbud-Green-Harris conjecture to Betti numbers:

Conjecture 1.3.5 (Evans). *Let $f_1, \cdots, f_r \in S = k[x_1, \cdots, x_n]$ be a homogeneous regular sequence in increasing degrees $e_1 \leq \cdots \leq e_r$, and set $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. Let I be any homogeneous ideal containing $(f_1, \cdots, f_r)$, and suppose that there exists a lex ideal L such that the lex-plus-powers ideal $L + P$ has the same Hilbert function as I. Then $b_{i,j}(L + P) \geq b_{i,j}(I)$ for all i, j.*

The lex-plus-powers conjecture is known only in a few very special cases (cf. [FR]). In chapter 7, we prove it in the case that k has characteristic at most 2 and $f_1, \cdots, f_r = x_1^2, \cdots, x_n^2$.

Chapter 2

Preliminaries

2.1 Notation

Instead of the cumbersome notation $\mathrm{Hilb}_M^S(d)$ for the Hilbert function of a monomial ideal M, we will frequently write $|M|_d^S$, or, when the ring is understood, $|M_d|$ or $|M|_d$. When the degree is understood as well, we write simply $|M|$.

We frequently wish to consider the set of monomials appearing in an ideal M; this will consistently be denoted $\{M\}$.

If $M \subset S$ is a monomial ideal, and R is another ring containing the variables of S, then every monomial of S may be naturally associated to either zero or a unique monomial of R ($m \in S$ may be uniquely written in the form $m = \prod x_i^{e_i}$; we associate to this the product $\prod x_i^{e_i} \in R$). We will frequently wish to study the monomial ideal of R generated by the monomials of $\{M\}$; when we say "M considered as an ideal of R", this is what we mean. In particular, if $S = k[x_1, \cdots, x_n]$ and $R = k'[x_1, \cdots, x_n]$ where k and k' are different fields, the Hilbert function of M does not depend on whether it is considered as an ideal of S or of R. Since it is usually convenient to work over an infinite field of characteristic zero, we will often change fields in this way.

The homogeneous maximal ideal of S, $(x_1, \cdots, x_n)$, will be referred to simply as $\mathbf{m}$.

A *graded vector subspace* is a k-vector subspace of S which is spanned by homogeneous elements. It is a *monomial vector space* if it is spanned by monomials. A monomial vector space is *lex* if it satisfies property 1.2.3. The definition of the Hilbert function extends naturally to graded vector spaces.

The two most important graded vector spaces are $\mathbf{m}_1 = (x_1, \cdots, x_n)$ and M_d, for a homogeneous ideal M.

A graded vector space $V = \oplus V_d$, with $V_d = \{v \in V$ homogeneous of degree $d\}$, is an ideal if and only if we have $\mathbf{m}_1 V_d \subset V_{d+1}$ for all d.

Many statements about Hilbert functions of ideals in S may be recast in terms of monomial vector spaces. These restatements are always harder to read, but often easier to prove. For example, Macaulay's theorem may be restated as follows:

Theorem 2.1.1 (Macaulay)**.** *Let $V \subset S_d$ be a vector subspace, and let L be the vector subspace of S_d spanned by the lex-first $|V|$ monomials in degree d (so that L is a lex vector space). Then we have $|\mathbf{m}_1 L| \leq |\mathbf{m}_1 V|$.*

We routinely induct using the following ordering on the monomial vector spaces with a fixed Hilbert function:

Definition 2.1.2. Let $V = \oplus V_d$ and $W = \oplus W_d$ be monomial vector spaces with the same Hilbert function. We say that V is *lexicographically greater than* W if the following condition holds:

> Let d be minimal such that $V_d \neq W_d$, and order the monomials of V_d and W_d so that v_i is lex-before v_j (respectively, w_i lex-before w_j) whenever $i < j$. Let i be minimal such that $v_i \neq w_i$. Then v_i is lex-before w_i.

It is a simple consequence of Macaulay's theorem that there are only finitely many monomial ideals with a fixed Hilbert function. Because we will need it in our proofs, we prove the following weaker statement without reference to Macaulay's theorem:

Lemma 2.1.3. *"Lexicographically greater than" is a well-ordering on the set of monomial ideals with a fixed Hilbert function.*

Proof. Let $\mathcal{M}$ be any collection of monomial ideals all having the same Hilbert function F. We will show that $\mathcal{M}$ has a lexicographically greatest element.

For every degree D, let $F_{\leq D}$ be the truncation of F below D: $F_{\leq D}(d) = F(d)$ if $d \leq D$ and $F_{\leq D}(d) = 0$ if $d > D$. For each $M \in \mathcal{M}$, let $M_{\leq D}$ be the vector space spanned by the monomials of M which have degree at most D. Then the $M_{\leq D}$ all have Hilbert function $F_{\leq D}$. There are only finitely many monomials with degree at most d, hence there are only finitely many monomial vector spaces with this Hilbert function, and so the $M_{\leq D}$ have a lexicographically greatest element, which we denote $N_{\leq D}$.

Let N_D be the monomial ideal generated by $N_{\leq D}$. We have the infinite ascending chain $N_1 \subset N_2 \cdots \subset N_D \subset N_{D+1} \subset \cdots$, which must stabilize, say at N_p, by the Hilbert Basis Theorem. Then N_p has Hilbert function F, is in $\mathcal{M}$, and is the lexicographically greatest ideal of $\mathcal{M}$. $\square$

2.2 Tools

We introduce several tools that we use frequently. With the exception of compression (section 2.2.3), these are treated in considerably more detail in many texts on commutative algebra (cf. [Ei]).

2.2.1 Borel Ideals

Definition 2.2.1. We say that a monomial ideal M is *strongly stable* or *0-Borel fixed* if it satisfies the following condition:

Let $u \in M$ be a monomial, and suppose that x_j divides u and $i < j$. Then $\frac{x_i}{x_j}u \in M$.

These ideals are called 0-Borel fixed (or, by abuse, simply *Borel*) because, in characteristic zero, they are fixed by the action of the Borel group (of upper triangular matrices) on S. There is also a combinatorial characterization of p-Borel fixed ideals, describing those which are fixed by the action of the Borel group in characteristic p, but it is considerably more complex.

Because strongly stable ideals are combinatorially very well-behaved, their Hilbert functions are easier to work with than those of general ideals. For example, we have the following formula, due to Bigatti [Bi]:

Proposition 2.2.2 (Bigatti). *Let $S = k[x_1, \cdots, x_n]$, and $M \subset S$ be a strongly stable ideal. Then we have* $|\mathbf{m}_1 M_d| = \sum_{i=1}^{n} |M_d \cap k[x_1, \cdots, x_i]|$.

Proof. Since M is strongly stable, every monomial of $\mathbf{m}_1 M_d$ may be written uniquely in the form $x_i m$, where $m \in M_d$ and x_i is lex-later than (or equal to) every variable dividing m, and hence $m \in k[x_1, \cdots, x_i]$; similarly, every monomial of this form is in $\mathbf{m}_1 M_d$. $\square$

Thus, comparing the Hilbert function growth of Lex and Borel ideals reduces to computing intersections with these subrings; this is one of our primary activities in chapter 4.

Given any homogeneous ideal I, it is useful to show that there is a Borel ideal M with the same Hilbert function as I; the usual technique is to take a generic initial ideal.

2.2.2 Generic Initial Ideals

Definition 2.2.3. Let $\succ$ be any monomial order, and $I \subset S = k[x_1, \cdots, x_n]$ any homogeneous ideal. The ideal $\mathrm{in}(I)$ is the monomial ideal generated by the leading terms (with respect to $\succ$) of all elements of I: $\mathrm{in}(I) = (\{\mathrm{lt}(f) : f \in I\})$.

By Gröbner basis theory, $\mathrm{in}(I)$ has the same Hilbert function as I and larger graded Betti numbers.

Now, suppose that k is an infinite field of characteristic zero, and let $f : x_i \mapsto \sum a_{i,j}x_j$ be a linear change of coordinates on S. Then $f(I)$ has the same Hilbert function and Betti numbers as I, and we have the following (cf. [Ei, Chapter 15.9]):

Definition/Theorem 2.2.4. There exists an ideal $\mathrm{gin}(I)$, called the *generic initial ideal* of I, such that, for generic f, we have $\mathrm{gin}(I) = \mathrm{in}(f(I))$. Furthermore, $\mathrm{gin}(I)$ is strongly stable.

2.2.3 Compression

Compression is a technique for approaching the lex ideal by induction on the dimension n of S.

Its use dates back at least to Macaulay, who used $\{x_2, \cdots, x_n\}$-compression in the original proof of Macaulay's theorem [Ma]. The original proof of Clements-Lindström's theorem [CL] uses compression with respect to three sets of cardinality $n-1$; the name "compression" is due to Clements and Lindström.

We define compression here over the ring $S = k[x_1, \cdots, x_n]$, although the results of this section hold in somewhat more generality. In particular, the same proofs apply if we work over the ring $R = S/(x_1^{e_1}, \cdots, x_n^{e_n})$ (with $e_1 \leq \cdots \leq e_n$), where

Clements-Lindström's Theorem replaces Macaulay's Theorem.

Fix a proper subset $\mathcal{A}$ of the variables, $\mathcal{A} \subset \{x_1, \cdots, x_n\}$. By $k[\mathcal{A}]$ and $k[\mathcal{A}^c]$ we mean $k[x_i : x_i \in \mathcal{A}]$ and $k[x_i : x_i \notin \mathcal{A}]$, respectively.

Construction 2.2.5. Let M be a monomial ideal. Then M decomposes as a direct sum of vector spaces

$$M = \bigoplus_{\substack{f \in k[\mathcal{A}^c] \\ \text{monomial}}} f M_f,$$

where f runs over all monomials not involving the variables of $\mathcal{A}$ and each M_f is an ideal of $k[\mathcal{A}]$.

Definition/Construction 2.2.6. If every M_f is a lex ideal of $k[\mathcal{A}]$, we say that M is $\mathcal{A}$-*compressed*. By Macaulay's theorem, there exist lex ideals $T_f \subset k[\mathcal{A}]$ having the same Hilbert functions as the M_f. The vector space $T = \oplus f T_f$ is called the $\mathcal{A}$-*compression* of M.

Following Clements and Lindström, when $\mathcal{A} = \{x_1, \cdots, \hat{x}_i, \cdots, x_n\}$ is missing only the variable x_i, we sometimes say *i-compression* rather than $\mathcal{A}$-compression.

We have the following:

Lemma 2.2.7. *Let T be the $\mathcal{A}$-compression of M. Then:*

1. *T is an ideal of S.*

2. *T has the same Hilbert function as M.*

3. *T is lexicographically greater than M.*

Proof. (2) and (3) are immediate; we prove (1). It suffices to show that, for any x_i, we have $x_i f T_f \subset T$. If $x_i \in \mathcal{A}$, we have $x_i f T_f \subset f T_f$ because T_f is an ideal of $k[\mathcal{A}]$. If $x_i \notin \mathcal{A}$, we have $M_f \subset M_{x_i f}$ since M is an ideal; consequently,

$|T_f|_d = |M_f|_d \leq |M_{x_if}|_d = |T_{x_if}|_d$ for all d. Thus $T_f \subset T_{x_if}$ since these are lex ideals of $k[\mathcal{A}]$, and so $x_ifT_f \subset x_ifT_{x_if} \subset T$. $\square$

The properties of compression and compressed ideals are studied in detail in chapters 5 and 6. As a simple application, we use the decomposition of construction 2.2.5 to derive the Macaulay representation.

Theorem 2.2.8. *Let $L \subset S = k[x_1, \cdots, x_n]$ be a lex ideal.*

1. *If $|L|_d = \sum_{j=1}^{n-1} \binom{a_j}{n-j}$ with the a_j monotonically decreasing, then $|L|_{d+1} \geq \sum \binom{a_j+1}{n-j}$, with equality if L has no generators in degree $d+1$.*

2. *If $|S/L|_d = \sum_{j=0}^{d} \binom{a_j}{d-j}$ with the a_j monotonically decreasing, then $|S/L|_{d+1} \leq \sum \binom{a_j+1}{d-j+1}$, with equality if L has no generators in degree $d+1$.*

Proof. Take $\mathcal{A} = \{x_2, \cdots, x_n\}$, and write

$$\begin{aligned} L_d &= L_1 \oplus x_1 L_{x_1} \oplus \cdots \oplus x_1^d L_{x_1^d} \\ &= 0 \oplus x_1^i L_{x_1^i} \oplus x_1^{i+1}\mathbf{m}_{d-i-1}. \end{aligned}$$

Now $|x_1^{i+1}\mathbf{m}_{d+i-1}| = \binom{n+d-i-2}{n-1}$, and, by induction, $|x_1^i L_{x_1^i}| = \sum \binom{a_j}{n-j}$, with the a_j decreasing and less than $n+d-j-2$. If L has no generators in degree $d+1$, we have $L_{d+1} = x_1^i L_{x_1^i}(x_2, \cdots, x_n) \oplus x_1^{i+1}\mathbf{m}_{d-i}$. The second summand has dimension $\binom{n+d-i-1}{n-1}$; the first has dimension $\sum \binom{a_j+1}{n-j}$ by induction. This proves (1).

For (2), write $L_d = x_1 L'_{x_1} \oplus \cdots \oplus x_n L'_{x_n}$, with L'_{x_j} lex in $k[x_j, \cdots, x_n]$. Then we have $(S/L)_d = (k[x_i, \cdots, x_n]/L'_i)_{d-1} \oplus (k[x_{i+1} \cdots, x_n])_d$. Now $(k[x_{i+1} \cdots, x_n])_d$ has dimension $\binom{n+d-i-1}{d}$ and $(k[x_i, \cdots, x_n]/L'_i)_{d-1}$ has dimension $\sum \binom{a_j}{d-j}$ by induction. If L has no generators in degree $d+1$, we have $(S/L)_{d+1} = (k[x_i, \cdots, x_n]/L'_i)_d \oplus (k[x_{i+1} \cdots, x_n])_{d+1}$. The second summand has dimension $\binom{n+d+i}{d+1}$; the first has dimension $\sum \binom{a_j+1}{d-j}$ by induction. $\square$

Chapter 3

Lexifying Ideals*

3.1 Introduction

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i.

Let M be a monomial ideal. We say that a graded ideal in S/M is *lexifiable* if there exists a lexicographic ideal in S/M with the same Hilbert function. We call M and S/M *Macaulay-Lex* if every graded ideal in S/M is lexifiable. The following results are well known: Macaulay's Theorem [Ma] says that 0 is a Macaulay-Lex ideal, Kruskal-Katona's Theorem [Ka, Kr] says that $(x_1^2, \dots, x_n^2)$ is a Macaulay-Lex ideal, and Clements-Lindström's Theorem [CL] says that $(x_1^{e_1}, \dots, x_n^{e_n})$ is a Macaulay-Lex ideal if $e_1 \leq \dots \leq e_n \leq \infty$. These theorems are well-known and have many applications in Commutative Algebra, Combinatorics, and Algebraic Geometry.

It is easy to construct examples like Example 3.2.8, where problems occur in the degrees of the minimal generators of M. This motivated us to slightly weaken the definition: Let q be the maximal degree of a minimal monomial generator of M; we call M and S/M *pro-lex* if every graded ideal generated in degrees $\geq q$ in S/M is lexifiable. There exist examples of non pro-lex rings; see Example 3.3.15. The main goal in this chapter is to open a new direction of research along the lines

* This chapter is adapted with permission from the paper "Lexifying Ideals" by Jeff Mermin and Irena Peeva, which will appear in *Mathematical Research Letters*. It has been modified to fit this dissertation in the following ways: Some introductory material, which overlaps the introductory chapters of the thesis, has been deleted. Some minor adjustments have been made to the notation, and some typesetting has been changed in order to conform to the thesis guidelines.

of the following problem.

Problem 3.1.1. *Find classes of pro-lex monomial ideals.*

Theorem 3.5.1 shows that if M is Macaulay-Lex and N is lexicographic, then $M+N$ is Macaulay-Lex. Theorem 3.4.1 shows that if M is Macaulay-Lex, then it stays Macaulay-Lex after we add extra variables to the ring S. In Section 3.3 we prove:

Theorem 3.1.2. . *Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^\infty = 0$), and M be a compressed monomial ideal in S/P generated in degrees $\leq p$. If $n = 2$, assume that M is (S/P)-lex. Set $\Upsilon = S/(M+P)$. Then Υ is pro-lex above p, that is, for every graded ideal Γ in Υ generated in degrees $\geq p$ there exists an Υ-lex ideal Θ with the same Hilbert function.*

In the case when $M = P = 0$, Theorem 3.1.2 is Macaulay's Theorem [Ma]; in the case when $M = 0$, Theorem 3.1.2 is Clements-Lindström's Theorem [CL]. Examples 3.3.14 and 3.3.15 show that there are obstructions to generalizing Theorem3.1.2

We make use of ideas of Bigatti [Bi], Clements and Lindström [CL], and Green [Gr2]. Our proofs are algebraic, and we avoid computations using generic forms (used in [Gr2]) and combinatorial counting (used in [CL]). In Section 3.2 we introduce definitions and notation used throughout the chapter.

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3.2 Lexification

The notation in this section will be used throughout the chapter. We introduce several definitions.

Let k be a field and $S = k[x_1, \ldots, x_n]$ be graded by $\deg(x_i) = 1$ for all i. We denote by S_d the k-vector space spanned by all monomials of degree d. Denote $\mathbf{m} = (x_1, \ldots, x_n)_1$ the k-vector space spanned by the variables. We order the variables lexicographically by $x_1 > \cdots > x_n$, and we denote by $>_{lex}$ the homogeneous lexicographic order on the monomials. We say that an ideal is *p-generated* if it has a system of generators of degree p.

A monomial $x_1^{a_1} \ldots x_n^{a_n}$ has *exponent vector* $\mathbf{a} = (a_1, \ldots, a_n)$, and is sometimes denoted by $\mathbf{x}^{\mathbf{a}}$. An ideal is called *monomial* if it can be generated by monomials; such an ideal has a unique minimal system of monomial generators.

Notation 3.2.1. Let M be a monomial ideal. Set $\Upsilon = S/M$. Vector spaces in Υ (and sometimes ideals) are denoted by greek letters. For example, we denote by C_d a subspace of S_d, and we denote by τ_d a subspace of Υ_d.

Definition 3.2.2. The *Υ_d-lex-segment* $\lambda_{d,p}$ of length p in degree d is defined as the k-vector space spanned by the lexicographically first (greatest) p monomials in Υ_d. We say that λ_d is a *lex-segment in* Υ_d if there exists a p such that $\lambda_d = \lambda_{d,p}$. For a Υ_d-monomial space τ_d, we say that $\lambda_{d,|\tau_d|}$ is its *Υ_d-lexification.*

For simplicity, we sometimes say lex instead of Υ-lex if it is clear over which ring we work.

Example 3.2.3. The ideal $(a^2,\, ab,\, b^2)$ is lex in the ring $k[a, b, c, d]/(ac,\, ad)$, and its generators span a lex-segment. The k-vector space spanned by a^2, ab, b^2 is the lexification of the k-vector space spanned by b^2, c^2, cd. However, the ideal is not lex in $k[a, b, c, d]$.

Proposition 3.2.4. *If τ_d is an Υ_d-lex-segment, then $\mathbf{m}\tau_d$ is an Υ_{d+1}-lex-segment.*

Definition 3.2.5. We say that an Υ_d-monomial space τ_d is Υ_d-*lexifiable* if its lexification λ_d has the property that $|\mathbf{m}\lambda_d| \leq |\mathbf{m}\tau_d|$. The monomial ideal M and the quotient ring $\Upsilon = S/M$ are called *d-pro-lex*, if every Υ_d-monomial space is Υ_d-lexifiable.

Definition 3.2.6. We say that a graded ideal R in Υ is *lexifiable* if there exists an Υ-lex ideal with the same Hilbert function as R. The monomial ideal M and the quotient ring $\Upsilon = S/M$ are called *Macaulay-Lex* if every graded ideal in Υ is lexifiable.

Example 3.2.7. This example shows that the order of the variables can make a difference. The ideal (ab) is not lexifiable in the ring $k[a,b]/(ab^2)$ for the lex order with $a > b$, but it is lexifiable for the lex order with $b > a$.

Example 3.2.8. The ideal (ab) is not lexifiable in the ring $k[a,b]/(a^2b,\ ab^2)$ in any lex order.

It is easy to construct many examples like Example 3.2.8. This observation suggests that in order to obtain positive results we need to slightly relax Definition 3.2.6:

Definition 3.2.9. Let q be the maximal degree of a minimal monomial generator of M. The monomial ideal M and the quotient ring $\Upsilon = S/M$ are called *pro-lex* if every graded ideal generated in degrees $\geq q$ in Υ is lexifiable.

In the examples we usually denote the variables by a, b, c, d for simplicity.

3.3 Compression

The following definition generalizes a definition introduced by Clements and Lindström [CL], who used it over a quotient of a polynomial ring modulo pure powers of the variables.

Definition 3.3.1. Let E be a monomial ideal in S. A $(S/E)_d$-monomial space τ_d is called *i-compressed* (or i-compressed in $(S/E)_d$) if it is $\{x_1, \cdots, \hat{x}_i, \cdots, x_n\}$-compressed in the sense of Definition 2.2.6 We say that a k-vector space τ_d is *$(S/E)_d$-compressed* (or compressed) if it is a $(S/E)_d$-monomial space and is i-compressed for all $1 \leq i \leq n$. A monomial ideal T in S/E is called *compressed* if T_d is compressed for all $d \geq 0$.

Example 3.3.2. The ideal

$$(a^3,\ a^2b,\ a^2c,\ ab^2,\ abc,\ b^3,\ b^2c)$$

is compressed in the ring $k[a, b, c]$.

Lemma 3.3.3. *If τ_d is i-compressed in $(S/E)_d$, then $\mathbf{m}\tau_d$ is i-compressed in $(S/E)_{d+1}$. If τ_d is $(S/E)_d$-lex, then it is $(S/E)_d$-compressed.*

Definition 3.3.4. A S-monomial ideal K is called *compressed-plus-powers* if $K = M + P$, where $P = (x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ and the monomial ideal M is compressed in S/P. Sometimes, when we need to be more precise, we say that K is *compressed-plus-P*. Furthermore, we say that K is *lex-plus-P* if M is lex in S/P.

Notation 3.3.5. Throughout this section we use the following notation and make the following assumptions:

- $P = (x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$
- The ideal $K = M + P$ is a compressed-plus-P monomial ideal in S; here M is compressed in S/P.

- If $n = 2$ we assume in addition that K is lex-plus-P.

- We assume that M is p-generated.

- Set $\Upsilon = S/K$.

- d is a degree such that $d \geq p$.

For a $(S/P)_d$-monomial space A_d set

$$\begin{aligned} t_i(A_d) &= \Big| \{ m \in \{A_d\} \mid \max(m) \leq i \} \Big| \\ s_i(A_d) &= \Big| \{ m \in \{A_d\} \mid \max(m) = i \ \text{ and } \ x_i^{e_i - 1} \text{ divides } m \} \Big| \\ r_{i,j}(A_d) &= \Big| \{ m \in \{A_d\} \mid \max(m) \leq i \ \text{ and } \ x_i^j \text{ does not divide } m \} \Big| . \end{aligned}$$

The formula in the following lemma is a generalization of a formula introduced by Bigatti [Bi], who used it for S-Borel ideals.

Lemma 3.3.6. *Let A_d be a $(S/P)_d$-monomial space.*

(1) If A_d is compressed and $n \geq 3$, then A_d is $(S/P)_d$-Borel.

(2) If A_d is $(S/P)_d$-Borel, then

$$\Big| \{\mathbf{m}A_d\} \Big| = \sum_{i=1}^{n} t_i(A_d) - s_i(A_d) = \sum_{i=1}^{n} r_{i,e_i - 1}(A_d) \, .$$

Proof. First, we prove (1). Let $m \in \{A_d\}$ and m' be a $(S/P)_d$–monomial in its big shadow. Hence $m' = \dfrac{x_i \, m}{x_j}$ for some x_j dividing m and some $i \leq j$. There exists an index $1 \leq q \leq n$ such that $q \neq i, j$. Note that that m and m' have the same q-exponents. Since A_d is q-compressed and $m' >_{lex} m$, it follows that $m' \in \{A_d\}$. Therefore, A_d is $(S/P)_d$–Borel.

Now, we prove (2). We will show that $\{\mathbf{m}A_d\}$ is equal to the set

$$\coprod_{i=1}^{n} x_i \,\{\, m \in \{A_d\} \,|\, \max(m) \le i \,\} \setminus \coprod_{i=1}^{n} x_i \,\{\, m \in \{A_d\} \,|\, \max(m) = i \text{ and } x_i^{e_i - 1} \text{ divides } m \,\}.$$

Denote by $\mathcal{P}$ the set above. Let $w \in A_d$. For $j \ge \max(w)$ we have that $x_j w \in \mathcal{P}$. Let $j < \max(w)$. Then $v = x_j \dfrac{w}{x_{\max(w)}} \in A_d$. So, $x_j w = x_{\max(w)} v \in \mathcal{P}$. $\square$

Lemma 3.3.7 is a generalization of a result by M. Green [Gr2], who proved a particular case of it it over a polynomial ring (in the case $M = 0$). Green's proof is entirely different than ours; he makes a computation with generic linear forms. It is not clear how to apply his computation to the case $M \ne 0$.

Lemma 3.3.7. *Let τ_d be an n-compressed Borel Υ_d-monomial space, and let λ_d be a lex-segment in Υ_d with $|\{\lambda_d\}| \le |\{\tau_d\}|$. Let L_d and T_d be the $(S/P)_d$-monomial spaces such that $\{L_d\} = \{\lambda_d\} \coprod \{M_d\}$ and $\{T_d\} = \{\tau_d\} \coprod \{M_d\}$. For each $1 \le i \le n$ and each $1 \le j \le e_i$ we have*

$$r_{i,j}(L_d) \le r_{i,j}(T_d)\,.$$

Proof. Set $R = S/P$. By Lemma 3.3.6, M_d is R_d-Borel. Therefore, both L_d and T_d are R_d-Borel and n-compressed.

First, we consider the case $i = n$. Clearly, $r_{n,e_n}(L_d) = |L_d| = |T_d| = r_{n,e_n}(T_d)$ (if $e_n = \infty$, then we consider $r_{n,d+1}$ here). We induct on j decreasingly. Suppose that $r_{i,j+1}(L_d) \le r_{i,j+1}(T_d)$ holds by induction.

If $\{T_d\}$ contains no monomial divisible by x_n^j then

$$r_{i,j}(L_d) \le r_{i,j+1}(L_d) \le r_{i,j+1}(T_d) = r_{i,j}(T_d)\,.$$

Suppose that $\{T_d\}$ contains a monomial divisible by x_n^j. Denote by $e = x_1^{b_1} \ldots x_n^{b_n}$, with $b_n \ge j$, the lex-smallest monomial in T_d that is divisible by x_n^j. Let $0 \le$

$q \leq j-1$. Since T_d is R_d-Borel, it follows that $c_q = x_{n-1}^{b_n - q} \dfrac{e}{x_n^{b_n - q}} \in T_d$. This is the lex-smallest monomial that is lex-greater than e and x_n divides it at power q. Let the monomial $a = x_1^{a_1} \dots x_{n-1}^{a_{n-1}} x_n^q \in R_d$ be lex-greater than e. Since T_d is n-compressed and a is lex-greater (or equal) than c_q, it follows that $a \in T_d$.

For a monomial u, we denote by $x_n \notin u$ the property that x_n^j does not divide u. By what we proved above, it follows that

$$\Big|\{u \in \{T_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}\big| = \big|\{u \in \{R_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}\Big| . \tag{3.3.8}$$

Therefore,

$$\begin{aligned}
r_{i,j}(L_d) &= |\{u \in \{L_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}| + |\{u \in \{L_d\} \,|\, x_n \notin u,\ u <_{lex} e\,\}| \\
&\leq |\{u \in \{R_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}| + |\{u \in \{L_d\} \,|\, x_n \notin u,\ u <_{lex} e\,\}| \\
&\leq |\{u \in \{R_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}| + |\{u \in \{L_d\} \,|\, u <_{lex} e\,\}| \\
&\leq |\{u \in \{R_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}| + |\{u \in \{T_d\} \,|\, u <_{lex} e\,\}| \\
&= |\{u \in \{R_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}| + |\{u \in \{T_d\} \,|\, x_n \notin u,\ u <_{lex} e\,\}| \\
&= |\{u \in \{T_d\} \,|\, x_n \notin u,\ u >_{lex} e\,\}| + |\{u \in \{T_d\} \,|\, x_n \notin u,\ u <_{lex} e\,\}| \\
&= r_{i,j}(T_d)\,;
\end{aligned}$$

for the third inequality we used the fact that λ_d is a lex-segment in Υ_d with $|\{\lambda_d\}| \leq |\{\tau_d\}|$; for the equality after that we used the definition of e; for the next equality we used (3.3.8). Thus, we have the desired inequality in the case $i = n$.

In particular, we proved that

$$t_{n-1}(L_d) = r_{n,1}(L_d) \leq r_{n,1}(T_d) = t_{n-1}(T_d)\,. \tag{3.3.9}$$

Finally, we prove the lemma for all $i < n$. Both $\{\tau_d/x_n\}$ and $\{\lambda_d/x_n\}$ are lex-segments in Υ_d/x_n since τ_d is n-compressed. By (3.3.9) the inequality $t_{n-1}(L_d) \leq$

$t_{n-1}(T_d)$ holds, and it implies the inclusion $\{\tau_d/x_n\} \supseteq \{\lambda_d/x_n\}$. The desired inequalities follow since

$$r_{i,j}(T_d) = r_{i,j}\big(T_d/(x_{i+1},\ldots,x_n)\big) = r_{i,j}\big(\{\tau_d/(x_{i+1},\ldots,x_n)\}\coprod\{M_d/(x_{i+1},\ldots,x_n)\}\big)$$
$$r_{i,j}(L_d) = r_{i,j}\big(L_d/(x_{i+1},\ldots,x_n)\big) = r_{i,j}\big(\{\lambda_d/(x_{i+1},\ldots,x_n)\}\coprod\{M_d/(x_{i+1},\ldots,x_n)\}\big)$$

□

Lemma 3.3.10. *Let υ_d be a Υ_d-monomial space. There exists a compressed monomial space τ_d in Υ_d such that $|\tau_d| = |\upsilon_d|$ and $|\mathbf{m}\tau_d| \le |\mathbf{m}\upsilon_d|$.*

Proof. Suppose that υ_d is not i-compressed. Set $z = x_i$. Since M is z-compressed in S/P, we have the disjoint union

$$\{M_d\} = \coprod_{0\le j\le d} z^{d-j}\{N_j\}\,,$$

where each N_j is a $(S/(z,P))_j$-lex-segment.

We also have the disjoint union

$$\{\upsilon_d\} = \coprod_{0\le j\le d} z^{d-j}\{\nu_j\}$$

where each ν_j is a monomial space in $S/(z,P,N_j)$. Let γ_j be the lexification of the space ν_j in $S/(z,P,N_j)$. Consider the Υ_d-monomial space τ_d defined by

$$\{\tau_d\} = \coprod_{0\le j\le d} z^{d-j}\{\gamma_j\}\,.$$

Clearly, $|\tau_d| = |\upsilon_d|$.

Consider the $(S/P)_d$-monomial spaces V_d and T_d such that

$$\{V_d\} = \{\upsilon_d\}\coprod\{M_d\} \quad\text{and}\quad \{T_d\} = \{\tau_d\}\coprod\{M_d\}\,.$$

Set $R = S/P$. The short exact sequence of k-vector subspaces of $(S/P)_{d+1}$

$$0 \to \mathbf{m}M_d \to \mathbf{m}T_d \longrightarrow \mathbf{m}T_d/\mathbf{m}M_d \;=\; \mathbf{m}\tau_d/\big(\mathbf{m}\tau_d \cap \mathbf{m}M_d\big) \to 0$$

shows that $|\mathbf{m}\tau_d| = |\mathbf{m}T_d| - |\mathbf{m}M_d|$ (here we mean $|\mathbf{m}\tau_d|^{\Upsilon} = |\mathbf{m}T_d|^{S/P} - |\mathbf{m}M_d|^{S/P}$). Similarly, the short exact sequence of k-vector subspaces of $(S/P)_{d+1}$

$$0 \to \mathbf{m}M_d \to \mathbf{m}V_d \longrightarrow \mathbf{m}V_d/\mathbf{m}M_d \;=\; \mathbf{m}\upsilon_d/\big(\mathbf{m}\upsilon_d \cap \mathbf{m}M_d\big) \to 0$$

shows that $|\mathbf{m}\upsilon_d| = |\mathbf{m}V_d| - |\mathbf{m}M_d|$. Therefore, the desired inequality $|\mathbf{m}\tau_d| \leq |\mathbf{m}\upsilon_d|$ is equivalent to the inequality

$$|\mathbf{m}T_d| \leq |\mathbf{m}V_d| \,.$$

We will prove the latter inequality.

We have the disjoint unions

$$\{V_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{U_j\} \quad \text{and} \quad \{T_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{F_j\}\,, \quad \text{where}$$

$$\{U_j\} = \{\nu_j\}\coprod\{N_j\} \quad \text{and} \quad \{F_j\} = \{\gamma_j\}\coprod\{N_j\} \quad \text{in the ring } S/(z,P)\,.$$

Note that each F_j is a $(S/(z,P))_j$-lex-segment. Furthermore, we have the disjoint unions

$$\{\mathbf{m}V_d\} = \coprod_{0\leq j\leq d} z^{d-j+1}\{U_j + \mathbf{n}U_{j-1}\}$$

$$\{\mathbf{m}T_d\} = \coprod_{0\leq j\leq d} z^{d-j+1}\{F_j + \mathbf{n}F_{j-1}\}\,,$$

where $\mathbf{n} = \mathbf{m}/z$. We will show that

$$|F_j + \mathbf{n}F_{j-1}| = \max\Big\{|F_j|,\, |\mathbf{n}F_{j-1}|\Big\} \leq \max\Big\{|U_j|,\, |\mathbf{n}U_{j-1}|\Big\} \leq |U_j + \mathbf{n}U_{j-1}|\,.$$

The first equality above holds because both F_j and $\mathbf{n}F_{j-1}$ are $(S/(z,P))_j$-lex-segments, so $F_j + \mathbf{n}F_{j-1}$ is the longer of these two lex-segments. The last inequality is obvious. It remains to prove the middle inequality. Using the short exact sequences of k-vector subspaces of $(S/P)_j$

$$0 \to \mathbf{n}N_{j-1} \to \mathbf{n}F_{j-1} \longrightarrow \mathbf{n}F_{j-1}/\mathbf{n}N_{j-1} \;=\; \mathbf{n}\gamma_{j-1}/\big(\mathbf{n}\gamma_{j-1} \cap \mathbf{n}N_{j-1}\big) \to 0$$

$$0 \to \mathbf{n}N_{j-1} \to \mathbf{n}U_{j-1} \longrightarrow \mathbf{n}U_{j-1}/\mathbf{n}N_{j-1} \;=\; \mathbf{n}\nu_{j-1}/\big(\mathbf{n}\nu_{j-1} \cap \mathbf{n}N_{j-1}\big) \to 0$$

we get $|\mathbf{n}\gamma_{j-1}| = |\mathbf{n}F_{j-1}| - |\mathbf{n}N_{j-1}|$ and $|\mathbf{n}\nu_{j-1}| = |\mathbf{n}U_{j-1}| - |\mathbf{n}N_{j-1}|$. Therefore, the desired inequality $|\mathbf{n}F_{j-1}| \leq |\mathbf{n}U_{j-1}|$ is equivalent to the inequality $|\mathbf{n}\gamma_{j-1}| \leq |\mathbf{n}\nu_{j-1}|$. The latter inequality holds since by construction γ_{j-1} is the lexification of ν_{j-1}, so $|\gamma_{j-1}| = |\nu_{j-1}|$ and by induction on the number of variables we can apply Theorem 3.3.11 to the ring $S/(z, P, N_j)$.

Thus, $|F_j + \mathbf{n}F_{j-1}| \leq |U_j + \mathbf{n}U_{j-1}|$. Multiplication by z^{d-j+1} is injective if $d - j + 1 \leq e_i - 1$ and is zero otherwise, therefore we conclude that

$$\left| z^{d-j+1}(F_j + \mathbf{n}F_{j-1}) \right| \leq \left| z^{d-j+1}(U_j + \mathbf{n}U_{j-1}) \right|.$$

This implies the desired inequality $|\mathbf{m}T_d| \leq |\mathbf{m}V_d|$.

Note that $\{\tau_d\}$ is greater lexicographically than $\{\upsilon_d\}$. If τ_d is not compressed, we can apply the argument above. After finitely many steps in this way, the process must terminate because at each step we construct a lex-greater monomial space. Thus, after finitely many steps, we reach a compressed monomial space. $\square$

Theorem 3.3.11. *Let υ_d be a Υ_d-monomial space and λ_d be its lexification in Υ_d. Then $|\,\mathbf{m}\lambda_d\,| \leq |\,\mathbf{m}\upsilon_d\,|$.*

Proof. The theorem clearly holds if $n = 1$. Suppose that $n = 2$. An easy calculation shows that the theorem holds, provided we do not have $e_2 \leq d + 1 < e_1$. By the assumption on the ordering of the exponents, this does not hold and we are fine.

Suppose that $n \geq 3$. First, we apply Lemma 3.3.10 to reduce to the compressed case. We obtain a compressed Υ_d-monomial space τ_d such that $|\tau_d| = |\upsilon_d|$ and $|\mathbf{m}\tau_d| \leq |\mathbf{m}\upsilon_d|$. Let L_d and T_d be the $(S/P)_d$-monomial spaces such that $\{L_d\} = \{\lambda_d\} \cup \{M_d\}$ and $\{T_d\} = \{\tau_d\} \cup \{M_d\}$, where the disjoint unions take place in S/P. Both L_d and T_d are $(S/P)_d$-compressed. We apply Lemma 3.3.6 to conclude that

$$\Big|\{\mathbf{m}T_d\}\Big| = \sum_{i=1}^{n} t_i(T_d) - \sum_{i=1}^{n} s_i(T_d) \quad \text{and} \quad \Big|\{\mathbf{m}L_d\}\Big| = \sum_{i=1}^{n} t_i(L_d) - \sum_{i=1}^{n} s_i(L_d)\,.$$

Finally, we apply Lemma 3.3.7 and conclude that $\big|\{\mathbf{m}L_d\}\big| \leq \big|\{\mathbf{m}T_d\}\big|$. This inequality and short exact sequences, as in the proof of Lemma 3.3.10, imply the desired $|\mathbf{m}\lambda_d| \leq |\mathbf{m}\upsilon_d|$. $\square$

Equivalently, we obtain the following theorem, stated in the introduction:

Theorem 3.3.12. *Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^{\infty} = 0$), and M be a compressed monomial ideal in S/P generated in degrees $\leq p$. If $n = 2$, assume that M is (S/P)-lex. Set $\Upsilon = S/(M + P)$. Then Υ is pro-lex above p, that is, for every graded ideal Γ in Υ generated in degrees $\geq p$ there exists an Υ-lex ideal Θ with the same Hilbert function.*

Proof. We can assume that Γ is a monomial ideal by Gröbner basis theory. For each $d \geq p$, let λ_d be the lexification of Γ_d. By Theorem 3.3.11, it follows that $\Theta = \oplus_{d \geq p} \lambda_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as Γ in all degrees greater than or equal to p. $\square$

Remark 3.3.13. In the case when $M = P = 0$, Theorem 3.3.12 is the well-known Macaulay's Theorem [Ma]. In the case $M = 0$, Theorem 3.3.12 is the Clements-Lindström's Theorem [CL].

Example 3.3.14. It is natural to ask if a compressed ideal is Macaulay-Lex. This example shows that the answer is negative. Take $P = 0$. The ideal

$$M = (\, a^3,\ a^2b,\ a^2c,\ ab^2,\ abc,\ b^3,\ b^2c\,)$$

is compressed (and Borel) in the ring $k[a, b, c]$. The ideal (a^2, ab, b^2) in $k[a, b, c]/M$ is not lexifiable. □

Example 3.3.15. It is natural to ask if Theorem 3.3.12 holds in the case when M is a S-Borel ideal. It does not. Take $P = 0$. The ideal

$$M = (\, a^3,\, a^2b,\, a^2c,\, a^2d,\, ab^2,\, abc,\, abd,\, b^3,\, b^2c\,)$$

is Borel in the ring $k[a, b, c, d]$. However it is not pro-lex because the ideal (b^2d) is not lexifiable in $k[a, b, c, d]/M$. □

3.4 Adding new variables

Theorem 3.4.1. *If S/M is Macaulay-Lex then $S[y]/M$ is Macaulay-Lex.*

In this section, $W = S[y]/M$, $\mathbf{m}$ is the k-vector space spanned by the variables in S (as in Section 3.2), and $\mathbf{q}$ is the k-vector space spanned by $\mathbf{m}$ and y.

Lemma 3.4.2. *Let V_d be a W_d-monomial space, and let T_d be its y-compression. Then $|T_d| = |V_d|$ and $|\mathbf{q}T_d| \le |\mathbf{q}V_d|$.*

Proof. The proof is based on the same idea as the proof of Lemma 3.3.10. We write $\{V_d\} = \coprod_{0\le j\le d} y^{d-j}\{U_j\}$ and $T_d = \coprod_{0\le j\le d} y^{d-j}\{F_j\}$, where the F_j are S/M-lex satisfying $|F_j| = |U_j|$. Then, as in the proof of Lemma 3.3.10, we have the disjoint unions

$$\{\mathbf{q}V_d\} = \coprod_{0\leq j\leq d} y^{d-i+1}\{U_j + \mathbf{m}U_{j-1}\}$$

$$\{\mathbf{q}T_d\} = \coprod_{0\leq j\leq d} y^{d-i+1}\{F_i + \mathbf{m}F_{j-1}\},$$

and we have the inequalities

$$|F_i + \mathbf{m}F_{j-1}| = \max\{|F_j|, |\mathbf{m}F_{j-1}|\} \leq \max\{|U_j|, |\mathbf{m}U_{j-1}|\} \leq |U_j + \mathbf{m}U_{j-1}|,$$

where the middle inequality holds because S/M is Macaulay-Lex. Since multiplication by y is injective, we get

$$|y^{d-i+1}(F_i + \mathbf{m}F_{j-1})| \leq |y^{d-i+1}(U_j + \mathbf{m}U_{j-1})|. \qquad \square$$

Lemma 3.4.3. *Let T_d be a y-compressed W_d-monomial space. Then either T_d is W_d-lex, or there exists a W_d-monomial space F_d, such that F_d is strictly lexicographically greater than T_d, $|F_d| = |T_d|$, and $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$.*

Proof. Let r be as large as possible among the numbers for which we can write

$$T_d = y^{d-r}P \oplus \left(\bigoplus_{i>r} y^{d-i}L_i\right)$$

with P a lex segment of W_d. Such an r always exists, as we can if necessary take $r = 0$.

If $r = d$, then T_d is W_d-lex and we are done. If not, then $yP + L_{r+1}$ is not lex in W. Let m be the lex-greatest monomial of W_{r+1} such that $m \notin yP + L_{r+1}$. We consider two cases depending on whether y divides m or not.

Suppose that y divides m. Let u be the lex-least monomial of $yP + L_{r+1}$. Since P is lex and y does not divide m, it follows that y does not divide u. Let Q be the

k-vector space spanned by $\{Q\}$, defined by

$$\{Q\} = \Big(\{yP\} \cup \{L_{r+1}\} \cup \{m\}\Big) \setminus \{u\}\,.$$

Set

$$F_d = y^{d-r-1}Q \oplus \left(\bigoplus_{i>r+1} y^{d-i}L_i\right).$$

Now, $\{F_d\}\setminus y^{d-r-1}m = \{T_d\}\setminus y^{d-r-1}u$. Hence, $\{F_d\}$ is strictly lexicographically greater than T_d. We will compare $\{\mathbf{q}F_d\}$ and $\{\mathbf{q}T_d\}$. The set $\{\mathbf{m}y^{d-r-1}m\}$ is contained in $\{\mathbf{q}T_d\}$, so we have $\mathbf{q}F_d \setminus (\mathbf{q}F_d \cap \mathbf{q}T_d) \subseteq \{y^{d-r}m\}$. Furthermore, we will show that $y^{d-r}u \notin \{\mathbf{q}F_d\}$. Suppose the opposite. Hence, there exists a q such that $y^{d-r}u = x_q\left(y^{d-r}\frac{u}{x_q}\right)$, where $\frac{u}{x_q} \in P$. But $y\frac{u}{x_q} \in yP$ is lex-smaller than u; this contradicts the choice of u. Hence $\{\mathbf{q}T_d\} \setminus (\mathbf{q}F_d \cap \mathbf{q}T_d) \supseteq \{y^{d-r}u\}$. Therefore, we have the desired inequality $|\mathbf{q}F_d| \leq |\mathbf{q}T_d|$. Thus, the lemma is proved in this case.

It remains to consider the case when m is not divisible by y. In this case, m is the lex-greatest monomial not divisible by y that is lex-smaller than all the monomials in $\{L_{r+1}\}$. Set $z = x_{\max(m)}$. In our construction we will use the set

$$N = \left\{u \in yP \,\middle|\, u <_{lex} m \text{ and } \left(\frac{z}{y}\right)^{e_u} u \neq 0 \text{ in } B/M\right\},$$

where e_u is the largest power of y dividing u. We will show that $N \neq \emptyset$ because $\frac{y}{z}m \in N$. Since m is the lex-greatest monomial missing in $m \notin yP + L_{r+1}$, it follows that there exists a monomial $ym' \in yP$ that is lex-smaller than m. Therefore, m' is (non-strictly) lex-smaller than $\frac{m}{z}$. As $m' \in P$ and P is lex, it follows that $\frac{m}{z} \in P$. Thus, $\frac{y}{z}m \in N$ as desired.

We will need three of the properties of N:

Claim.

(1) m is (non-strictly) lex-greater than all the monomials in $\dfrac{z}{y}N$.

(2) $\dfrac{z}{y}N \cap \{L_{r+1}\} = \emptyset$.

(3) $\dfrac{z}{y}N \cap \{yP\} \subseteq N$.

We will prove the claim. (3) is clear. (2) follows from (1) and the fact that in the considered case m is the lex-greatest monomial not divisible by y that is lex-smaller than all the monomials in $\{L_{r+1}\}$. We will prove (1). Write

$$m = x_1^{a_1} x_2^{a_2} \dots z^{a_z} \quad \text{and} \quad u = x_1^{b_1} x_2^{b_2} \dots z^{b_z} w y^{b_y}\,,$$

where w is not divisible by $x_1, \dots, z$ or by y. Suppose that $\frac{z}{y}u = x_1^{b_1} x_2^{b_2} \dots z^{b_z+1} w y^{b_y-1}$ is lex-greater than m. On the other hand, m is lex-greater than u. It follows that $a_j = b_j$ for $j < \max(m)$ and $b_z < a_z \leq b_z + 1$. Since the monomials have the same degree, it follows that $a_z = b_z + 1$, $w = 1$, and $b_y = 1$. Hence $m = \frac{z}{y}u$. The claim is proved.

Let Q be the k-vector space such that

$$\{Q\} = \Big(\{yP + L_{r+1}\} \setminus N\Big) \; \cup \frac{z}{y}N\,.$$

By the claim above, it follows that we have the disjoint union $\{Q\} = \{L_{r+1}\} \coprod yP \setminus N \coprod \dfrac{z}{y}N$. Clearly, $|Q| = |L_{r+1} \oplus yP|$.

We consider the set

$$F_d = y^{d-r-1}Q \oplus \left(\bigoplus_{i>r+1} y^{d-i} L_i \right).$$

It is clear that $|F_d| = |T_d|$. Since $y^{d-r-1}m \in F_d$, we see that F_d is strictly lexicographically greater than T_d. We will show that the inequality $|\mathbf{q}F_d| \le |\mathbf{q}T_d|$ holds. Set $U = L_{r+1} \oplus yP$ and $V = \oplus_{i>r+1} y^{d-i} L_i$.

Since

$$|\mathbf{q}Q| - |\mathbf{q}U| = -\left|\left\{\ t \in \mathbf{q}N \setminus (\mathbf{q}N \cap \mathbf{q}(U \setminus N)\ \Big|\ \frac{z}{y}t = 0\ \right\}\right| \le 0)$$

it follows that $|\mathbf{q}Q| \le |\mathbf{q}U|$. Furthermore, we have

$$\begin{aligned}
|\mathbf{q}F_d| &= |\mathbf{q}\, y^{d-r-1}Q| + |\mathbf{q}V| - |\mathbf{q}V \cap \mathbf{q}y^{d-r-1}Q| \\
&= |\mathbf{q}\, y^{d-r-1}Q| + |\mathbf{q}V| - \left|y^{d-r-1}\big(L_{r+2} \cap \mathbf{m}\{v \in Q \big| y \text{ does not divide } v\}\big)\right| \\
&\le |\mathbf{q}\, y^{d-r-1}U| + |\mathbf{q}V| - \left|y^{d-r-1}\big(L_{r+2} \cap \mathbf{m}\{v \in Q \big| y \text{ does not divide } v\}\big)\right| \\
&\le |\mathbf{q}\, y^{d-r-1}U| + |\mathbf{q}V| - \left|y^{d-r-1}\big(L_{r+2} \cap \mathbf{m}\{v \in U \big| y \text{ does not divide } v\}\big)\right| \\
&= |\mathbf{q}T_d|;
\end{aligned}$$

the first inequality holds because multiplication by y is injective, the second holds by set containment. □

Proof of Theorem 3.4.1: Let V_d be a W_d-monomial space. If V_d is not W-lex, apply Lemmas 3.4.2 and 3.4.3 to obtain a y-compressed W_d-monomial space F_d which is strictly greater lexicographically than V_d and satisfies $|F_d| = |V_d|$ and $|\mathbf{q}F_d| \le |\mathbf{q}V_d|$. If F_d is not W-lex, we can apply the lemmas again. After finitely many steps, the process must terminate in a lexicographic monomial space. Hence W is d-pro-lex for all degrees $d \ge 0$, and so is Macaulay-Lex. □

3.5 Lexicographic quotients

Theorem 3.5.1. *If M is Macaulay-Lex and N is a S/M-lex ideal, then $M + N$ is Macaulay-Lex.*

The theorem follows immediately from the following result:

Proposition 3.5.2. *Fix a degree* $d \geq 1$*. If* M *is* $(d-1)$ *-pro-lex and* N *is a* S/M*-lex ideal, then* $M+N$ *is* $(d-1)$ *-pro-lex.*

Proof. Throughout this proof, for a monomial space $\bar{V}$ in $S/(M+N)$, we denote by V the k-vector space spanned by $\{\bar{V}\}$ in S/M.

Let $\bar{W}_{d-1}$ be a monomial space in $\Big(S/(M+N)\Big)_{d-1}$. Let $\bar{L}_{d-1}$ be the $S/(M+N)$-lexification of $\bar{W}_{d-1}$. Set $\bar{L}_d$ to be the k-vector space spanned by $\mathbf{m}\{\bar{L}_{d-1}\}$ and $\bar{W}_d$ be the k-vector space spanned by $\mathbf{m}\{\bar{W}_{d-1}\}$. We will prove that

$$|\bar{L}_d|^{S/(M+N)} \leq |\bar{W}_d|^{S/(M+N)}.$$

First, we assume that the ideal N has no minimal generators in degree d.

Note that $N_{d-1}+L_{d-1}$ is a S/M-lex-segment. Therefore, $N_{d-1}+L_{d-1}$ is the S/M-lexification of $N_{d-1}+W_{d-1}$ in the ring S/M. Since M is $(d-1)$-pro-lex, the following inequality holds:

$$|N_d+L_d|^{S/M} \leq |N_d+W_d|^{S/M}.$$

On the other hand,

$$|N_d+L_d|^{S/M} = |N_d|^{S/M} + |L_d|^{S/M} - |N_d \cap L_d|^{S/M}$$

$$|N_d+W_d|^{S/M} = |N_d|^{S/M} + |W_d|^{S/M} - |N_d \cap W_d|^{S/M}$$

Therefore, we obtain the inequality

$$|L_d|^{S/M} - |N_d \cap L_d|^{S/M} \leq |W_d|^{S/M} - |N_d \cap W_d|^{S/M}.$$

Note that the left hand-side is equal to $|L_d|^{S/(M+N)}$ whereas the right-hand side is equal to $|W_d|^{S/(M+N)}$. Thus, we get the desired inequality

$$|L_d|^{S/(M+N)} \leq |W_d|^{S/(M+N)}.$$

Now, suppose that N has minimal monomial generators in degree d.

If $L_d \subseteq N_d$, then

$$0 = |L_d|^{S/(M+N)} \leq |W_d|^{S/(M+N)}.$$

Suppose that $L_d \not\subseteq N_d$. Set $Q = \{N_d\} \setminus \{\mathbf{m}N_{d-1}\}$. Since both $\mathbf{m}N_{d-1} + L_d$ and N_d are S/M-lex-segments, it follows that one of them contains the other. Hence $\{L_d\} \supseteq Q$, and therefore

$$|L_d|^{S/(M+N)} = |L_d|^{S/(M+(N_{d-1}))} - |Q|.$$

The argument above (for the case when the ideal is $(d-1)$-generated) can be applied to N_{d-1}, and it yields

$$|L_d|^{S/(M+(N_{d-1}))} \leq |W_d|^{S/(M+(N_{d-1}))}.$$

Therefore we have

$$\begin{aligned}
|L_d|^{S/(M+N)} &= |L_d|^{S/(M+(N_{d-1}))} - |Q| \\
&\leq |W_d|^{S/(M+(N_{d-1}))} - |Q| \leq |W_d|^{S/(M+(N_{d-1}))} - |Q \cap \{W_d\}| \\
&= |W_d|^{S/(M+N)}.
\end{aligned}$$

□

Macaulay's Theorem [Ma] says that 0 is pro-lex. Hence, Theorem 3.5.1 applied to $M = 0$ yields the following:

Corollary 3.5.3. *If U is a S-lex ideal then it is Macaulay-Lex.*

Remark 3.5.4. Following [Sh], we say that a monomial ideal M in S is *piecewise lex* if, whenever $\mathbf{x}^{\mathbf{a}} \in M$, $\mathbf{x}^{\mathbf{b}} >_{lex} \mathbf{x}^{\mathbf{a}}$, and $\max(\mathbf{x}^{\mathbf{b}}) \leq \max(\mathbf{x}^{\mathbf{a}})$, we have $\mathbf{x}^{\mathbf{b}} \in M$. Shakin [Sh] proved that if M is a piecewise lex ideal in S, then it is Macaulay-Lex. This result can be proved differently using our technique as follows: We induct on n. Let $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_r}$ be the minimal monomial generators of M divisible by x_n. So the lex segment L_j ending in $\mathbf{x}^{\mathbf{a}_j}$ must be contained in M. Set $N = M \cap k[x_1, \cdots, x_{n-1}]$. Then N is piecewise lex and so by induction is Macaulay-Lex in $k[x_1, \cdots, x_{n-1}]$. By Theorem 3.4.1, N is Macaulay-Lex in S. By induction on j, we conclude that $(N + L_1 + \cdots + L_{j-1}) + L_j$ is Macaulay-Lex by Theorem 3.5.1. Hence, $M = N + L_1 + \cdots + L_r$ is Macaulay-Lex as well. $\square$

Chapter 4

Hilbert Functions and Lex Ideals*

4.1 Introduction

There are many papers on Hilbert functions or using them. In many of the recent papers and books, Hilbert functions are described using Macaulay's representation (which has nothing to do with Macaulay) with binomials. Thus, the arguments consist of very clever computations with binomials. We have intentionally avoided computations with binomials. One of our main goals is to go back to Macaulay's original idea in 1927 [Ma]: *there exist highly structured monomial ideals - lex ideals - that attain all possible Hilbert functions.* The open problems that we discuss are very natural questions on the role of lex ideals. It seems to us that Problems 4.3.6 and 4.3.8 are very basic and natural problems; the only reason why these problems have not been explored is probably that it is messy to formulate them in terms of binomials.

Throughout the chapter $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i. Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^{\infty} = 0$) and set $R = S/P$. The Clements-Lindström Theorem [CL] characterizes the possible Hilbert functions of graded ideals in the quotient ring R; Macaulay's Theorem [Ma] covers the particular case when $R = S$. In Section 4.2, we present an algebraic proof of the Clements-Lindström Theorem combining ideas of Bigatti [Bi], Clements and Lindström [CL], and Green [Gr2]. The proof is based on the argument in Chapter 3. One of our main results is the Comparison Theorem

* This chapter is modified from the paper "Hilbert functions and lex ideals", which has been submitted for publication. It is joint work with Irena Peeva.

4.2.9 which was inspired by Green's Theorem. Note that the Comparison Theorem 4.2.9 holds in the ring R. As an immediate corollary we obtain the Generalized Green's Theorem 4.2.10. Green's Theorem 4.2.11 is over the ring S and is just a particular case of Theorem 4.2.10. Theorem 4.2.11 was first proved by Green [Gr1] for linear forms, then it was extended to non-linear forms by Gasharov, Herzog, and Popescu [Ga, HP].

In Section 4.3, we raise problems and conjectures which are natural extensions of:

∘ Macaulay's Theorem and Clements-Lindström's Theorem

∘ Evans' Conjecture on lex-plus-powers ideals

∘ conjectures by Gasharov, Herzog, Hibi, and Peeva.

We also very briefly discuss Eisenbud-Green-Harris's Conjecture. All the problems focus on the role of lex ideals.

By Macaulay's Theorem [Ma] lex sequences of monomials have the minimal possible growth of the Hilbert function. There exist many other monomial sequences with this property. The study of such sequences was started by Mermin [Me2]; they are called lexlike sequences. In Section 4.4, we introduce lexlike ideals and prove an extension of Macaulay's Theorem for lexlike ideals. By Macaulay's Theorem, every Hilbert function is attained by a (unique up to reordering of the variables) lex ideal. One of our main results, Theorem 4.4.11, shows that it is also attained by (usually many) lexlike ideals; this is illustrated in Example 4.4.12. Furthermore, we extend the result of Bigatti, Hulett, Pardue, that lex ideals have maximal graded Betti numbers among all ideals with a fixed Hilbert function. We show in Theorem 4.4.14 that lexlike ideals have maximal graded Betti numbers among all ideals with a fixed Hilbert function.

In the last section 4.5, we discuss multigraded Hilbert functions and introduce multilex ideals.

4.2 The Clements-Lindström and Green's Theorems

Throughout this section we use the following notation. Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i. Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, with $e_1 \leq e_2 \leq \cdots \leq e_n \leq \infty$ (here $x_i^\infty = 0$) and set $R = S/P$. We denote by R_d the k-vector space spanned by all monomials in R of degree d. Denote $\mathbf{m} = (x_1, \ldots, x_n)_1$ the k-vector space spanned by the variables. We order the variables $x_1 > \cdots > x_n$, and we denote by $<_{\text{Lex}}$ the homogeneous lexicographic order on the monomials. For a monomial m, set $\max(m) = \max\{i|\, x_i \text{ divides } m\}$.

We say that A_d is a R_d*-monomial space* if it can be spanned by monomials of degree d. We denote by $\{A_d\}$ the set of monomials (non-zero monomials in R_d) contained in A_d. The cardinality of this set is $|A_d| = \dim_k A_d$. By $\mathbf{m}A_d$ we mean the k-vector subspace $\Big(\mathbf{m}\,(A_d)\Big)_{d+1}$ of R_{d+1}.

Compressed ideals were introduced by Clements and Lindström [CL]. They play an important role in the proof of the theorem.

Definition 4.2.1. We say that an R_d-monomial space C_d is i*-compressed* if it is $\{x_1, \cdots, \hat{x}_i, \cdots, x_n\}$-compressed, in the sense of Definition 2.2.6. We say that a k-vector space C_d is R_d*-compressed* (or compressed) if it is a R_d-monomial space and is i-compressed for all $1 \leq i \leq n$. A monomial ideal M in R is called *compressed* if M_d is compressed for all $d \geq 0$.

Definition 4.2.2. We say that an R_d-monomial space B_d is R_d*-Borel* if it satisfies the following condition:

Let $u \in B_d$ be a nonzero monomial, and suppose that x_j divides u and $i < j$. Then $\frac{x_i}{x_j} u \in B_d$.

For a R_d-monomial space A_d set

$$r_{i,j}(A_d) = \Big| \{ \, m \in \{A_d\} \mid \max(m) \leq i \ \text{ and } \ x_i^j \text{ does not divide } m \, \} \Big| .$$

The following lemma is a generalization of a result by Bigatti [Bi].

Lemma 4.2.3. *If a monomial space B_d is R_d-Borel, then*

$$\Big| \{\mathbf{m}B_d\} \Big| = \sum_{i=1}^{n} r_{i,e_i-1}(B_d) \, .$$

Proof. We will show that $\{\mathbf{m}B_d\}$ is equal to the set

$$\coprod_{i=1}^{n} x_i \{ \, m \in \{B_d\} \mid \max(m) \leq i \, \} \ \setminus \ \coprod_{i=1}^{n} x_i \left\{ m \in \{B_d\} \, \middle| \, \begin{matrix} \max(m) = i \text{ and} \\ x_i^{e_i - 1} \text{ divides } \ m \end{matrix} \right\}.$$

Denote by $\mathcal{P}$ the set above. Let $w \in B_d$. For $j \geq \max(w)$ we have that $x_j w \in \mathcal{P}$. Let $j < \max(w)$. Then $v = x_j \dfrac{w}{x_{\max(w)}} \in B_d$. So, $x_j w = x_{\max(w)} v \in \mathcal{P}$. ☐

Lemma 4.2.4. *If L_d is a lex-segment, then it is Borel and R_d-compressed.*

The main work for proving a generalized Green's theorem is in the following lemma:

Lemma 4.2.5. *Let C_d be an n-compressed Borel R_d-monomial space, and let L_d be a lex-segment in R_d with $|L_d| \leq |C_d|$. For each $1 \leq i \leq n$ and each $1 \leq j \leq e_i$ we have*

$$r_{i,j}(L_d) \leq r_{i,j}(C_d) \, .$$

Proof. Note that both L_d and C_d are R_d-Borel and n-compressed.

First, we consider the case $i = n$. Clearly, $r_{n,e_n}(L_d) = |L_d| = |C_d| = r_{n,e_n}(C_d)$ (if $e_n = \infty$, then we consider $r_{n,d+1}$ here). We induct on j decreasingly. Suppose that $r_{i,j+1}(L_d) \leq r_{i,j+1}(C_d)$ holds by induction.

If $\{C_d\}$ contains no monomial divisible by x_n^j then

$$r_{i,j}(L_d) \leq r_{i,j+1}(L_d) \leq r_{i,j+1}(C_d) = r_{i,j}(C_d)\,.$$

Suppose that $\{C_d\}$ contains a monomial divisible by x_n^j. Denote by $e = x_1^{b_1}\dots x_n^{b_n}$, with $b_n \geq j$, the lex-smallest monomial in C_d that is divisible by x_n^j. Let $0 \leq q \leq j-1$. Since C_d is R_d-Borel, it follows that $c_q = x_{n-1}^{b_n-q}\dfrac{e}{x_n^{b_n-q}} \in C_d$. This is the lex-smallest monomial that is lex-greater than e and x_n divides it at power q. Let the monomial $a = x_1^{a_1}\dots x_{n-1}^{a_{n-1}}x_n^q \in R_d$ be lex-greater than e. Since C_d is n-compressed and a is lex-greater (or equal) than c_q, it follows that $a \in C_d$.

For a monomial u, we denote by $x_n \notin u$ the property that x_n^j does not divide u. By what we proved above, it follows that

$$\Big|\{u \in \{C_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}\big| = \big|\{u \in \{R_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}\Big|\,. \tag{4.2.6}$$

Therefore,

$$\begin{aligned}
r_{i,j}(L_d) &= |\{u \in \{L_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}| + |\{u \in \{L_d\} \mid x_n \notin u,\ u >_{\text{Lex}} e\}| \\
&\leq |\{u \in \{R_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}| + |\{u \in \{L_d\} \mid x_n \notin u,\ u >_{\text{Lex}} e\}| \\
&\leq |\{u \in \{R_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}| + |\{u \in \{L_d\} \mid u >_{\text{Lex}} e\}| \\
&\leq |\{u \in \{R_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}| + |\{u \in \{C_d\} \mid u >_{\text{Lex}} e\}| \\
&= |\{u \in \{R_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}| + |\{u \in \{C_d\} \mid x_n \notin u,\ u >_{\text{Lex}} e\}| \\
&= |\{u \in \{C_d\} \mid x_n \notin u,\ u <_{\text{Lex}} e\}| + |\{u \in \{C_d\} \mid x_n \notin u,\ u >_{\text{Lex}} e\}| \\
&= r_{i,j}(C_d)\,;
\end{aligned}$$

for the third inequality we used the fact that L_d is a lex-segment in R_d with $|L_d| \leq |C_d|$; for the equality after that we used the definition of e; for the next equality we used (4.2.6). Thus, we have the desired inequality in the case $i = n$.

In particular, we proved that

$$r_{n,1}(L_d) \leq r_{n,1}(C_d)\,. \tag{4.2.7}$$

Finally, we prove the lemma for all $i < n$. Both $\{C_d/x_n\}$ and $\{L_d/x_n\}$ are lex-segments in R_d/x_n since C_d is n-compressed. By (4.2.7) the inequality $r_{n,1}(L_d) \leq r_{n,1}(C_d)$ holds, and it implies the inclusion $\{C_d/x_n\} \supseteq \{L_d/x_n\}$. The desired inequalities follow since

$$\begin{aligned} r_{i,j}(C_d) &= r_{i,j}\big(C_d/(x_{i+1},\ldots,x_n)\big) \\ r_{i,j}(L_d) &= r_{i,j}\big(L_d/(x_{i+1},\ldots,x_n)\big)\,. \end{aligned}$$

□

Let B_d be a Borel monomial space in R_d. Set $z = x_n$. We have the disjoint union

$$\{B_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{U_j\}$$

where each U_j is a monomial space in R/z. Let F_j be the lexification of the space U_j in R/z. Consider the R_d-monomial space T_d defined by

$$\{T_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{F_j\}\,.$$

Clearly, $|T_d| = |B_d|$. We call T_d the *n-compression* of B_d.

Lemma 4.2.8. *Let B_d be a Borel monomial space in R_d. Its n-compression T_d is Borel.*

Proof. Consider the disjoint unions

$$\{B_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{U_j\}$$

$$\{T_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{F_j\}\,.$$

Since B_d is Borel, it follows that $\mathbf{n}U_j \subseteq U_{j+1}$. Since $|F_j| = |U_j|$, we can apply Theorem 4.2.12(1) by induction on the number of the variables, and it follows that $|\mathbf{n}F_j| \leq |\mathbf{n}U_j| \leq |U_{j+1}| = |F_{j+1}|$. As both $\mathbf{n}F_j$ and F_{j+1} are lex-segments, we conclude that $\mathbf{n}F_j \subseteq F_{j+1}$. If $x_n^{d-j}m$ is a monomial in T_d and $m \in F_j$, then for each $1 \leq i < n$ we have that $x_i m \in \mathbf{n}F_j \subseteq F_{j+1}$, so $x_n^{d-j-1}x_i m \in T_d$. If x_p divides m, then for each $1 \leq q \leq p$ we have that $\frac{x_q m}{x_p} \in F_j \subset T_d$ since F_j is lex. We proved that T_d is Borel. $\square$

Comparison Theorem 4.2.9. *Let B_d be a Borel monomial space in R_d. Let L_d be a lex-segment in R_d with $|L_d| \leq |B_d|$. The following inequalities hold:*

$$r_{i,j}(L_d) \leq r_{i,j}(B_d)\,.$$

for each $1 \leq i \leq n$ and each $1 \leq j \leq e_i$.

Proof. We prove the inequalities by decreasing induction on the number of variables n. Let T_d be the n-compression of B_d. Since T_d is Borel and n-compressed by Lemma 4.2.8, we can apply Lemma 3.3.6 and we get

$$r_{i,j}(L_d) \leq r_{i,j}(T_d)$$

for each $1 \leq i \leq n$ and each $1 \leq j \leq e_i$. It remains to compare $r_{i,j}(T_d)$ and $r_{i,j}(B_d)$. For $i = n$, we have equalities $r_{n,j}(T_d) = r_{n,j}(B_d)$. Let $i < n$. Then $r_{i,j}(T_d) = r_{i,j}(T_d/x_n)$ and $r_{i,j}(B_d) = r_{i,j}(B_d/x_n)$, where $T_d/x_n = L_d$ is a lex-segment and $B_d/x_n = U_d$ is Borel. So, by induction the desired inequalities hold. $\square$

Generalized Green's Theorem 4.2.10. *Let B_d be a Borel monomial space in R_d. Let L_d be a lex-segment in R_d with $|L_d| \leq |B_d|$. The following inequalities hold:*

$$\dim\Big(R_d/(L_d \oplus \mathbf{n}^{(d-j)}x_n^j)\Big) \geq \dim\Big(R_d/(B_d \oplus \mathbf{n}^{(d-j)}x_n^j)\Big)$$

for each each $1 \leq j \leq e_n,\ j \leq d$.

Proof. Note that the desired inequality is equivalent to

$$r_{n,j}(L_d) \leq r_{n,j}(B_d)\,.$$

It holds by Theorem 4.2.9. ☐

The following result is a straightforward corollary of Theorem 4.2.10 since x_n^j is a generic form for every Borel ideal in S.

Green's Theorem 4.2.11. *Let B_d be a Borel monomial space in S_d. Let L_d be a lex-segment in S_d with $|L_d| \leq |B_d|$. Let g be a generic homogeneous form of degree $j \geq 1$. The following inequality holds:*

$$\dim\Big(S_d/(L_d \oplus \mathbf{m}^{(d-j)}g)\Big) \geq \dim\Big(S_d/(B_d \oplus \mathbf{m}^{(d-j)}g)\Big)\,.$$

Remark 4.2.12. Theorem 4.2.11 in the case when $j = 1$ was proved by Green [Gr1]. Theorem 4.2.11 in the case when $j > 1$ was proved by Gasharov, Herzog, and Popescu [Ga, HP]. Theorem 4.2.10 in the case when $j = 1$ was proved by Gasharov [Ga2, Theorem 2.1].

We are ready to prove Macaulay's Theorem [Ma] which characterizes the possible Hilbert functions of graded ideals in S. There are several different proofs of this theorem, cf. Green [Gr2].

Macaulay's Theorem 4.2.13. *The following two properties are equivalent, and they hold:*

(1) Let A_d be an S_d-monomial space and L_d be its lexification in S_d. Then $|\,\mathbf{m}L_d\,| \leq |\,\mathbf{m}A_d\,|$.

(2) For every graded ideal J in S there exists a lex ideal L with the same Hilbert function.

Proof. First, we will prove that (1) holds. Since A_d and L_d are monomial spaces, (1) does not depend on the field k. Thus, we can replace the field if necessary and assume that k has characteristic zero. This makes it possible to use Gröbner basis theory to reduce to the Borel case, cf. [Ei, Chapter 15]. We obtain a Borel S_d-monomial space B_d such that $|B_d| = |A_d|$ and $|\mathbf{m}B_d| \leq |\mathbf{m}A_d|$. For an S_d-monomial space D_d set $t_i(D_d) = r_{i+1,1}(D_d) = \Big|\{\, m \in \{D_d\} \mid \max(m) \leq i \,\}\Big|$. We apply Lemma 4.2.3 to conclude that

$$\Big|\{\mathbf{m}B_d\}\Big| = \sum_{i=1}^{n} t_i(B_d) \quad \text{and} \quad \Big|\{\mathbf{m}L_d\}\Big| = \sum_{i=1}^{n} t_i(L_d)\,.$$

Finally, we apply Theorem 4.2.10 and get the inequality $\big|\{\mathbf{m}L_d\}\big| \leq \big|\{\mathbf{m}B_d\}\big|$. We proved (1).

Now, we prove that (1) and (2) are equivalent. Clearly, (2) implies (1). We assume that (1) holds and will prove (2). We can assume that J is a monomial ideal by Gröbner basis theory. For each $d \geq 0$, let L_d be the lexification of J_d. By (1), it follows that $L = \oplus_{d\geq 0} L_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as J in all degrees. $\square$

We continue with the proof of Clements-Lindström's Theorem.

Lemma 4.2.14. *Let A_d be an R_d-monomial space. There exists a compressed monomial space C_d in R_d such that $|C_d| = |A_d|$ and $|\mathbf{m}C_d| \leq |\mathbf{m}A_d|$.*

Proof. Suppose that A_d is not i-compressed. Set $z = x_i$.

We have the disjoint union

$$\{A_d\} = \coprod_{0\leq j\leq d} z^{d-j}\{U_j\}$$

where each U_j is a monomial space in R/z. Let F_j be the lexification of the space U_j in R/z. Consider the R_d-monomial space T_d defined by

$$\{T_d\} = \coprod_{0 \le j \le d} z^{d-j}\{F_j\}\,.$$

Clearly, $|T_d| = |A_d|$. We will prove that

$$|\mathbf{m}T_d| \le |\mathbf{m}A_d|\,.$$

We have the disjoint unions

$$\{\mathbf{m}A_d\} = \coprod_{0 \le j \le d} z^{d-j+1}\{U_j + \mathbf{n}U_{j-1}\}$$

$$\{\mathbf{m}T_d\} = \coprod_{0 \le j \le d} z^{d-j+1}\{F_j + \mathbf{n}F_{j-1}\}\,,$$

where $\mathbf{n} = \mathbf{m}/z$. We will show that

$$|F_j + \mathbf{n}F_{j-1}| = \max\Big\{|F_j|,\, |\mathbf{n}F_{j-1}|\Big\} \le \max\Big\{|U_j|,\, |\mathbf{n}U_{j-1}|\Big\} \le |U_j + \mathbf{n}U_{j-1}|\,.$$

The first equality above holds because both F_j and $\mathbf{n}F_{j-1}$ are $(R/z)_j$-lex-segments, so $F_j + \mathbf{n}F_{j-1}$ is the longer of these two lex-segments. The last inequality is obvious. The middle inequality holds since by construction F_{j-1} is the lexification of U_{j-1}, so $|F_{j-1}| = |U_{j-1}|$ and by induction on the number of variables we can apply Theorem 4.2.15(1) to the ring R/z.

Thus, $|F_j + \mathbf{n}F_{j-1}| \le |U_j + \mathbf{n}U_{j-1}|$. Multiplication by z^{d-j+1} is injective if $d - j + 1 \le e_i - 1$ and is zero otherwise, therefore we conclude that

$$\Big|z^{d-j+1}(F_j + \mathbf{n}F_{j-1})\Big| \le \Big|z^{d-j+1}(U_j + \mathbf{n}U_{j-1})\Big|\,.$$

This implies the desired inequality $|\mathbf{m}T_d| \le |\mathbf{m}A_d|$.

If T_d is not compressed, we can apply the argument above. After finitely many steps in this way, the process must terminate because at each step we construct a lex-greater monomial space. Thus, after finitely many steps, we reach a compressed monomial space. □

The Clements and Lindström Theorem [CL] is:

Clements and Lindström's Theorem 4.2.15. *The following two properties are equivalent, and they hold:*

(1) Let A_d be an R_d-monomial space and L_d be its lexification in R_d. Then $|\,\mathbf{m}L_d\,| \le |\,\mathbf{m}A_d\,|$.

(2) For every graded ideal J in R there exists a lex ideal L with the same Hilbert function.

Proof. First, we will prove that (1) holds. The theorem clearly holds if $n = 1$. Suppose that $n = 2$. An easy calculation shows that the theorem holds, provided we do not have $e_2 \le d+1 < e_1$. By the assumption on the ordering of the exponents, this does not hold and we are fine.

Suppose that $n \ge 3$. First, we apply Lemma 4.2.14 to reduce to the compressed case. We obtain a compressed R_d-monomial space C_d such that $|C_d| = |A_d|$ and $|\mathbf{m}C_d| \le |\mathbf{m}A_d|$. Both L_d and C_d are $(S/P)_d$-compressed. We apply Lemma 4.2.3 to conclude that

$$\Big|\,\{\mathbf{m}C_d\}\,\Big| = \sum_{i=1}^{n} r_{i,e_i-1}(C_d)$$
$$\Big|\,\{\mathbf{m}L_d\}\,\Big| = \sum_{i=1}^{n} r_{i,e_i-1}(L_d).$$

Finally, we apply Lemma 4.2.5 and obtain the inequality $\big|\,\{\mathbf{m}L_d\}\,\big| \le \big|\,\{\mathbf{m}C_d\}\,\big|$. We proved (1).

Now, we prove that (1) and (2) are equivalent. Clearly, (2) implies (1). We assume that (1) holds and will prove (2). We can assume that J is a monomial ideal by Gröbner basis theory. For each $d \geq 0$, let L_d be the lexification of J_d. By (1), it follows that $L = \oplus_{d \geq 0} L_d$ is an ideal. By construction, it is a lex-ideal and has the same Hilbert function as J in all degrees. $\square$

Lexicographic ideals are highly structured and it is easy to derive the inequalities characterizing their possible Hilbert functions.

4.3 Open Problems

Throughout this section, M is a monomial ideal in S.

4.3.1 Hilbert functions in quotient rings

We focus on the problem to identify rings, other than S and S/P, where Macaulay's and Clements-Lindström's theorems hold. First, we recall the necessary definitions.

Definition 4.3.1. A homogeneous ideal of S/M is *lexifiable* if there exists a lex ideal with the same Hilbert function. We say that M and S/M are *Macaulay-Lex* if every homogeneous ideal of S/M is lexifiable.

The following problem is a natural extension of Macaulay's and Clements-Lindström's Theorems:

Problem 4.3.2. *Identify monomial ideals which are Macaulay-Lex.*

Macaulay's Theorem [Ma] says that 0 is Macaulay-Lex. Clements-Lindström's Theorem [CL] says that $(x_1^{e_1}, \cdots, x_n^{e_n})$ is Macaulay-Lex when $e_1 \leq \cdots \leq e_n \leq \infty$. We recall the following theorems from chapter 3.

Theorem 4.3.3. *Let M be Macaulay-Lex and L be lex. Then $M + L$ is Macaulay-Lex.*

Theorem 4.3.4. *Let S/M be Macaulay-Lex. Then $(S/M)[y]$ is Macaulay-Lex.*

Macaulay-Lex ideals appear to be rare, however. For example, Shakin [Sh] has recently shown that a Borel ideal M is Macaulay-Lex if and only if it is *piecewise lex*, that is, if M may be written $M = \sum L_i$ with L_i generated by a lex segment of $k[x_1, \cdots, x_i]$.

It is easy to construct examples like Example 3.2.8, where a given Hilbert function is not attained by any lexicographic ideal in the degrees of the minimal generators of M. This suggests that our definitions should be relaxed somewhat. In chapter 3 we introduced the following definition:

Definition 4.3.5. We say that M and S/M are *pro-lex above* q if every homogeneous ideal of S/M generated in degrees $\geq q$ is lexifiable. Let d be the maximal degree of a minimal generator of M. We say that M and S/M are *pro-lex* if they are pro-lex above d.

We have the following variation of Problem 4.3.2:

Problem 4.3.6. *Identify monomial ideals which are pro-lex.*

As a first step in this direction, we show in chapter 3:

Theorem 4.3.7. *Let $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, where $e_1 \leq \cdots e_n \leq \infty$ (here $x_i^{\infty} = 0$). Let K be a compressed monomial ideal of S/P, and let d be the maximal degree of a minimal monomial generator of K. If $n = 2$, assume that K is lex. The ideal $K + P$ is pro-lex above d.*

It is natural to try to extend to non-monomial ideals:

Problem 4.3.8. *Find other graded rings where the notion of lex ideal makes sense and which are pro-lex.*

Of particular interest are the coordinate rings of projective toric varieties. Toric varieties are an important class of varieties which occur at the intersection of Algebraic Geometry, Commutative Algebra, and Combinatorics. They might provide many examples of interesting rings in which all Hilbert functions are attained by lex ideals.

Problem 4.3.9. *Find projective toric rings which are pro-lex (or Macaulay-Lex).*

The coordinate rings of toric varieties admit a natural multigraded structure which refines the usual grading and which yields a multigraded Hilbert function; this is studied in Section 4.5.

4.3.2 The Eisenbud-Green-Harris Conjecture

The most exciting currently open conjecture on Hilbert functions is given by Eisenbud, Green, and Harris in [EGH1, EGH2]. The conjecture is wide open.

Conjecture 4.3.10. *Let N be a homogeneous ideal containing a homogeneous regular sequence in degrees $e_1 \leq \cdots \leq e_r$. There is a monomial ideal T such that N and $T + (x_1^{e_1}, \cdots, x_r^{e_r})$ have the same Hilbert function.*

The original conjecture differs from 4.3.10 in the following two minor aspects:

∘ In the original conjecture $r = n$.

∘ The original conjecture gives a numerical characterization of the possible Hilbert functions of N. It is well known that this numerical characterization is equivalent to the fact that there exists a lex ideal L such that $L + (x_1^{e_1}, \cdots, x_r^{e_r})$ has the same

Hilbert function as N. By Clements-Lindström's Theorem, this is equivalent to Conjecture 4.3.10.

4.3.3 Betti numbers

The study of Hilbert functions is often closely related to the study of free resolutions. We focus on problems based on the idea that the lex ideal has the greatest Betti numbers among all ideals with a fixed Hilbert function.

Conjecture 4.3.11. *Let k be an infinite field (possibly, one should also assume that k has characteristic 0). Suppose that S/M is pro-lex above d, J is a homogeneous ideal in S/M, generated in degrees $\geq d$, and L is the lex ideal with the same Hilbert function as J. Then:*

(1) *The Betti numbers of J over R are less than or equal to those of L.*

(2) *The Betti numbers of $J+M$ over S are less than or equal to those of $L+M$.*

Note that the first part of the conjecture is about infinite resolutions (unless M is generated by linear forms), whereas the second part is about finite ones.

In the case $M=0$, Conjecture 4.3.11 holds by a result of Bigatti [Bi], Hulett [Hu], and Pardue [Pa]. Also, Conjecture 4.3.11(1) holds by a result of Aramova, Herzog, and Hibi [AHH] over an exterior algebra. Furthermore, Conjecture 4.3.11(2) was inspired by work of Graham Evans and his conjecture, cf. [FR]:

Conjecture 4.3.12 (Evans)**.** *Suppose that a homogeneous ideal I contains a regular sequence of homogeneous elements of degrees $a_1, \dots, a_n$ in S. Suppose that there exists a lex-plus-powers ideal L with the same Hilbert function as I. Then the Betti numbers of L are greater than or equal to those of I.*

Conjecture 4.3.12 was inspired by the Eisenbud-Green-Harris Conjecture 4.3.11.

When the regular sequence in Conjecture 4.3.12 consists of powers of the variables, Conjecture 4.3.12 coincides with Conjecture 4.3.11(2). Also, in the case when M is generated by powers of the variables, Conjecture 4.3.11(1) coincides with a conjecture of Gasharov, Hibi, and Peeva [GHP], and in the case when M is generated by squares of the variables Conjecture 4.3.11(2) coincides with a conjecture of Herzog and Hibi.

Remark 4.3.13. It is natural to wonder whether Conjecture 4.3.11 should have part (3) that states that the Betti numbers of J over S are less or equal to those of L. There is a counterexample in [GHP]: take $J = (x^2, y^2)$ in $k[x, y]/(x^3, y^3)$ and $L = (x^2, xy)$, then the graded Betti numbers of L over S are not greater or equal to those of J over S. It should be noticed that J and L do not have the same Hilbert function as ideals in S.

4.4 Lex-like ideals

In this section we work over the polynomial ring $S = k[x_1, \cdots, x_n]$. Macaulay's Theorem [Ma] has the following two equivalent formulations (given in Theorem 4.2.13).

Theorem 4.4.1. *Let A_d be a monomial space in degree d and L_d be the space spanned by a lex segment in degree d such that $|A_d| = |L_d|$. Then $|\,\mathbf{m}L_d\,| \leq |\,\mathbf{m}A_d\,|$.*

Theorem 4.4.2. *For every graded ideal J in S there exists a lex ideal L with the same Hilbert function.*

The goal of this section is to show that a generalization of Macaulay's Theorem holds for ideals generated by initial segments of lexlike sequences. Lexlike sequences were discovered by Mermin in [Me2]; we recall the definition.

Definition 4.4.3. A *monomial sequence* (of a fixed degree d) is a sequence X_d of all the monomials of $S = k[x_1, \cdots, x_n]$ of degree d. We denote by $X_d(i)$ the monomial space generated by the first i monomials in X_d. We say that X_d is *lexlike* if, for every i, and for every vector space V generated by i monomials of degree d, we have

$$|\mathbf{m}X_d(i)| < |\mathbf{m}V| .$$

The *lex sequence* in degree d consists of all the degree d monomials ordered lexicographically; it is denoted by Lex_d or simply Lex.

Lemma 4.4.4.

(1) Lex_d is a lexlike sequence.

(2) X_d is a lexlike sequence of degree d if and only if, for every i we have

$$|\mathbf{m}X_d(i)| = |\mathbf{m}\,\mathrm{Lex}_d(i)| .$$

Proof. (1) is Macaulay's Theorem 4.2.13. (2) follows from (1). $\square$

Thus, lexlike sequences have minimal Hilbert function growth, as lex sequences have.

By Definition 4.4.3 it follows immediately that the first formulation 4.1 of Macaulay's Theorem holds for lexlike sequences:

Theorem 4.4.5. *Let A_d be a monomial space in degree d and I_d be the space spanned by the initial segment of a lexlike sequence in degree d such that $|A_d| = |I_d|$. Then $|\,\mathbf{m}I_d\,| \leq |\,\mathbf{m}A_d\,|$.*

However, it is not immediately clear that the second formulation 4.4.2 of Macaulay's Theorem holds for lexlike sequences. The problem is that one has to construct lexlike ideals and show that they are well defined. Here is an outline of what we do in order to extend Theorem 4.4.2: In each degree d we have the lex sequence Lex_d. If L_d is spanned by an initial segment of Lex_d, then $\mathbf{m}L_d$ is spanned by an initial segment of Lex_{d+1}. This property is very easy to prove. It is very important, because it makes it possible to define lexicographic ideals. In [Me2, Corollary 3.18] Mermin proved that the same property holds for lexlike sequences. This makes it possible to introduce lexlike ideals in Definition 4.4.9. We prove in Theorem 4.4.10 that Macaulay's Theorem 4.4.2 for lex ideals holds for lexlike ideals as well.

First, we recall a definition in [Me2]: Let X_d be a sequence of all the monomials of S of degree d, and let X_{d-1} be a sequence of all the monomials of S of degree $d-1$. We say that X_d is *above* X_{d-1} if, for all i, there is a j such that $\mathbf{m}X_{d-1}(i) = X_d(j)$. By [Me2, Theorem 3.20], if X_d is a monomial sequence above a lexlike sequence X_{d-1}, then X_d is lexlike.

Lemma 4.4.6. *Let Y be a lexlike sequence in degree d. In every degree p, there exists a lexlike sequence X_p such that $X_d = Y$ and X_{p+1} is above X_p for all p. In particular, if a space V_p is spanned by an initial segment of X_p, then $\mathbf{m}V_p$ is spanned by an initial segment of X_{p+1}.*

Proof. Repeatedly apply [Me2, Theorem 3.21] to get X_p for $p < d$. Repeatedly apply [Me2, Theorem 3.20] to get X_p for $p > d$. □

Definition 4.4.7. Let $\mathbf{X}$ be a collection of lexlike sequences X_d in each degree d, such that X_{d+1} is above X_d for each d. We call $\mathbf{X}$ a *lexlike tower*.

If we multiply a monomial sequence X by a monomial m by termwise multiplication, then we denote the new monomial sequence by mX. If Y is another monomial sequence, we denote concatenation with a semicolon, so $X;Y$. Towers of monomial sequences are highly structured:

Theorem 4.4.8. *Let $\mathbf{X}$ be a lexlike tower. There exists a variable x_i, a lexlike tower $\mathbf{Y}$ of monomials in S, and a lexlike tower $\mathbf{Z}$ of monomials in S/x_i, such that*

$$\mathbf{X} = x_i\mathbf{Y};\mathbf{Z}\,.$$

Proof. The variable x_i is the first term of X_1. Each X_d begins with all the degree d monomials divisible by x_i. Writing $X_d = x_iY_{d-1};Z_d$ for each d, we have that $\mathbf{Y}$ is a lexlike tower and $\mathbf{Z}$ is a lexlike tower in $n-1$ variables. $\square$

Remark 4.4.9. The lexicographic tower is compatible with the lexicographic order in each degree. A lexlike tower $\mathbf{X}$ induces a total ordering $<_{\mathbf{X}}$ on the monomials of S which refines the partial order by degree. It is natural to ask what term orders occur this way. We show that the lexicographic order is the only one (up to reordering the variables): Suppose that $<_{\mathbf{X}}$ is a term order. Clearly X_1 is Lex for the corresponding order of the variables. Writing $X_2 = x_1Y_1;Z_1$, we apply $x_1x_i <_{\mathbf{X}} x_1x_j$ whenever $x_i <_{\mathbf{X}} x_j$ to see that Y_1 is Lex and induction on n to see that Z_1 is Lex. Thus X_2 is Lex. Now if $X_d = x_1Y_{d-1};Z_d$ is Lex, induction on d and n shows that Y_d and Z_{d+1}, and hence X_{d+1}, are Lex as well.

In the spirit of the definition of lex ideals, we introduce lexlike ideals as follows:

Definition 4.4.10. Let $\mathbf{X}$ be a lexlike tower. We say that a d-vector space is an $\mathbf{X}$-space if it is spanned by an initial segment of X_d. We say that a homogeneous ideal I is $\mathbf{X}$-lexlike if I_d is an $\mathbf{X}$-space for all d. We say that an ideal I is *lexlike* if there exists a lexlike tower $\mathbf{X}$ so that I is $\mathbf{X}$-lexlike.

A lex ideal is lexlike by Lemma 4.4.4(1).

Macaulay's Theorem for Lexlike Ideals 4.4.11. *Let $\mathbf{X}$ be a lexlike tower. Let J be a homogeneous ideal, and for each d let I_d be the $\mathbf{X}$-space spanned by the first $|J_d|$ monomials of X_d. Then $I = \bigoplus I_d$ is an $\mathbf{X}$-lexlike ideal and has the same Hilbert function as J.*

Proof. It suffices to show that I is an ideal, that is, that $\mathbf{m}I_d \subset I_{d+1}$ for each degree d. We have $|\mathbf{m}I_d| \leq |\mathbf{m}J_d| \leq |J_{d+1}| = |I_{d+1}|$, and $\mathbf{m}I_d$ and I_{d+1} are both spanned by initial segments of X_{d+1}. Since X_{d+1} is above X_d, it follows that $\mathbf{m}I_d \subset I_{d+1}$. $\square$

Thus, every Hilbert function is attained not only by a lex ideal (which is unique up to reordering of the variables) but also by (usually many) lexlike ideals. These distinct lexlike ideals are obtained by varying the lexlike tower $\mathbf{X}$. The following example illustrates this.

Example 4.4.12. The lexlike ideals $(ab, ac, a^3, a^2d, ad^3, b^2c)$ and $(ab, ac, ad^2, a^2d, a^4, b^4)$ have the same Hilbert function as the lex ideal $(a^2, ab, ac^2, acd, ad^3, b^4)$.

Proposition 4.4.13.

(1) *If I and I' are two lexlike ideals with the same Hilbert function, then they have the same number of minimal monomial generators in each degree.*

(2) *Among all ideals with the same Hilbert function, the lexlike ideals have the maximal number of minimal monomial generators (in each degree).*

Proof. First, we prove (1). Let L be the lexicographic ideal with the same Hilbert function as J. By Definition 4.4.1, it follows that I and L have the same number of minimal monomial generators (in each degree).

Now, we prove (2). Macaulay's Theorem implies that among all ideals with the same Hilbert function, the lex ideal has the maximal number of minimal monomial generators (in each degree). Apply (1). $\square$

The above theorem can be extended to all graded Betti numbers as follows:

Theorem 4.4.14.

(1) Let I be a lexlike ideal and L be a lex ideal with the same Hilbert function. The graded Betti numbers of I are equal to those of L.

(2) Among all ideals with the same Hilbert function, the lexlike ideals have the greatest graded Betti numbers.

This is an extension of the following well-known result by [Bi,Hu,Pa]:

Theorem 4.4.15 (Bigatti, Hulett, Pardue)**.** *Among all ideals with the same Hilbert function, the lex ideal has the greatest graded Betti numbers.*

Proof of Theorem 4.4.14: (2) follows from (1) and Theorem 4.4.15. We will prove (1).

Let p be the smallest degree in which L has a minimal monomial generator. For $d \geq p$, denote by $I(d)$ the ideal generated by all monomials in I of degree $\leq d$. Similarly, denote by $L(d)$ the ideal generated by all monomials in L of degree $\leq d$. By Lemma 4.4.4(2), for each $d \geq p$ the ideals $I(d)$ and $L(d)$ have the same Hilbert function. Furthermore, by Theorem 4.4.15 it follows that the graded Betti numbers of $S/L(d)$ are greater or equal to those of $S/I(d)$.

The following formula (cf. [Ei]) relates the graded Betti numbers $\beta_{i,j}(S/T)$ of a homogeneous ideal T and its Hilbert function:

$$\sum_{j=0}^{\infty} \dim_k (S/T)_j \, t^j = \frac{\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \beta_{i,j}(S/T)\, t^j}{(1-t)^n}\,.$$

Therefore, for each $d \geq p$ we have that

$$\sum_{j=0}^{\infty} \sum_{i=0}^{n} (-1)^i \Big(\beta_{i,j}(S/I(d)) - \beta_{i,j}(S/L(d)) \Big)\, t^j = 0\,. \tag{4.4.16}$$

By induction on d we will show that the graded Betti numbers of $S/L(d)$ are equal to those of $S/I(d)$.

First, consider the case when $d = p$. By the Eliahou-Kervaire resolution [EK], it follows that $L(p)$ has a linear minimal free resolution, that is, $\beta_{i,j}(S/L(p)) = 0$ for $j \neq i + p - 1$. Since the graded Betti numbers of $S/L(p)$ are greater or equal to those of $S/I(p)$, it follows that $\beta_{i,j}(S/I(p)) = 0$ for $j \neq i + p - 1$. By (4.4.16) it follows that

$$\beta_{i,j}(S/I(p)) = \beta_{i,j}(S/L(p)) \quad \text{for all } i, j\,.$$

Suppose that the claim is proved for d. Consider $L(d+1)$ and $I(d+1)$. For $j < i + d$, we have that

$$\beta_{i,j}(S/L(d+1)) = \beta_{i,j}(S/L(d)) = \beta_{i,j}(S/I(d))\,,$$

where the first equality follows from the Eliahou-Kervaire resolution [EK] and the second equality holds by induction hypothesis. As $I(d+1)_q = I(d)_q$ for $q \leq d$ and since $\beta_{i,j}(S/I(d)) = 0$ for $j \geq i + d$, it follows that $\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/I(d))$ for $j < i + d$. Therefore,

$\beta_{i,j}(S/L(d+1)) = \beta_{i,j}(S/I(d+1))$ for $j < i + d$

$\beta_{i,j}(S/L(d+1)) = 0$ for $j > i + d$, by the Eliahou-Kervaire resolution [EK].

Since the graded Betti numbers of $S/L(d+1)$ are greater or equal to those of $S/I(d+1)$, we conclude that

$$\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/L(d+1)) \text{ for } j < i+d$$
$$\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/L(d+1)) = 0 \text{ for } j > i+d\,.$$

By (4.4.16) it follows that

$$\beta_{i,j}(S/I(d+1)) = \beta_{i,j}(S/L(d+1)) \quad \text{for all } i, j\,,$$

as desired. $\square$

Remark 4.4.17. Let I be a lexlike ideal and L a lex ideal with the same Hilbert function. Since their graded Betti numbers are equal, one might wonder whether the minimal free resolution $\mathbf{F}_I$ of I is provided by Eliahou-Kervaire's construction [EK]. The leading terms in the differential of $\mathbf{F}_I$ are the same as in Eliahou-Kervaire's construction. However, the other terms could be quite different: there are examples in which the differential of $\mathbf{F}_I$ has more non-zero terms than the differential in Eliahou-Kervaire's construction.

4.5 Multigraded Hilbert functions

In this subsection we consider the polynomial ring S with a different grading, called multigrading. Such gradings are used for toric ideals. In this case, we have a multigraded Hilbert function.

Let $\mathcal{A} = \{a_1, \ldots, a_n\}$ be a subset of $\mathbb{N}^c \setminus \{\mathbf{0}\}$, A be the matrix with columns a_i, and suppose that $\operatorname{rank}(A) = c$. Consider the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k generated by variables $x_1, \ldots, x_n$ in $\mathbb{N}^c$-degrees $a_1, \ldots, a_n$ respectively.

We say that an ideal J is $\mathcal{A}$-*multigraded* if it is homogeneous with respect to this $\mathbb{N}^c$-grading. For simplicity, we often say multigraded instead of $\mathcal{A}$-multigraded.

The prime ideal $I_{\mathcal{A}}$, that is the kernel of the homomorphism

$$\varphi : \ k[x_1, \dots, x_n] \to k[t_1, \dots, t_c]$$
$$x_i \mapsto \mathbf{t}^{a_i} = t_1^{a_{i1}} \dots t_c^{a_{ic}}$$

is called the *toric ideal* associated to $\mathcal{A}$. For an integer vector $\mathbf{v} = (v_1, \dots, v_n)$ we set $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} \dots x_n^{v_n}$. Then $\varphi(\mathbf{x}^{\mathbf{v}}) = \mathbf{t}^{A\mathbf{v}}$. The *toric ring* associated to $\mathcal{A}$ is

$$S/I_{\mathcal{A}} \cong k[\mathbf{t}^{a_1}, \dots, \mathbf{t}^{a_n}] \cong \mathbb{N}\mathcal{A}, \tag{4.5.1}$$

where the former isomorphism is given by $\mathbf{x}^{\mathbf{v}} \mapsto \mathbf{t}^{A\mathbf{v}}$ and the latter isomorphism is given by $\mathbf{t}^{\mathbf{a}} \mapsto \mathbf{a}$.

The ideal $I_{\mathcal{A}}$ is $\mathcal{A}$-multigraded. By (4.5.1), it follows that we have the multigraded Hilbert function

$$\dim_k((S/I_{\mathcal{A}})_\alpha) = \begin{cases} 1 & \text{if } \alpha \in \mathbb{N}\mathcal{A} \\ 0 & \text{otherwise.} \end{cases} \tag{4.5.2}$$

There exists a minimal free resolution of $S/I_{\mathcal{A}}$ over S which is $\mathbb{N}^c$-graded.

For $\alpha \in \mathbb{N}^c$, the set of all monomials in S of degree α is called the *fiber* of α. We introduce multilex ideals generalizing the notion of lex ideal:

Definition 4.5.3. Order the monomials in each fiber lexicographically. An $\mathcal{A}$-*multilex segment* (or multilex segment) in multidegree α is a vector space spanned by an initial segment of the monomials in the fiber of α. We say that a monomial ideal L is $\mathcal{A}$-*multilex* (or multilex) if for every $\alpha \in \mathbb{N}^c$, the vector space L_α is a multilex segment.

Theorem 4.5.4. *There exists an $\mathcal{A}$-multilex ideal $L_{\mathcal{A}}$ with the same Hilbert function as the toric ideal $I_{\mathcal{A}}$. The Betti numbers of $L_{\mathcal{A}}$ are greater or equal to those of $I_{\mathcal{A}}$.*

Proof. Order the monomials in each fiber lexicographically. For $\alpha \in \mathbb{N}^c$, denote by m_α the last monomial in the fiber of α. Let L_α be the vector space spanned by all monomials in the fiber of α except m_α. Set $L_{\mathcal{A}} = \oplus_\alpha L_\alpha$, where we consider $L_{\mathcal{A}}$ as a vector space. By (4.5.2), it follows that $L_{\mathcal{A}}$ and $I_{\mathcal{A}}$ have the same Hilbert function.

Denote by $>_{\text{Lex}}$ the lex order on monomials. We will show that $L_{\mathcal{A}}$ is the initial ideal of $I_{\mathcal{A}}$ with respect to $>_{\text{Lex}}$; in particular, $L_{\mathcal{A}}$ is an ideal. Let m be a monomial in L_α. Then $m - m_\alpha \in I_{\mathcal{A}}$ and $m <_{\text{Lex}} m_\alpha$. Hence m is in the initial ideal of $I_{\mathcal{A}}$. Therefore, $L_{\mathcal{A}}$ is contained in the initial ideal. Since the multigraded Hilbert functions of $L_{\mathcal{A}}$ and $I_{\mathcal{A}}$ are the same, it follows that $L_{\mathcal{A}}$ is the initial ideal.

Clearly, $L_{\mathcal{A}}$ is multilex by construction. Since it is an initial ideal, it follows that the Betti numbers of $L_{\mathcal{A}}$ are greater or equal to those of $I_{\mathcal{A}}$. $\square$

Example 4.5.5. It should be noted that $L_{\mathcal{A}}$ depends not only on $\mathcal{A}$, but also on the choice of lexicographic order (that is, on the order of variables). For example, for the vanishing ideal $I_{\mathcal{A}} = (ad - bc,\ b^2 - ac,\ c^2 - bd)$ of the twisted cubic curve, one can get $L_{\mathcal{A}}$ to be (ac, ad, bd) if $a > b > c > d$ and (b^2, bc, bd, c^3) if $b > c > a > d$. These two multilex ideals have different Betti numbers.

Chapter 5

Monomial Regular Sequences*

5.1 Introduction

We study Hilbert functions of ideals containing a regular sequence of monomials.

Macaulay [Ma] showed in 1927 that all Hilbert functions over $S = k[x_1, \cdots, x_n]$ are attained by lex ideals. Over what quotients of S is this true? Let M be a monomial ideal of S. We say that M is *Lex-Macaulay* if every Hilbert function over S/M is attained by a lex ideal of S/M. Clements and Lindström [CL] proved that $(x_1^{e_1}, \cdots, x_n^{e_n})$ is Lex-Macaulay for $e_1 \leq \cdots \leq e_n$. $(x_1^{e_1}, \cdots, x_n^{e_n})$ is the most important example of a *monomial regular sequence*, a set of monomials $\{f_1, \cdots, f_r\}$ satisfying $\gcd(f_i, f_j) = 1$ for all $i \neq j$. In section 5.3 we classify the monomial regular sequences which are Lex-Macaulay: Theorem 5.3.8 says that a regular sequence of monomials is Lex-Macaulay if and only if it has the form $(x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$, with $e_1 \leq \cdots \leq e_r$ and $y = x_i$ for some $i \geq r$.

Eisenbud, Green, and Harris [EGH1, EGH2] made the following conjecture motivated by applications in algebraic geometry:

Conjecture 5.1.1 (Eisenbud-Green-Harris)**.** *Let N be any homogeneous ideal containing a regular sequence in degrees $e_1 \leq \cdots \leq e_r$. There is a lex ideal L such that N and $L + (x_1^{e_1}, \cdots, x_r^{e_r})$ have the same Hilbert function.*

In the original conjecture, $r = n$. The conjecture is wide open. We prove it for ideals containing a regular sequence of monomials in Theorem 5.2.9.

* This chapter is modified from the paper "Monomial regular sequences", which has been submitted for publication.

5.2 $\mathcal{A}$-compression

We recall from section 2.2.3 the definition of compressed ideals.

Definition 5.2.1. Let $\mathcal{A}$ be a subset of $\{x_1, \cdots, x_n\}$, and let M be a monomial ideal. Then M decomposes as a direct sum of vector spaces

$$M = \bigoplus_{\substack{f \in k[\mathcal{A}^c] \\ \text{monomial}}} f M_f,$$

where f runs over all monomials not involving the variables of $\mathcal{A}$ and each M_f is an ideal of $k[\mathcal{A}]$.

If every M_f is a lex ideal of $k[\mathcal{A}]$, we say that M is $\mathcal{A}$-*compressed*. By Macaulay's theorem, there exist lex ideals $T_f \subset k[\mathcal{A}]$ having the same Hilbert functions as the M_f. The vector space $T = \oplus f T_f$ is called the $\mathcal{A}$-*compression* of M.

Remark 5.2.2. What [CL,MP1,MP2] called i-compressed ideals (or monomial spaces) are $\{x_1, \cdots, \hat{x_i}, \cdots, x_n\}$-compressed in this notation. (Here the $\hat{x_i}$ means that x_i is omitted.) This reversal is for simplicity in several proofs below, and so that $\mathcal{A}$-compressed ideals will remain $\mathcal{A}$-compressed after new variables are added to the ring S.

Example 5.2.3. If $\mathcal{A}$ is a one-element set, every monomial space is $\mathcal{A}$-compressed.

Example 5.2.4. Let $M = (a^3, a^2b, a^2c, ab^2, abc, abd, b^3, b^2c)$. Then M is not $\{a, b\}$-compressed because $V_d = (ab)$ is not lex in $k[a, b]$, but M is $\mathcal{A}$-compressed for every other two-element set $\mathcal{A}$.

Definition 5.2.5. A monomial ideal M is $\mathcal{A}$-*compressed* if it is $\mathcal{A}$-compressed in every degree d.

It is easy to prove the following two propositions:

Proposition 5.2.6. *A lex ideal (or lex monomial space) is $\mathcal{A}$-compressed for every $\mathcal{A}$.*

Proposition 5.2.7. *Suppose $\mathcal{A} \supset \mathcal{B}$. Then every $\mathcal{A}$-compressed ideal (or $\mathcal{A}$-compressed monomial space) is $\mathcal{B}$-compressed.*

Example 5.2.8. Borel ideals are $\mathcal{A}$-compressed for every two-element set $\mathcal{A}$.

Due to the following lemma, which is a restatement of Lemma 2.2.7, $\mathcal{A}$-compressed ideals are useful in the study of Hilbert functions:

Lemma 5.2.9. *Let M_d be a d-monomial vector space, and let T_d be its $\mathcal{A}$-compression. Then $|\mathbf{m}T_d| \leq |\mathbf{m}N_d|$.*

Using Lemma 5.2.9 we prove Conjecture 5.1.1 for ideals containing a regular sequence of monomials:

Theorem 5.2.10. *Let $F = (f_1 \cdots, f_r)$ be a regular sequence of monomials with $\deg f_i = e_i$ and $e_1 \leq \cdots \leq e_r$. Let N be any homogeneous ideal containing F. Then there is a lex ideal L such that N and $L + (x_1^{e_1}, \cdots, x_r^{e_r})$ have the same Hilbert function.*

Proof. Set $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. It suffices to show that, for any d-vector space N_d containing F_d, there is a lex monomial space L_d such that $|L_d + P_d| = |N_d|$ and $|\mathbf{m}L_d + P_{d+1}| \leq |\mathbf{m}N_d|$.

Reorder the variables so that x_i divides f_i for all i. By Gröbner basis theory, we may assume (after taking an initial ideal if necessary) that N_d is a monomial space. Set $N(0) = N_d$. For each $i \leq r$, let $\mathcal{A}_i$ be the set of variables dividing f_i, and let $N(i)$ be the $\mathcal{A}_i$-compression of $N(i-1)$. Then $N(i)$ contains $x_i^{e_i}$ if $d \geq e_i$. Furthermore, $|N(r)| = |N_d|$ and $|\mathbf{m}N(r)| \leq |\mathbf{m}N_d|$ by Lemma 5.2.9. By

Clements-Lindström's theorem, there is a lex space L_d such that $|L_d+P_d| = |N(r)|$ and $|\mathbf{m}L_d + P_{d+1}| \leq |\mathbf{m}N(r)|$. $\square$

5.3 Lex-Macaulay monomial regular sequences

Throughout this section, let M be a monomial ideal of S. We recall the following definitions and results from chapter 3:

Definition 5.3.1. M is *Lex-Macaulay* if every Hilbert function in the quotient S/M is attained by a lex ideal. Equivalently, M is Lex-Macaulay if, for every d and every d-monomial vector space V_d, there exists a lex space L_d such that $|L_d + M_d| = |V_d + M_d|$ and $|\mathbf{m}L_d + M_{d+1}| \leq |\mathbf{m}V_d + M_{d+1}|$.

Theorem 5.3.2. *If M is Lex-Macaulay as an ideal of S, then it is Lex-Macaulay as an ideal of $S[y]$.*

Proposition 5.3.3. *If M is Lex-Macaulay and L is lex, then $L + M$ is Lex-Macaulay.*

Now, we will characterize the monomial regular sequences which are Lex-Macaulay.

Let $F = (f_1, \cdots, f_r)$ be a monomial regular sequence (that is, $\gcd(f_i, f_j) = 1$ for all $i \neq j$), and order these monomials so that $i < j$ if $\deg f_i < \deg f_j$, or if $\deg f_i = \deg f_j$ and $f_i >_{\text{Lex}} f_j$. Set $e_i = \deg f_i$, and suppose throughout that $e_i > 1$.

Lemma 5.3.4. *Suppose that F is Lex-Macaulay. Then $x_i^{e_i-1}$ divides f_i.*

Proof. By induction we have $x_j^{e_j-1}$ divides f_j for $j < i$, and by Proposition 5.3.3 $F + (x_1, \cdots, x_{i-1})$ is Lex-Macaulay, so we may assume without loss of generality that $i = 1$. Let g be any monomial in degree $e_1 - 1$ dividing f_1. Then, since F is Lex-Macaulay, we have $|(x_1^{e_1-1})_{e_1}| \leq |(g)_{e_1}| \leq n-1$, i.e., $x_1^{e_1-1}$ divides f_1. $\square$

Lemma 5.3.5. *Suppose that F is Lex-Macaulay, and write $f_i = x_i^{e_i-1}y_i$. Suppose that $y_i \neq x_i$. Then $i = r$.*

Proof. We may assume as in the proof of Lemma 5.3.4 that $i = 1$. Suppose $r \neq 1$ and $y_1 \neq x_1$. Set $g = x_1^{e_1-1}x_2^{e_2-1}$. Then $|(g)_{e_1+e_2-1}| = n-2$, while $|(x_1^{e_1+e_2-2})_{e_1+e_2-1}| = n-1$, so F is not Lex-Macaulay. $\square$

Thus, all Lex-Macaulay monomial regular sequences may be written in the form $(x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$. We will show conversely that all such sequences are Lex-Macaulay.

Lemma 5.3.6. *Suppose $r \neq n$, and let $F = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_n)$. Let N_d be a d-monomial vector space containing F_d. Then there exists a d-monomial vector space T_d containing Q_d, where $Q = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$ such that $|T_d| = |M_d|$ and $|\mathbf{m}T_d| \leq |\mathbf{m}M_d|$.*

Proof. Set $\mathcal{A} = \{x_n, x_{n-1}\}$, and take T_d to be the $\mathcal{A}$-compression of M_d. Apply Lemma 5.2.9, and note that Q is the $\mathcal{A}$-compression of F. $\square$

Lemma 5.3.7. *Suppose $r \neq n$, and let $F = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_n)$ and $Q = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}x_{n-1})$. Let L_d be a lex segment in S. Then there exists a lex d-monomial space T_d such that $|T_d+F_d| = |L_d+Q_d|$ and $|\mathbf{m}T_d+F_{d+1}| = |\mathbf{m}L_d+Q_{d+1}|$.*

Proof. Let T_d be the smallest lex segment such that $T_d + Q_d = L_d + Q_d$.

We claim that T_d satisfies the property: If g is a monomial such that $gx_r^{e_r-1}x_{n-1} \in T_d$, then $gx_r^{e_r-1}x_n \in T_d$. We will prove this claim. Observe that $gx_r^{e_r-1}x_{n-1} \in Q_d$, and that $gx_r^{e_r-1}x_n$ is the successor of $gx_r^{e_r-1}x_{n-1}$ in the graded lex order. By construction, if $gx_r^{e_r-1}x_{n-1} \in T_d$, we must have a monomial $v \in T_d$ which comes lexicographically after $gx_r^{e_r-1}x_{n-1}$ and $v \notin Q_d$. Then v is lexicographically after $gx_r^{e_r-1}x_n$ as well, and since T_d is a lex segment, we have $gx_r^{e_r-1}x_n \in T_d$, proving the claim.

Set $A = T_d + (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}})_d$ and B and C such that $T_d + Q_d = A \oplus B$ and $T_d + F_d = A \oplus C$. Then if $\{B\}$ and $\{C\}$ are the sets of monomials of B and C, respectively, we have:

$$\begin{aligned}
\{B\} &= \{gx_r^{e_r-1}x_{n-1} : gx_r^{e_r-1}x_{n-1} \notin A\} \\
\{C\} &= \{gx_r^{e_r-1}x_n : gx_r^{e_r-1}x_n \notin A\} \\
&= \{gx_r^{e_r-1}x_n : gx_r^{e_r-1}x_{n-1} \in B\}.
\end{aligned}$$

In particular, multiplication by $\frac{x_n}{x_{n-1}}$ is a bijection from $\{B\}$ to $\{C\}$. Thus $|\mathbf{m}B| = |\mathbf{m}C|$ and $|\mathbf{m}B \cap \mathbf{m}A| = |\mathbf{m}C \cap \mathbf{m}A|$, so we have $|\mathbf{m}(T_d + F_d)| = |\mathbf{m}(T_d + Q_d)| = |\mathbf{m}(L_d + Q_d)|$, the first equality by inclusion-exclusion, the second by construction.

Now $|\mathbf{m}T_d + F_{d+1}| = |\mathbf{m}(T_d + F_d)|$ unless F has minimal monomial generators in degree $d+1$ which are not in $\mathbf{m}T_d$; likewise $|\mathbf{m}L_d + Q_{d+1}| = |\mathbf{m}(L_d + Q_d)|$ unless Q has minimal monomial generators in degree $d+1$ which are not in $\mathbf{m}L_d$ and hence not in $\mathbf{m}T_d$. Since $x_r^d x_{n-1} \in \mathbf{m}T_d$ if and only if $x_r^d x_n \in \mathbf{m}T_d$, we obtain $|\mathbf{m}T_d + F_{d+1}| = |\mathbf{m}L_d + Q_{d+1}|$ as desired. $\square$

Theorem 5.3.8. *Let F be a regular sequence of monomials. Then F is Lex-Macaulay if and only if $F = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r-1}y)$, with $e_1 \leq \cdots \leq e_r$ and $y = x_i$ for some $i \geq r$.*

Proof. If F is Lex-Macaulay, apply Lemmas 5.3.4 and 5.3.5.

Conversely, suppose F has the form above. By Theorem 5.3.2, we may assume $y = x_n$. If $n = r$, this is Clements-Lindström's Theorem; otherwise, we induct on $n - r$. Set $Q = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r - 1}x_{n-1})$. Choose a degree d, and let N_d be any d-monomial space containing F_d. By Lemma 5.3.6, there is a d-monomial space T_d containing Q_d with $|T_d| = |N_d|$ and $|\mathbf{m}T_d| \leq |\mathbf{m}N_d|$. Q is Lex-Macaulay by induction, so there is a monomial space L_d containing Q_d with $|L_d| = |T_d|$ and $|\mathbf{m}L_d| \leq |\mathbf{m}T_d|$. By Lemma 5.3.7, there is a monomial space T_d containing F_d with $|T_d| = |L_d|$ and $|\mathbf{m}T_d| = |\mathbf{m}L_d|$. Thus F is Lex-Macaulay. $\square$

Chapter 6

Compressed Ideals*

6.1 Introduction

Lex ideals are important in the study of a polynomial ring $S = k[x_1, \cdots, x_n]$ because they can be used to classify the Hilbert functions of ideals in S. An important tool in the study of lex ideals, dating back to Macaulay [Ma], has been compression, which allows one to move carefully towards the lex ideal while controlling the Hilbert function. Compressed ideals are combinatorially very well-behaved, which allows us to compare their invariants to those of lex ideals in ways which are impossible for monomial or even Borel ideals. The goal of this chapter is to codify the theory of compression and show how it may be used to recover some classical results on Hilbert functions and Betti numbers.

Throughout this chapter, S is the polynomial ring $k[x_1, \cdots, x_n]$ and R is the quotient of S by the squares of the variables, $R = S/(x_1^2, \cdots, x_n^2)$.

In section 6.2, we study compressed ideals, culminating in the classification of compressed ideals of S and R, respectively, in Theorems 6.2.11 and 6.2.12. Theorem 6.2.9 reduces many questions about lex ideals to questions about lex ideals of $k[a, b, c]$ and an inductive step. This will be illustrated in sections 6.3 and 6.4.

In section 6.3, we use compressed ideals to give new proofs of the theorems of Macaulay [Ma] and Kruskal-Katona [Kr, Ka] that every Hilbert function in S (and, respectively, R), is attained by a lex ideal.

* This chapter is modified from the paper "Compressed Ideals", which has been submitted for publication.

In section 6.4, we show that Betti numbers are nondecreasing under compression. As an application, we recover the theorem of Bigatti, Hulett, and Pardue [Bi, Hu, Pa] that lex ideals have maximal graded Betti numbers in S.

In the short section 6.5, we make some comments about possible applications to the Hilbert scheme.

6.2 Structure of compressed ideals

We make a number of observations about $\mathcal{A}$-compressed ideals.

Proposition 6.2.1. *Suppose that $N \subset S$ is $\mathcal{A}$-compressed. Set $B = S[y]$. Then NB is $\mathcal{A}$-compressed as an ideal of B.*

Remark 6.2.2. Proposition 6.2.1 holds regardless of the position of y in the lexicographic order.

Proposition 6.2.3. *If N is lex, then N is $\mathcal{A}$-compressed.*

Proposition 6.2.4. *If N is $\mathcal{A}$-compressed and $\mathcal{A} \supset \mathcal{B}$, then N is $\mathcal{B}$-compressed.*

Proposition 6.2.5. *N is $\{x_i\}$-compressed for any x_i.*

Definition 6.2.6. Let r be a positive integer. If N is $\mathcal{A}$-compressed for every r-element set $\mathcal{A}$, we say that N is *r-compressed.* If N is $\mathcal{A}$-compressed for every proper subset $\mathcal{A}$ of $\{x_1, \cdots, x_n\}$, we simply say that N is *compressed.*

Proposition 6.2.7. *N is 2-compressed if and only if N is strongly stable.*

Remark 6.2.8. Up to this point, everything has held in somewhat more generality. The ring S could have been replaced by, for example, a quotient of the form $k[x_1, \cdots, x_n]/(x_1^{e_1}, \cdots, x_n^{e_n})$ with $e_1 \leq \cdots \leq e_n \leq \infty$ without meaningful modification to any of the statements or proofs. (Macaulay's theorem in such a ring is known as Clements-Lindström's theorem [CL].) Beginning with Theorem 6.2.9, however, it will be essential that we work over the correct ring.

Theorem 6.2.9. *Let N be a monomial ideal of S. N is 3-compressed if and only if N is lex.*

Theorem 6.2.9 is a corollary of the following sharper result:

Proposition 6.2.10. *Suppose that $N \subset S$ is 2-compressed and also $\mathcal{A}$-compressed for every set $\mathcal{A}$ of the form $\{x_i, x_{i+1}, x_n\}$. Then N is lex.*

Proof. Let $u \in N$ be a monomial of degree d, and let v be another monomial of degree d, lex-before u. We will show that $v \in N$.

Write $u = \prod x_i^{e_i}$ and $v = \prod x_i^{f_i}$. Let i be minimal such that $e_i \neq f_i$; we have $e_i < f_i$. Put $w = \prod_{j=1}^{i} x_i^{e_i}$, $u' = \frac{u}{w}$, and $v' = \frac{v}{x_i w}$. Set $D = \deg u'$, and observe that $u \in k[x_{i+1}, \cdots, x_n]$, $v \in k[x_i, \cdots, x_n]$.

Since $u = wu' \in N$ and N is strongly stable, we have $wx_{i+1}^D \in N$.

Since N is $\{x_i, x_{i+1}x_n\}$-compressed, we have $wx_i x_n^{D-1} \in N$.

Since N is strongly stable, we have $wx_i v' = v \in N$. $\square$

Propositions 6.2.5 and 6.2.7 and Theorem 6.2.9 combine to give us the following structure theorem for compressed ideals of S:

Theorem 6.2.11. *We classify the compressed ideals of S as follows:*

- *If $n < 3$, every monomial ideal is compressed.*
- *If $n = 3$, the compressed ideals are precisely the strongly stable ideals.*
- *If $n > 3$, the compressed ideals are precisely the lex ideals.*

We can also describe the compressed ideals of R, as follows:

Theorem 6.2.12. *Let N be a compressed ideal of R. Then:*

- *If n is odd, N is lex.*
- *If $n = 2r$ is even, the vector space N_d is lex for every $d \neq r$.*
- *If $n = 2r$ and $u \in N$, $v \notin N$ are degree r monomials with v lex-before u, then $u = x_2x_3 \cdots x_{r+1}$ and $v = x_1x_{r+2}x_{r+3} \cdots x_{2r}$.*

In particular, if N is not lex, then N_d is generated by $\{(x_1)_d\} \setminus \{v\}$ and u, where u and v are as above and $\{(x_1)_d\}$ denotes the monomials in $(x_1)_d$. That is, if N is not lex, N_d is generated by the lex segment terminating at u, with a single gap at v. Note that u is the successor of v in the lex order.

Proof. Suppose that $u \in N$ and $v \notin N$ both have degree r, and v is lex-before u. Write $u = \prod x_i^{e_i}$ and $v = \prod x_i^{f_i}$. Set $\mathcal{A} = \{x_i : e_i \neq f_i\}$.

N cannot be $\mathcal{A}$-compressed, so we must have $\mathcal{A} = \{x_1, \cdots, x_n\}$. On the other hand, $\mathcal{A} \subset \text{supp}(u) \cup \text{supp}(v)$. Thus $\mathcal{A} = \text{supp}(u) \cup \text{supp}(v)$ and $n = 2r$.

We have $\text{supp}(u) \cap \text{supp}(v) = \varnothing$, and x_1 divides v, since v is lex-before u. If $n = 2$ we are done, otherwise suppose that x_i divides v, for some $2 \leq i \leq r+1$. Then there exists x_j dividing u with $j > r+1$. Since N is $\{x_i, x_j\}$-compressed, we have $u\frac{x_i}{x_j} \in N$. Then, since N is $(\{x_1, \cdots, x_n\} \setminus \{x_i, x_j\})$-compressed, we have $v \in N$. Thus every $x_i, 2 \leq i \leq r+1$, must divide u, so $u = x_2x_3 \cdots x_{r+1}$ and $v = x_1x_{r+2}x_{r+3} \cdots x_{2r}$. ∎

6.3 Macaulay's Theorem

Macaulay's theorem classifies the Hilbert functions over S in terms of lex ideals:

Theorem 6.3.1 (Macaulay)**.** *Every Hilbert function over S is attained by a lex ideal. That is, let I be any homogeneous ideal of S. Then there exists a lex ideal L such that* $\mathrm{Hilb}_L^S = \mathrm{Hilb}_I^S$.

Macaulay's original proof is probably the first example of a compression argument (using $\mathcal{A} = \{x_2, \cdots, x_n\}$), but is sufficiently opaque that he felt it necessary to warn his readers away, saying:

> This proof of the theorem... is given only to place it on record. It is too long and complicated to provide any but the most tedious reading.

A number of other proofs have appeared since, most notably that of Green [Gr1]. In this section, we present one more.

Our proof is by induction on n, and so will make free use of Theorem 6.3.2, which was proved using Macaulay's theorem on the smaller ring $k[\mathcal{A}]$. We begin with some remarks on compression.

First, we recall the key result, proved in Lemma 2.2.7:

Theorem 6.3.2. *Let N be a monomial ideal, and let T be its $\mathcal{A}$-compression. Then T is an ideal as well.*

Lemma 6.3.3. *Let N be a monomial ideal, $\mathcal{A} \subsetneq \{x_1, \cdots, x_n\}$ any set of variables, and T the $\mathcal{A}$-compression of N. Then T has the same Hilbert function as N, and T is lexicographically greater than N.*

Proof. Every T_f has the same Hilbert function as N_f, and is lexicographically greater than N_f. □

Lemma 6.3.4. *Let N be any homogeneous ideal, and let $\mathfrak{A}$ be any collection of proper subsets of $\{x_1, \cdots, x_n\}$. Then there exists an ideal T, having the same Hilbert function as N, which is $\mathcal{A}$-compressed for all $\mathcal{A} \in \mathfrak{A}$.*

Proof. Set $T_0 = N$ and proceed inductively as follows: If $\mathcal{A} \in \mathfrak{A}$ is such that T_i is not $\mathcal{A}$-compressed, let T_{i+1} be the $\mathcal{A}$-compression of T_i. Then all T_i have the same Hilbert function, and T_{i+1} is lexicographically greater than T_i for all i. Since "lexicographically greater than" is a well-ordering on the (finite) sets of monomials in S, this process must stabilize in degree less than or equal to d, say at $T_{s(d)}$, for all d. Let T be the ideal whose degree-d components are the $T_{s(d)}$. □

Corollary 6.3.5. *Let N be any homogeneous ideal. Then there exists a compressed ideal T having the same Hilbert function as N.*

We are now ready for the proof of Macaulay's theorem:

Proof of Macaulay's Theorem 6.3.1. We may assume by Gröbner basis theory that N is monomial. If $n = 1$ or 2, the theorem is now obvious.

Otherwise, by corollary 6.3.5, we may assume that N is compressed. If $n \geq 4$, Theorem 6.2.11 shows that N is lex.

This leaves the case that $S = k[a, b, c]$ and N is strongly stable. It suffices to show for every degree d that, if L_d is the vector space spanned by the lex-first $|N_d|$ monomials of S, we have $|(a, b, c)L_d| \leq |(a, b, c)N_d|$. Since N_d is strongly stable, we have

$$|(a, b, c)N_d| = |N_d| + |N_d \cap k[a, b]| + |N_d \cap k[a]|$$

and likewise

$$|(a, b, c)L_d| = |L_d| + |L_d \cap k[a, b]| + |L_d \cap k[a]|.$$

$|N_d| = |L_d|$ by construction, and $N_d \cap k[a] = L_d \cap k[a] = (a^d)$, so it suffices to show that $|N_d \cap k[a, b]| \geq |L_d \cap k[a, b]|$.

Suppose that u, v are degree d monomials with $u \in N$, $v \notin N$, and v lex-before u. Then c divides v, as otherwise the strongly stable condition, applied to u, would require $v = (\frac{a}{b})^i(\frac{b}{c})^j u \in N$. In particular, any $v \in L_d \smallsetminus N_d$ is not in $k[a,b]$, yielding the desired inequality. $\square$

Macaulay's theorem is known to hold in many quotients of S [Kr, Ka, CL, Sh, MP1, MP2, Me1]. The first extension was due to Kruskal [Kr] and Katona [Ka], who showed that it holds in $R = S/(x_1^2, \cdots, x_n^2)$:

Theorem 6.3.6 (Kruskal, Katona)**.** *Macaulay's theorem holds in R. That is, if N is any homogeneous ideal of R, there exists a lex ideal L such that L and N have the same Hilbert function.*

We prove Kruskal-Katona's theorem, in the spirit of our proof of Macaulay's theorem:

Proof. We may assume by Gröbner basis theory that N is monomial. If $n = 1$ or 2, the theorem is now obvious.

Otherwise, we may assume by Corollary 6.3.5 that N is compressed. If N is lex we are done.

If N is not lex, we have by Theorem 6.2.12 that $n = 2r$ and that N fails to be lex only in degree r, and only by containing $u = x_2 \cdots x_{r+1}$ but not $v = x_1 x_{r+2} \cdots x_n$. Let $\{N\}$ be the set of monomials of N, and let L be the vector space spanned by $\{N\} \smallsetminus \{u\} \cup \{v\}$. Clearly, L has the same Hilbert function as N; we claim that it is an ideal. Indeed N (hence L) contains every multiple of v (as $x_i v = x_n(\frac{x_i}{x_n}v)$) and no divisor of u (as $\frac{u}{x_i} \in N$ would force $\frac{x_n}{x_i}u \in N$). $\square$

6.4 Betti numbers

Definition 6.4.1. If I is a homogeneous S-module, a *free resolution* of I is an exact sequence $\mathbb{F} : \cdots \to F_1 \to F_0 \to I \to 0$, with each F_i a free module. $\mathbb{F}$ is *minimal* if each F_i has minimum possible rank. The F_i may be graded $F_i = \oplus S(-d)^{b_{i,d}}$ so that each map $F_{i+1} \to F_i$ is homogeneous of degree zero. If a minimal free resolution is graded in this way, the *graded Betti numbers* of I are the $b_{i,d}$.

Bigatti [Bi], Hulett [Hu], and Pardue [Pa] showed that the graded Betti numbers of lex ideals of S are maximal among those of all homogeneous ideals with a fixed Hilbert function:

Theorem 6.4.2 (Bigatti, Hulett, Pardue)**.** *Let I be a homogeneous ideal of S, and let L be the lex ideal having the same Hilbert function as I. Then, for all i, d, we have $b_{i,d}(L) \geq b_{i,d}(I)$.*

Analogous results are known in many rings which satisfy Macaulay's theorem [AHH, GHP, MPS], and are widely conjectured in general [FR].

In this section we show that the graded Betti numbers are nondecreasing under $\mathcal{A}$-compression, and provide a new proof of the results of Bigatti, Hulett, and Pardue over S. Our argument is in the spirit of Hartshorne's proof that the Hilbert scheme is connected [Ha].

Lemma 6.4.3. *Let k have characteristic zero, let $S = k[a, b]$, and let N be a monomial ideal of S. Let L be the lex ideal with the same Hilbert function as N. Then there exists a change of variables f such that the initial ideal of $f(N)$ with respect to the lex order is L, and the initial ideal of $f(N)$ with respect to the inverse lex order is N.*

Proof. Set $f(a) = a$ and $f(b) = a + b$. (Alternatively, a generic linear form for $f(b)$ will work.) $\square$

Lemma 6.4.4. *Let $S = k[a, b, c]$ (over an arbitrary infinite field k), and let N be strongly stable but not lex. Then there exists an ideal $\tilde{N}$ such that:*

- *N and $\tilde{N}$ have the same Hilbert function.*

- *N is the initial ideal of $\tilde{N}$ with respect to the grevlex order.*

- *The initial ideal of $\tilde{N}$ with respect to the lex order is strictly lexicographically greater than N.*

- *N and $\tilde{N}$ have the same graded Betti numbers.*

In particular, if N' is the lex initial ideal of $\tilde{N}$, we have $b_{i,j}(N') \geq b_{i,j}(\tilde{N}) = b_{i,j}(N)$ for all i, j.

Proof. Let S^{po} and N^{po} be the polarizations of S and N with respect to the variable b. (So if N is minimally generated by $\{a^{e_{i,1}}b^{e_{i,2}}c^{e_{i,3}}\}$, we have N^{po} generated by $\{a^{e_{i,1}}b_1b_2\cdots b_{e_{i,2}}c^{e_{i,3}}\}$). Let $f : S^{\mathrm{po}} \to S$ be defined by sending each b_j to a generic linear function of the form $f_j = f_{1,j}a + f_{2,j}b + f_{3,j}c$. Let $\tilde{N}$ be the image of N^{po} under f.

That N and $\tilde{N}$ have the same Hilbert function and graded Betti numbers is immediate from the theory of polarization (e.g., [Pe2]) (in particular, the operation $\tilde{(\cdot)}$ extends to any graded free resolution of N: a straightforward argument in the degree of the b-variables shows that $b_j - f_j$ and $b_j - b$ are non-zero-divisors on S/N. Then, by a well-known Tor argument, the Betti numbers of N and $\tilde{N}$ over S are equal to those of N^{po} over S^{po}.)

To see that N is the grevlex initial ideal of $\tilde{N}$, observe that, if $m = a^{e_1}b^{e_2}c^{e_3}$ is any monomial of N, then we have $\tilde{m} = c^{e_3}(\sum_{i=0}^{e_2} g_i a^{e_1+e_2-i}b^i) + c^{e_3+1}(\text{other terms}) \in \tilde{N}$. Subtracting an appropriate linear combination of $\tilde{n}$, for $n = a^{e_1+i}b^{e_2-i}c^{e_3} \in N$, gives us $m + c^{e_3+1}(\text{other terms}) \in \tilde{N}$, whose initial term is m.

To see that the lex initial ideal of $\tilde{N}$ is lexicographically greater than N, let $u \in N$ be the lex-first degree d monomial such that there exists $v \notin N$ lex-before

u; choose v to be the first such. Since the f are generic, there exists a linear combination $\sum \tilde{m}$, for $m \in N$ the monomials lex-before or equal to u, with leading term v: since N is strongly stable, we have $u = a^{e_1}b^{e_2}$, and so $v = a^{e_1+1}b^i c^{e_2-i-1}$; we could have chosen $f_j(b) = \alpha_j a + b$ for $j \leq i$ and $f_j(b) = \alpha_j a + b + \gamma_j c$ for $j > i$. ☐

Remark 6.4.5. We believe, based on some extremely limited computer experiments, that if k has characteristic zero, the functions $f_1(b) = a + b$ and $f_i(b) = a + b + c$, for $i > 1$, produce $\tilde{N}$ satisfying the desired conditions.

Remark 6.4.6. We believe that the functions f_i may be chosen so that the lex initial ideal of $\tilde{N}$ is the lexicographic ideal L. Unfortunately, we have no proof at this time.

Remark 6.4.7. A similar argument in two variables shows that lemma 6.4.3 holds in arbitrary characteristic.

Remark 6.4.8. The operation $\tilde{(\cdot)}$ defined in the proof of lemma 6.4.4 does not depend on the dimension of S. The proof continues to hold if $\{a, b, c\}$ is a subset of $\{x_1, \cdots, x_n\}$.

Theorem 6.4.9. *Let N be any monomial ideal of S, and let $\mathcal{A}$ be any subset of variables. If T is the $\mathcal{A}$-compression of N, then $b_{i,j}(T) \geq b_{i,j}(N)$ for all i, j.*

Proof. By induction on the cardinality of $\mathcal{A}$, we may assume that N is $\mathcal{B}$-compressed for all proper subsets $\mathcal{B}$ of $\mathcal{A}$. If $|\mathcal{A}| \geq 4$, we have $N = T$, by Theorem 6.2.11. If $|\mathcal{A}| = 2$ or 3, the proof of lemma 6.4.3 or 6.4.4, respectively, gives us a monomial ideal N' satisfying:

- The $\mathcal{A}$-compression of N' is T.

- N' is lexicographically greater than N

- $b_{i,j}(N') \geq b_{i,j}(N)$.

Since the monomial ideals with a fixed Hilbert function are well-ordered by "lexicographically greater than", we are done by induction. ☐

In fact, combining the proof of Theorem 6.4.9 and the proof in [Pe1] we obtain the following stronger result:

Theorem 6.4.10. *Under the assumptions of Theorem 6.4.9, the Betti numbers $b_{i,j}(N)$ of N can be obtained from the $b_{i,j}(T)$ by a sequence of consecutive cancellations.*

We obtain as a corollary the result of Bigatti [Bi], Hulett [Hu], and Pardue [Pa].

Theorem 6.4.11 (Bigatti, Hulett, Pardue)**.** *Let I be any homogeneous ideal of S, and let L be the lex ideal with the same Hilbert function as I. Then $b_{i,j}(L) \geq b_{i,j}(I)$ for all i, j.*

Proof. Let N be the initial ideal of I in any order, so $b_{i,j}(N) \geq b_{i,j}(I)$. Now apply Theorem 6.4.9 to the ideal N, with $\mathcal{A} = \{x_1, \cdots, x_n\}$. ☐

Remark 6.4.12 (On multidegrees)**.** Fix $\mathcal{A}$, and endow S with a multigraded structure as follows: If $m = fg$ is a degree d monomial with $f \in k[\mathcal{A}^c]$ and $g \in k[\mathcal{A}]$, set the *coarse multidegree* of m to be $\operatorname{cmdeg}(m) = (f, d)$. Then we may define coarsely homogeneous ideals, coarse Hilbert functions, etc., by analogy to the usual definitions in a graded ring. With this notation, an $\mathcal{A}$-compressed ideal is coarsely lex, and our results may be restated in more familiar terms:

- Theorem 6.3.2 states that every coarse Hilbert function is attained by a coarsely lex ideal.

- Theorem 6.4.9 states that, if N is coarsely homogeneous, and T is coarsely lex with the same coarse Hilbert function, then the coarse Betti numbers of T are greater than or equal to those of N.

- Lemmas 6.4.3 and 6.4.4, together with Theorem 6.2.11, show that the coarse Hilbert scheme is connected.

6.5 Remarks on the Hilbert scheme

We close with some remarks about possible applications to the Hilbert scheme, which parametrizes all ideals with a fixed Hilbert function.

It is known that the Hilbert scheme is connected. Hartshorne proves this [Ha] by showing that there is a path to the lex ideal from any point on the Hilbert scheme. Reeves [Re] and Pardue [Pa] have shown that there exists a path of length at most $d+2$, where $d \leq n$ is the degree of the Hilbert polynomial.

In section 6.4, we have shown that one can walk to lex by walking to a sequence of compressions. These moves are much simpler than those defined in [Re], which involved Borel fans. It is natural to ask how many of our "compression steps" are necessary to reach the lex ideal from any monomial ideal. There might be a nice bound in terms of $(n-2)$ and the radii of Hilbert schemes in $k[a, b, c]$ (since by proposition 6.2.10 it suffices to be simultaneously Borel and compressed with respect to the $n-2$ sets $\{x_i, x_{i+1}, x_n\}$.)

In a slightly different direction, it should be possible to perform multiple such compressions at once, as the coordinate changes involved in the compressions with respect to, say, $\{x_i, x_{i+1}, x_n\}$ and $\{x_{i+2}, x_{i+3}, x_n\}$, might not interact harmfully.

In exploring these questions, we would like it to be the case that if N is $\mathcal{B}$-compressed and T is its $\mathcal{A}$-compression, then T is $\mathcal{B}$-compressed as well. Unfortunately, this is not true in general; in fact it can be impossible to find $\mathcal{A}$ for which this holds:

Example 6.5.1. Let $N = (a^2, ab, ac, b^2, bc) \subset k[a, b, c, d]$. Then N is compressed with respect to every proper subset of $\{a, b, c, d\}$ except $\{a, b, d\}$. Its $\{a, b, d\}$-compression is $T = (a^2, ab, ac, ad, bc)$, which is not $\{b, c\}$-compressed.

In Lemma 4.2.8 we show that compression with respect to the set $\{x_1, \cdots, x_{n-1}\}$ is well-behaved in the sense that it takes strongly stable ideals to strongly stable ideals. More research in this direction might prove productive.

Chapter 7

Ideals Containing the Squares of the Variables*

7.1 Introduction

Throughout the chapter $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i, and $P = (x_1^2, \ldots, x_n^2)$ is the ideal generated by the squares of the variables.

It is well known how the Hilbert function changes when we add P to a squarefree monomial ideal I; this is given by the relation between the f-vector and the h-vector, cf. [BH]. It has been an open question how the Betti numbers change. We answer this question in Theorem 7.2.1, which provides a relation between the Betti numbers of I and those of $I + P$. In Theorem 7.3.3, we describe a basis of the minimal free resolution of $I + P$ in the case when I is Borel.

By Kruskal-Katona's Theorem [Kr,Ka], there exists a squarefree lex ideal L such that $L + P$ has the same Hilbert function as $I + P$. The ideal $L + P$ is called lex-plus-squares. It was conjectured by Herzog and Hibi that the graded Betti numbers of $L + P$ are greater than or equal to those of $I + P$. Later, Graham Evans conjectured the more general lex-plus-powers conjecture that, among all graded ideals with a fixed Hilbert function and containing a homogeneous regular sequence in fixed degrees, the lex-plus-powers ideal has greatest graded Betti numbers in characteristic 0. This conjecture is very difficult and wide open. Some

* This chapter is modified from the paper "Ideals containing the squares of the variables", which has been submitted for publication. It is joint work with Irena Peeva and Mike Stillman.

special cases are proved by G. Evans, C. Francisco, B. Richert, and S. Sabourin [ER,Fr1,Fr2,Ri,RS]. An expository paper describing the current status of the conjecture is [FR]. In 7.5.1 we prove the following:

Theorem 7.1.1. *Suppose that* $\operatorname{char}(k) = 0$. *Let* F *be a graded ideal containing* $P = (x_1^2, \ldots, x_n^2)$. *Let* L *be the squarefree lex ideal such that* F *and the lex-plus-squares ideal* $L + P$ *have the same Hilbert function. The graded Betti numbers of the lex-plus-squares ideal* $L + P$ *are greater than or equal to those of* F.

The methods used in [Bi,Hu,Pa,CGP] to show that the lex ideal has greatest Betti numbers are not applicable; see Examples 7.3.10 and 7.3.11. We use the technique of compression. Compression was introduced by Macaulay [Ma], and used by [Ma, CL,MP1,MP2,Me1,Me3] to study Hilbert functions. It is not known in general how Betti numbers behave under compression, but it is reasonable to expect that they increase. We address this in Section 7.4.

The proof of Theorem 7.1.1 consists of the following steps:

∘ In Section 7.5 (the proof of Theorem 7.5.1.), we reduce to the case of a squarefree Borel ideal (plus squares); this is not immediate because a generic change of variables does not preserve P.

∘ In Section 7.3, we reduce to the case of a squarefree $\{x_1, \ldots, x_{n-1}\}$-compressed Borel ideal (plus squares).

∘ In Section 7.4, we deal with squarefree $\{x_1, \ldots, x_{n-1}\}$-compressed Borel ideals.

Given the intricacy of the proof in the Borel case (Section 7.3 and 7.4), we think that the following particular case of the lex-plus-powers conjecture is of interest:

Conjecture 7.1.2. *The lex-plus-powers ideal has greatest graded Betti numbers among all Borel-plus-powers monomial ideals with the same Hilbert function.*

A refinement of the lex-plus-powers conjecture is to study consecutive cancellations in Betti numbers. In view of the result in [Pe1], it is natural to ask:

Problem 7.1.3. *Under the assumptions of Theorem 7.1.1, is it true that the Betti numbers of $L+P$ and those of F differ by consecutive cancellations?*

7.2 Squarefree monomial ideals plus squares

A monomial ideal is called *squarefree* if it is generated by squarefree monomials. If I is squarefree, then $I+P$ is called *squarefree-plus-squares.*

For a monomial m, let $\max(m)$ be the index of the lex-last variable dividing m, that is, $\max(m) = \max\{i \mid x_i \text{ divides } m\}$.

The ring S is standardly graded by $\deg(x_i) = 1$ for each i. In addition, S is $\mathbb{N}^n$-graded by setting the multidegree of x_i to be the i'th standard vector in $\mathbb{N}^n$. Usually we say that S is *multigraded* instead of $\mathbb{N}^n$-graded, and we say *multidegree* instead of $\mathbb{N}^n$-degree. For every vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$ there exists a unique monomial of degree $\mathbf{a}$, namely $\mathbf{x}^{\mathbf{a}} = x_1^{a_1} \cdots x_n^{a_n}$. If an element g (say in a module) has $\mathbb{N}^n$-degree $\mathbf{a}$, then we say that is has *multidegree* $\mathbf{x}^{\mathbf{a}}$ and denote $\deg(g) = \mathbf{x}^{\mathbf{a}}$. Denote by $S(-\mathbf{x}^{\mathbf{a}})$ the free S-module generated by one element in multidegree $\mathbf{x}^{\mathbf{a}}$. Every monomial ideal is multihomogeneous, so it has a multigraded minimal free resolution. Thus, the minimal free resolutions of S/I and $S/(I+P)$ are graded and multigraded. We will use both gradings.

For a subset $\sigma \subseteq \{1, \dots, n\}$, let $|\sigma|$ denote the number of elements in σ. We will abuse notation to sometimes identify a subset with the squarefree monomial supported on it, so σ may stand for $\prod_{j \in \sigma} x_j$. It will always be clear from context what is meant. By $S(-2\sigma)$ we denote the free S-module generated in multidegree $\prod_{i\in\sigma} x_i^2$.

Theorem 7.2.1. *Let I be a squarefree monomial ideal.*

(1) Set $F_i = \bigoplus_{|\sigma|=i} S/(I:\sigma)\big(-2\sigma\big)$, where $\sigma \subseteq \{1,\dots,n\}$. We have the long exact sequence

$$0 \to F_n \xrightarrow{\varphi_n} \cdots \xrightarrow{\varphi_2} F_1 \xrightarrow{\varphi_1} F_0 = S/I \to S/(I+P) \to 0 \tag{7.2.2}$$

with maps φ_i the Koszul maps for the sequence $x_1^2,\dots,x_n^2$.

(2) Each of the ideals $(I:\sigma)$ in (1) is a squarefree monomial ideal.

(3) $S/(I+P)$ is minimally resolved by the iterated mapping cones from (7.2.2).

(4) For the graded Betti numbers of $S/(I+P)$ we have

$$b_{p,s}(S/(I+P)) = \sum_{i=0}^{n}\left(\sum_{|\sigma|=i} b_{p+i,s+2i}(S/(I:\sigma))\right).$$

Proof. Since the ideal I is squarefree, it follows that $(I:\sigma^2) = (I:\sigma)$ is squarefree.

We have the exact Koszul complex $\mathbf{K}$ for the sequence $x_1^2,\dots,x_n^2$:

$$0 \to \bigoplus_{|\sigma|=n} S \xrightarrow{\varphi_n} \ldots \to \bigoplus_{|\sigma|=i} S \xrightarrow{\varphi_i} \bigoplus_{|\sigma|=i-1} S \to \ldots$$
$$\ldots \to \bigoplus_{|\sigma|=1} S = \bigoplus_{j=1}^{n} S \xrightarrow{\varphi_1} \bigoplus_{|\sigma|=0} S = S \to S/P \to 0\,.$$

We can write $\mathbf{K} = \mathbf{K}' \oplus \mathbf{K}''$, where $\mathbf{K}'$ consists of the components of $\mathbf{K}$ in all multidegrees $m \notin I$, and $\mathbf{K}''$ consists of the components of $\mathbf{K}$ in all multidegrees $m \in I$. Both $\mathbf{K}'$ and $\mathbf{K}''$ are exact. We will show that (7.2.2) coincides with $\mathbf{K}'$. Consider $\mathbf{K}$ as an exterior algebra on basis $e_1, \dots e_n$. The multidegree of the variable e_j is x_j^2. Let $f = me_{j_1} \wedge \cdots \wedge e_{j_i}$ be an element in $\mathbf{K}_i$ and m be a monomial. The multidegree of f is $mx_{j_1}^2 \dots x_{j_i}^2$. We have that $f \in \mathbf{K}'$ if and only if $mx_{j_1}^2 \dots x_{j_i}^2 \notin I$, if and only if $mx_{j_1} \dots x_{j_t} \notin I$, if and only if $m \notin (I : x_{j_1} \dots x_{j_i})$. Therefore, we have the vector space isomorphism

$$\mathbf{K}_i' \to \bigoplus_{|\sigma|=i} S/(I : \sigma)$$

$$me_{j_1} \wedge \cdots \wedge e_{j_i} \mapsto m \in S/(I : x_{j_1} \dots x_{j_i})\,.$$

This proves (1).

We will prove (3) by induction on $n - i$. Denote by K_i the kernel of φ_i. We have the short exact sequence

$$0 \to K_i \to \bigoplus_{|\sigma|=i} S/(I : \sigma) \to K_{i-1} \to 0\,.$$

Each of the ideals $(I : \sigma)$ is squarefree. By Taylor's resolution, it follows that the Betti numbers of $\bigoplus_{|\sigma|=i} S/(I : \sigma)$ are concentrated in squarefree multidegrees. On the other hand, the entries in the matrix of the map φ_i are squares of the variables. Therefore, there can be no cancellations in the mapping cone. Hence, the mapping cone yields a minimal free resolution of K_{i-1}.

(4) follows from (3). $\square$

The Hilbert function of a graded finitely generated module T is

$$\mathrm{Hilb}_T(i) = \dim_k (T_i).$$

For squarefree ideals, we consider also the squarefree Hilbert function, sHilb, that counts only squarefree monomials. It is well-known that if I and J are squarefree ideals, then S/I and S/J have the same Hilbert function if and only if $S/(I+P)$ and $S/(J+P)$ have the same Hilbert function; thus, I and J have the same Hilbert function if and only if they have the same squarefree Hilbert function.

Proposition 7.2.3. *Let I and J be squarefree monomial ideals with the same Hilbert function. Fix an integer $1 \leq p \leq n$. The graded modules $\bigoplus_{|\sigma|=p} (I : \sigma)$ and $\bigoplus_{|\sigma|=p} (J : \sigma)$ have the same Hilbert function and the same squarefree Hilbert function.*

Proof. We consider squarefree Hilbert functions. Set $I(p) = \bigoplus_{|\sigma|=p} (I : \sigma)$. Let τ be a squarefree monomial of degree d in $(I : \nu)$. Then $\nu\tau \in I_{d+p}$. If $|\nu \cap \tau| = s$, choose μ so that $\mu = \mathrm{lcm}(\nu, \tau)$ is a squarefree monomial in I_{d+p-s}.

Let μ be a squarefree monomial in I_{d+p-s}. We can choose ν in $\binom{d+p-s}{p}$ ways so that $|\nu| = p$ and ν divides μ. For each so chosen ν, we can choose τ in $\binom{p}{s}$ ways so that $|\nu \cap \tau| = s$ and τ divides μ. Therefore, the monomial μ contributes $\binom{d+p-s}{p}\binom{p}{s}$ monomials in $\big(I(p)\big)_d$. For such a monomial, we say that it is coming from I_{d+p-s}, or that its source is I_{d+p-s}.

Suppose that one element in $\big(I(p)\big)_d$ can be obtained in two different ways by this procedure. Since we have the same element in $\big(I(p)\big)_d$, it follows that ν and τ are fixed. But then, μ and s are uniquely determined. Hence, both the source and μ are uniquely determined. Therefore, one and the same element in $\big(I(p)\big)_d$ cannot be obtained in two different ways by the above procedure.

For a vector space Q spanned by monomials, we denote by $\mathrm{sdim}(Q)$ the number of squarefree monomials in Q. We have shown that

$$\mathrm{sdim}\left(\bigoplus_{|\sigma|=p} (I : \sigma)\right)_d = \sum_{x=0}^{n} \binom{d+p-s}{p}\binom{p}{s} \mathrm{sdim}(I_{d+p-s})\,. \tag{2.4}$$

The same formula holds for J as well. Now, the proposition follows from the well-known fact that for each $j \geq 0$ we have that $\mathrm{sdim}(I_j) = \mathrm{sdim}(J_j)$ since I and J are squarefree ideals with the same Hilbert function. $\square$

Let I be a squarefree monomial ideal, and Δ be its Stanley-Reisner simplicial complex. Let $\sigma \subseteq \{1,\dots,n\}$. We remark that it is well-known that the Stanley-Reisner simplicial complex of $(I:\sigma)$ is $\mathrm{star}_\Delta(\sigma) = \{\tau \in \Delta \,|\, \tau \cup \sigma \in \Delta\}$.

7.3 Squarefree Borel ideals plus squares

A squarefree monomial ideal N is *squarefree Borel* if, whenever $mx_j \in N$ is a squarefree monomial, and $i < j$ and mx_i is squarefree, we have $mx_i \in N$ as well. A squarefree monomial ideal L is *squarefree lex* if, whenever $m \in L$ is a squarefree monomial and m' is a squarefree monomial lexicographically greater than m, we have $m' \in L$ as well.

If N is squarefree Borel, then by Kruskal-Katona's Theorem [Kr,Ka], there exists a squarefree lex ideal L with the same Hilbert function.

Lemma 7.3.1.

- *(1) Let N be a squarefree Borel ideal. For any $\sigma \subseteq \{1,\dots,n\}$, the ideal $(N:\sigma)$ is squarefree Borel in the ring $S/(\{x_i|i \in \sigma\})$.*

- *(2) Let L be a squarefree lex ideal. For any $\sigma \subseteq \{1,\dots,n\}$, the ideal $(L:\sigma)$ is squarefree lex in the ring $S/(\{x_i|i \in \sigma\})$.*

The ideals $(N:\sigma)$ and $(L:\sigma)$ are generated by monomials in the smaller ring $S/(\{x_i|i \in \sigma\})$, so we may view them as ideals of $S/(\{x_i|i \in \sigma\})$.

For a monomial ideal M, we denote by $\mathrm{gens}(M)$ the set of monomials that generate M minimally.

Construction 7.3.2. If N is squarefree Borel, then the minimal free resolution of S/N is the squarefree Eliahou-Kervaire resolution [AHH] with basis denoted

$$\{1\} \cup \left\{ (h,\alpha) \,|\, h \in \mathrm{gens}(N),\ \alpha \subset \{1,\dots,n\},\ h\alpha \text{ is squarefree},\ \max(\alpha) < \max(h) \right\}.$$

The basis element (h, α) has homological degree $\deg(\alpha) + 1$, degree $\deg(h) + \deg(\alpha)$, and multidegree $h\alpha$; the basis element 1 has homological degree 0 and degree 0. In order to describe a basis of the minimal free resolution of $S/(N + P)$ we introduce EK-triples. For $\sigma \subseteq \{1, \ldots, n\}$, we say that $(\sigma,\ h,\ \alpha)$ is an *EK-triple* if $(h,\ \alpha)$ is a basis element in the minimal free resolution of $S/(N : \sigma)$. By Lemma 7.3.1, it follows that $(\sigma,\ h,\ \alpha)$ is an EK-triple $(\sigma,\ h,\ \alpha)$ if and only if:

$\circ$ $h \in \mathrm{gens}(N : \sigma)$

$\circ$ $\alpha = \{j_1, \ldots, j_t\}$ is an increasing sequence of numbers in the set $\{i \,|\, i \notin \sigma\}$, such that $1 \leq j_1 < \cdots < j_t < \max(h)$

$\circ$ $\sigma h\alpha$ is squarefree.

By Theorem 7.2.1, Lemma 7.3.1, and Construction 7.3.2, it follows that:

Theorem 7.3.3. *Let N be a squarefree Borel ideal. The minimal free resolution of $S/(N + P)$ has basis consisting of $\{1\}$ and the EK-triples. An EK-triple $(\sigma,\ h,\ \alpha)$ has homological degree $|\sigma| + |\alpha|$ and degree $2|\sigma| + |\alpha| + |h|$; it has multidegree $\sigma^2 h\alpha$. In particular, for all $p, s \geq 0$, the graded Betti number $b_{p,s}(S/(N + P))$ is equal to the number of EK-triples such that $p = |\sigma| + |\alpha|$ and $s = 2|\sigma| + |\alpha| + |h|$.*

We will prove:

Theorem 7.3.4. *Let N be a squarefree Borel and L be the squarefree lex with the same Hilbert function, (equivalently, let $N + P$ and $L + P$ have the same Hilbert function). For all p, s, the graded Betti numbers satisfy*

$$b_{p,s}(S/(L + P)) \geq b_{p,s}(S/(N + P)) \, .$$

For the proof, we need the notion of compression. Compression of ideals was introduced by Macaulay [Ma], and was used by Clements-Lindström [CL], Macaulay [Ma], Mermin [Me1,Me3], Mermin-Peeva [MP1,MP2] to study Hilbert functions.

Definition 7.3.5. Let I be a squarefree monomial ideal. We denote by $\bar{I}$ the monomial ideal in S/P generated by the squarefree monomials generating I. Fix a subset $\mathcal{A}$ of the variables, and let $\bar{T}$ be the $\mathcal{A}$-compression of $\bar{I}$ in S/P. Let T be the squarefree ideal of S generated by the nonzero monomials of $\bar{T}$. We say that T is the *squarefree compression* of I; by abuse of notation we will say "compression" instead of "squarefree compression" throughout the chapter.

We need the following lemmas:

Lemma 7.3.6. *Let I be a squarefree ideal. If I is $(2i-2)$-compressed for some $i \geq 1$, then I is $(2i-1)$-compressed.*

Proof. Let $v \in I$ be a squarefree monomial. Suppose that u is a squarefree monomial of the same degree such that $u > v$. Set $w = \gcd(u, v)$, so that we can write $u = u'w$, $v = v'w$ with $\gcd(u', v') = 1$. Suppose $|u'v'| \leq 2i-1$. Denote by $\mathcal{B}$ the set of variables that appear in exactly one of the monomials u and v. Since u and v have the same degree, it follows that the number of variables in $\mathcal{B}$ is even. Since $|\mathcal{B}| \leq 2i-1$, we have $|\mathcal{B}| \leq 2i-2$. Hence, I is $\mathcal{B}$-compressed. Therefore, $v \in I$ implies that $u \in I$. $\square$

Lemma 7.3.7. *Let N be a squarefree Borel ideal. Its $\{x_1, \ldots, x_{n-1}\}$-compression J is a squarefree Borel ideal.*

Proof. We have to show that $\bar{J}$ is a squarefree Borel ideal. Consider the direct sums

$$\bar{N} = x_n \bar{V}_{x_n} \oplus \bar{V}_1 \quad \text{and} \quad \bar{J} = x_n \bar{W}_{x_n} \oplus \bar{W}_1 \,.$$

Set $\mathbf{n} = (x_1, \ldots, x_{n-1})$. Since $\bar{N}$ is squarefree Borel, it follows that $\mathbf{n}\bar{V}_{x_n} \subseteq \bar{V}_1$. By Kruskal-Katona's Theorem [Kr, Ka] it follows that $\mathbf{n}\bar{W}_{x_n} \subseteq \bar{W}_1$. If $x_n m$ is a monomial in $x_n\bar{W}_{x_n}$, then for each $1 \leq i < n$ we have that $x_i m \in \bar{W}_1$. If x_j divides a monomial $m \in \bar{W}_1$ (respectively, $\bar{W}_{x_n}$), then for each $1 \leq i \leq j$ we have that $\frac{x_i m}{x_j} \in \bar{W}_1$ (respectively, $\bar{W}_{x_n}$) since $\bar{W}_1$ (respectively, $\bar{W}_{x_n}$) is squarefree lex. Thus, $\bar{J}$ is squarefree Borel. $\square$

Lemma 7.3.8. *Let N be a squarefree Borel ideal and J be its $\{x_1, \ldots, x_{n-1}\}$-compression. For all p, s, the graded Betti numbers satisfy*

$$b_{p,s}(S/(J+P)) \geq b_{p,s}(S/(N+P)) .$$

Proof. Set $A = S/x_n$. We assume, by induction on the number of variables, that Theorem 7.3.4 holds over the polynomial ring A.

Consider the EK-triples (σ, h, α) for N in degree (p, s). Let $c_{p,s}(N)$ be the number of triples such that x_n divides σ; let $d_{p,s}(N)$ be the number of triples such that x_n divides h; let $e_{p,s}(N)$ be the number of triples such that x_n does not divide $\sigma h \alpha$. Since x_n cannot divide α by Construction 7.3.2, it follows by Theorem 7.3.3 that

$$b_{p,s}(S/(N+P)) = c_{p,s}(N) + d_{p,s}(N) + e_{p,s}(N) .$$

Similarly, we introduce the numbers $c_{p,s}(J)$, $d_{p,s}(J)$, $e_{p,s}(J)$ and get

$$b_{p,s}(S/(J+P)) = c_{p,s}(J) + d_{p,s}(J) + e_{p,s}(J) .$$

We will show that the above introduced numbers for J are greater than or equal to the corresponding numbers for N.

As in the proof of Lemma 7.3.7, we write $\bar{N} = \bar{V}_1 \oplus x_n\bar{V}_{x_n}$ and $\bar{J} = \bar{W}_1 \oplus x_n\bar{W}_{x_n}$.

First, we consider the number $c_{p,s}(N)$. Note that $\bar{V}_{x_n} \supset \bar{V}_1$. Therefore, $(\bar{N} : x_n\tau) = (\bar{V}_{x_n} : \tau)$ for $x_n \notin \tau$. Hence, the EK-triples counted by $c_{p,s}(N)$ correspond bijectively to the EK-triples for V_{x_n} of degree $(p-1, s-2)$ by the correspondence $(x_n\tau, h, \alpha) \iff (\tau, h, \alpha)$. Thus,

$$c_{p,s}(N) = b_{p-1,s-2}(A/(V_{x_n} + (x_1^2, \dots, x_{n-1}^2)))\,.$$

By Lemma 7.3.7, J is squarefree Borel, so we get the same formula for J. By the construction of compression, the ideal W_{x_n} is the squarefree lex ideal in the polynomial ring A with the same Hilbert function as V_{x_n}. Since Theorem 7.3.4 holds over the ring A by the induction hypothesis, we conclude that $c_{p,s}(J) \geq c_{p,s}(N)$.

Now we consider the number $e_{p,s}(N)$. The EK-triples counted by $e_{p,s}(N)$ are exactly the EK-triples for V_1. Hence, $e_{p,s}(N) = b_{p,s}(A/(V_1 + (x_1^2, \dots, x_{n-1}^2)))$. The same equality holds for $e_{p,s}(J)$. By the construction of compression, the ideal W_1 is the squarefree lex ideal in the polynomial ring A with the same Hilbert function as V_1. Since Theorem 7.3.4 holds over the ring A by the induction hypothesis, we conclude that $e_{p,s}(J) \geq e_{p,s}(N)$.

It remains to consider $d_{p,s}(N)$. Since N is squarefree Borel, it follows that $\bar{V}_1 \supseteq \mathbf{n}\bar{V}_{x_n}$. Therefore, for each degree j

$$\{\mathrm{gens}(N:\sigma)_j \text{ that are divisible by } x_n\} = \{(\bar{x}_n(V_{x_n}:\sigma))_j\} \setminus \{(x_n(\bar{V}_1:\sigma))_j\}\,.$$

Hence, for each degree j, the number of minimal monomial generators of degree j of $(N:\sigma)$ that are divisible by x_n is

$$\dim_k(\bar{V}_{x_n}:\sigma)_{j-1} - \dim_k(\bar{V}_1:\sigma)_{j-1}\,.$$

For each such minimal monomial generator h, we have that $\max(h) = n$. Since α is prime to σ and $\mathrm{supp}(h)$, by Construction 7.3.2 we see that there are $\binom{n-|h|-|\sigma|}{|\alpha|}$

$= \binom{n-j-|\sigma|}{p-|\sigma|}$ possibilities for α in the EK-triples. By Theorem 7.3.3, we conclude that

$$d_{p,s}(N) = \sum_{n \notin \sigma} \binom{n-j-|\sigma|}{p-|\sigma|} \Big(\dim_k(\bar{V}_{x_n} : \sigma)_{s-p-|\sigma|-1} - \dim_k(\bar{V}_1 : \sigma)_{s-p-|\sigma|-1} \Big) .$$

As the ideal J is squarefree Borel by Lemma 7.3.6, the same formula holds for J. By the construction of compression, $\bar{V}_1$ and $\bar{W}_1$ have the same Hilbert function, as do $\bar{V}_{x_n}$ and $\bar{W}_{x_n}$. By the displayed formula for $d_{p,s}$ above and Proposition 7.2.3, the number $d_{p,s}$ depends only on these Hilbert functions. Therefore, $d_{p,s}(J) = d_{p,s}(N)$. □

Main Lemma 7.3.9. *Let N be a squarefree Borel $\{x_1, \ldots, x_{n-1}\}$-compressed ideal. Suppose that N is not squarefree lex. There exists a squarefree Borel ideal T such that:*

- *T has the same Hilbert function as N*
- *T is lexicographically greater than N (here "lexicographically greater" means that for each $d \geq 0$ we order the monomials in N_d and T_d lexicographically, and then compare the two ordered sets lexicographically)*
- *for all p, s, the graded Betti numbers satisfy*

$$b_{p,s}(S/(T+P)) \geq b_{p,s}(S/(N+P)) .$$

The proof of Lemma 7.3.9 is long and very technical. We present it in the next section.

Proof of Theorem 7.3.4: Let N be a squarefree Borel ideal. By Lemma 7.3.7, we can assume that N is $\{x_1, \ldots, x_{n-1}\}$-compressed. Lemma 7.3.9 implies that we can replace N by a squarefree Borel ideal which is lexicographically greater.

We proceed in this way until we reach the squarefree lex ideal L. This process is finite since there exist only finitely many squarefree Borel ideals with a fixed Hilbert function. □

Example 7.3.10. It is natural to ask if Green's Theorem can be used, as in [CGP], in order to obtain a short proof of Theorem 7.3.4. Unfortunately, in the example $N = (ab, ac, bc)$, $L = (ab, ac, ad, bcd)$ in $k[a, b, c, d]$, one of the inequalities needed for the proof does not hold. Thus, the short proof in [CGP] cannot be generalized to cover Theorem 7.3.4.

Furthermore, the inequality

$$\sum_{|\sigma|=i} b_{r,s}(S/(N : \sigma)) \leq \sum_{|\sigma|=i} b_{r,s}(S/(L : \sigma))$$

may not hold. For example, it fails for $S = k[a, b, c, d, e]$ and

$$N = (abc, abd, acd, bcd)$$

and $i = 2$. In this case we have $L = (abc, abd, abe, acd, bcde)$. Computer computation gives

$$\sum_{|\sigma|=2} b_{1,2}(S/(N : \sigma)) = 12 \quad \text{while} \quad \sum_{|\sigma|=2} b_{1,2}(S/(L : \sigma)) = 11\,,$$

and

$$\sum_{|\sigma|=2} b_{2,2}(S/(N : \sigma)) = 6 \quad \text{while} \quad \sum_{|\sigma|=2} b_{2,2}(S/(L : \sigma)) = 5\,.$$

Example 7.3.11. Let N be squarefree Borel and L be squarefree lex with the same Hilbert function (equivalently, let $N + P$ and $L + P$ have the same Hilbert function). It is natural to ask:

Question: Are the graded Betti numbers of $S/\big(L+(x_1^2,\dots,x_i^2)\big)$ *greater or equal to those of* $S/\big(N+(x_1^2,\dots,x_i^2)\big)$*, for each* i*?*

This question is closely related to a result proved by Charalambous and Evans [CE]. Let M be a squarefree Borel ideal. Set $P(i)=(x_1^2,\dots,x_i^2)$ and $P(0)=0$. By [CE], for each $0\le i<n$, the mapping cone of the short exact sequence

$$0\to S/\big((M+P(i)):x_{i+1}\big)\to S/\big(M+P(i)\big)\to S/\big(M+P(i+1)\big)\to 0$$

yields a minimal free resolution of $S/\big(M+P(i+1)\big)$.

The following example gives a negative answer to the above question. Let $A=k[a,b,c,d,e,f]$ and T be the ideal generated by the squarefree cubic monomials. The ideal $N=(ab,ac,ad,bc,bd)+T$ is squarefree Borel. The ideal $L=(ab,ac,ad,ae,af)+T$ is squarefree lex. The ideals N and L have the same Hilbert function. The graded Betti numbers of $S/\big(L+(x_1^2)\big)$ are not greater or equal to those of $S/\big(N+(x_1^2)\big)$. For example,

$$b_{5,7}\Big(S/\big(L+(x_1^2)\big)\Big))=0 \quad\text{and}\quad b_{5,7}\Big(S/\big(N+(x_1^2)\big)\Big)=1\,.$$

7.4 Proof of the Main Lemma 7.3.9

Throughout this section, we make the following assumptions:

Assumptions 7.4.1. N is a squarefree Borel $\{x_1,\cdots,x_{n-1}\}$-compressed ideal in $S=k[x_1,\cdots,x_n]/P$, and is not squarefree lex.

Construction 7.4.2. Since every squarefree Borel ideal in two variables is squarefree lex, it follows that the ideal N is $\mathcal{B}$-compressed for every set $\mathcal{B}$ of two variables. Let $r\ge 2$ be maximal such that N is $(2r-2)$-compressed. By Lemma 7.3.6, we have that N is $(2r-1)$-compressed. There exists a set $\mathcal{A}$ of $2r$ variables such that N is not $\mathcal{A}$-compressed. Choose w lex-first such that there exist variables $w > y_1 > \cdots > y_r > z_2 > \cdots > z_r$ for which N is not $\{w, y_1, \ldots, y_r, z_2, \ldots, z_r\}$-compressed. Then choose $\{y_1 > \cdots > y_r\}$ lex-last such that there exist $z_2, \ldots, z_r$ with $y_r > z_2 > \cdots > z_r$ such that N is not $\{w, y_1, \ldots, y_r, z_2, \ldots, z_r\}$-compressed. Finally, choose $z_2 > \cdots > z_r$ lex-first such that N is not $\{w, y_1, \ldots, y_r, z_2, \cdots, z_r\}$-compressed. Set $\mathcal{A} = \{w, y_1, \ldots, y_r, z_2, \ldots, z_r\}$. We make this choice so that we can show in Lemma 7.4.6 that the $\mathcal{A}$-compression of N is still Borel.

If $z_r \neq x_n$, then N is $\mathcal{A}$-compressed because $\mathcal{A} \subseteq \{x_1, \ldots, x_{n-1}\}$. Therefore, $z_r = x_n$.

Following the notation of Definition 7.3.5, write

$$\bar{N} = \bigoplus_f f\, \bar{N}_f \, .$$

Each $\bar{N}_f$ is an ideal in $k[\mathcal{A}]$. For simplicity we will write N, N_f instead of $\bar{N}, \bar{N}_f$, that is, we will abuse notation and regard N (resp. N_f) as both a squarefree ideal of S (resp. $k[\mathcal{A}]$) and an ideal of S/P (resp. $k[\mathcal{A}]/(P \cap k[\mathcal{A}])$).

Our assumptions imply the existence of a squarefree monomial f such that N_f is not squarefree lex.

Notation 7.4.3. In this section, f stands for a squarefree monomial in $k[\mathcal{A}^c]$.

Set $\mathbf{y} = y_1 \ldots y_r$ and $\mathbf{z} = z_2 \ldots z_r$.

We denote by a, c variables in $k[\mathcal{A}^c]$ (usually, these are variables dividing f).

We denote by m, u, v squarefree monomials.

The following lemma gives some properties of N_f:

Lemma 7.4.4. *Suppose that f is a squarefree monomial such that the ideal N_f is not squarefree lex.*

(1) The vector space $(N_f)_j$ is lex for every degree $j \neq r$.

(2) The vector space $(N_f)_r$ contains precisely the monomials $\{m \mid m \geq \mathbf{y},\ m \neq w\mathbf{z}\}$, that is, $(N_f)_r$ is spanned by the initial squarefree lex segment ending at $\mathbf{y}$ with one gap at $w\mathbf{z}$.

Remark. The proof of Lemma 7.4.4 uses only that N is squarefree Borel and $(2r-2)$-compressed, and that N_f is not lex. Thus, lemma 7.4.4 holds for every ideal Y satisfying these properties.

Proof. (1) Let $u > v$ be two squarefree monomials of degree j in the variables in $\mathcal{A}$. Let $m = \gcd(u, v)$, and $u = mu'$, $v = mv'$. It follows that $\deg(u') = \deg(v') \leq r$. Suppose that $\deg(u') < r$. Denote by $\mathcal{B}$ the set of variables that appear in exactly one of the two monomials. The number of variables in $\mathcal{B}$ is an even number $< 2r$. Therefore, N is $\mathcal{B}$-compressed. If $fv = fmv' \in N$, then $fu = fmu' \in N$. Hence, if $(N_f)_j$ is not squarefree lex, then $\deg(u') = \deg(v') = r$, so $m = 1$ and $j = r$.

(2) Let $u > v$ be two squarefree monomials of degree r in the variables in $\mathcal{A}$ such that $v \in N_f$ but $u \notin N_f$. The above argument shows that $\deg(u) = \deg(v) = r$, and $uv = w\mathbf{yz}$. Since $u > v$, we conclude that w divides u.

Suppose that u is divisible by some y_i. Hence v is divisible by some z_j. As N is squarefree Borel, $fv \in N$ implies that $f\dfrac{vy_i}{z_j} \in N$. The ideal N is $(\mathcal{A} \setminus \{y_i, z_j\})$-compressed. Therefore, $fy_im \in N$ for every squarefree monomial $m \in k[\mathcal{A}]$ such that $m > \dfrac{v}{z_j}$. We obtain the contradiction that $fu \in N$. Hence, u is not divisible by any of the variables y_i.

It follows that $u = w\mathbf{z}$ and $v = \mathbf{y}$. ∎

Construction 7.4.5. Denote by T the $\mathcal{A}$-compression of N.

The following lemma gives some properties of T:

Lemma 7.4.6.

(1) $(T_f)_j = (N_f)_j$ *for* $j \neq r$ *and every* f*.*

(2) The sets of monomials in $(T_f)_r$ *and in* $(N_f)_r$ *differ only in that* $(T_f)_r$ *contains* $w\mathbf{z}$ *instead of* $\mathbf{y}$*, in the case when* $(N_f)_r$ *is not squarefree lex. Note that* $w\mathbf{z} > \mathbf{y}$ *are consecutive monomials in the lexicographic order in* $k[\mathcal{A}]$*.*

(3) Denote by $\mathcal{F}$ *the set of minimal, with respect to divisibility, monomials* f *in the variables* $\mathcal{A}^c$ *such that* N_f *is not squarefree lex. We have that*

$$\operatorname{gens}(N) \setminus \operatorname{gens}(T) = \{ f\mathbf{y} \mid f \in \mathcal{F} \} ,$$

$$\operatorname{gens}(T) \supseteq \Big\{ \operatorname{gens}(N) \setminus \{ f\mathbf{y} | f \in \mathcal{F} \} \Big\} \cup \{ fw\mathbf{z} \, | f \in \mathcal{F} \} \\ \cup \{ f\mathbf{y}z_j | f \in \mathcal{F},\ 2 \leq j \leq r,\ \max(f) < \max(z_j) \} .$$

(4) The ideal T *has the same Hilbert function as* N*.*

(5) The ideal T *is lexicographically greater than* N*.*

(6) The ideal T *is squarefree Borel.*

Remark. It is possible to show that $\mathcal{F}$ is the set of all f such that N_f is not lex.

Proof. (1) and (2) hold by Lemma 7.4.4, and (4) and (5) hold by the construction of compression.

(3) Denote by $\{T\}$ and $\{N\}$ the sets of monomials in T and in N, respectively. By (1) and (2), we have that

$$\{N\} \setminus \{T\} = \{ f\mathbf{y} \mid N_f \text{ is not squarefree lex } \}.$$

It follows that $\text{gens}(N) \setminus \text{gens}(T) \supseteq \{f\mathbf{y} \,|\, f \in \mathcal{F}\}$.

We will show that equality holds. Suppose that m is a generator of N but not of T, and does not have the form $f\mathbf{y}$. Then, by (2), $m = fw\mathbf{z}x_i$ where $f\mathbf{y} \in N, fw\mathbf{z} \notin N$. N is $(\{x_i\} \cup \mathcal{A} \setminus \{z_r\})$-compressed, because this set does not contain $z_r = x_n$. Thus $f\mathbf{y} \in N$ implies that $fw\dfrac{\mathbf{z}}{z_r}x_i \in N$, and so $fw\mathbf{z}x_i \notin \text{gens}(N)$. Hence, any multiple of $fw\mathbf{z}$ is not a minimal monomial generator of N. The equality follows.

Now, we will prove that

$$\begin{aligned} \text{gens}(T) \supseteq &\Big\{\text{gens}(N) \setminus \{f\mathbf{y}|f \in \mathcal{F}\}\Big\} \cup \{fw\mathbf{z}\,|f \in \mathcal{F}\} \\ &\cup \{f\mathbf{y}z_j|f \in \mathcal{F},\ 2 \le j \le r, \max(f) < z_j\}\,. \end{aligned}$$

The inclusion $\text{gens}(T) \supseteq \Big\{\text{gens}(N)\setminus\{f\mathbf{y}|f \in \mathcal{F}\}\Big\}$ follows from $\text{gens}(N)\setminus\text{gens}(T) = \{f\mathbf{y}\,|\,f \in \mathcal{F}\}$. By (1) and (2), we also have that

$$\{T\} \setminus \{N\} = \{fw\mathbf{z}\,|\,N_f \text{ is not squarefree lex}\}.$$

Therefore, $\text{gens}(T) \supseteq \{fw\mathbf{z}\,|\,f \in \mathcal{F}\}$. The inclusion

$$\text{gens}(T) \supseteq \{f\mathbf{y}z_j|f \in \mathcal{F},\ 2 \le j \le r,\ \max(f) < z_j\}$$

holds because if $f\dfrac{\mathbf{y}}{z_j}a \in T$ for some variable a then, since T is squarefree Borel, we get the contradiction $f\mathbf{y} \in T$.

It remains to prove (6). Fix an f such that N_f is not squarefree lex. In view of (1), we need to consider only $(T_f)_r$. By (2), we conclude that we have to check two properties: We have to show that, if a squarefree monomial m is obtained from $fw\mathbf{z}$ by replacing a variable with a lex-greater variable, then m is in T. We also have to show that, if a squarefree monomial u is obtained from $f\mathbf{y}$ by replacing a variable with a lex-smaller variable, then u is not in T.

There are several possibilities for m and u. First, we consider four cases for m.

Suppose $m = \frac{fc}{a}w\mathbf{z}$, where a divides f and $c \in k[\mathcal{A}^c]$. Since $f\mathbf{y} \in N$ and N is squarefree Borel, we have that $\frac{fc}{a}\mathbf{y} \in N$. Hence, $\mathbf{y} \in N_{\frac{fc}{a}}$. Therefore $w\mathbf{z} \in T_{\frac{fc}{a}}$, and so $m \in T$.

Suppose $m = \frac{fe}{a}w\mathbf{z}$, where a divides f and $e \in k[\mathcal{A}]$. It follows that $e = y_j$ for some j. So, $m = \frac{f}{a}(wy_j\mathbf{z})$. By Lemma 7.4.4, we have that $(N_{\frac{f}{a}})_{r+1} = (T_{\frac{f}{a}})_{r+1}$, so we have to prove that $m \in N$. The ideal N is $(\{a\} \cup \mathcal{A} \setminus \{y_j\})$-compressed by Construction 7.4.2 since $a < y_j$. Hence, $f\mathbf{y} = \frac{fy_j}{a}\frac{a\mathbf{y}}{y_j} \in N$ implies that $\frac{fy_j}{a}w\mathbf{z} = m \in N$.

Suppose $m = f\frac{w\mathbf{z}y_j}{e}$, where e divides $w\mathbf{z}$. Since T_f is squarefree lex in $k[\mathcal{A}]$, we conclude that $m \in T$.

Suppose $m = f\frac{w\mathbf{z}c}{e} = (fc)\frac{w\mathbf{z}}{e}$, where e divides $w\mathbf{z}$ and $c \in k[\mathcal{A}^c]$. By Lemma 7.4.4, we have that $(N_{fc})_{r-1} = (T_{fc})_{r-1}$, so we have to prove that $m \in N$. First, we consider the subcase when either $e = w$ or $c < y_r$. The ideal N is $(\{c\} \cup \mathcal{A} \setminus \{e\})$-compressed by Construction 7.4.2 since $c > e$. Hence, $f\mathbf{y} \in N$ implies that $m = f(\frac{w\mathbf{z}c}{e}) \in N$. Now, let $e = z_i$ for some i and $c > y_r$. Since N is squarefree Borel, $f\mathbf{y} \in N$ implies that $fc\frac{\mathbf{y}}{y_r} \in N$. As $(N_{fc})_{r-1}$ is squarefree lex, we get that $m \in N$.

Recall that we also have to show that every squarefree monomial u, obtained from $f\mathbf{y}$ by replacing a variable with a lex-smaller variable, is not in T. Similarly, we consider four cases for u. We assume the opposite, that is $u \in T$, and we will arrive at the contradiction that $fw\mathbf{z} \in N$.

Suppose $u = \frac{fc}{a}\mathbf{y}$, where a divides f and $c \in k[\mathcal{A}^c]$. Since $\mathbf{y} \in T_{\frac{fc}{a}}$, by Lemma 7.4.4 we conclude that $(T_{\frac{fc}{a}})_r = (N_{\frac{fc}{a}})_r$ is squarefree lex, and so $w\mathbf{z} \in N_{\frac{fc}{a}}$. As N is squarefree Borel, it follows that $\left(\frac{fc}{a}\right)\frac{a}{c}w\mathbf{z} = fw\mathbf{z} \in N$.

Suppose $u = \frac{f}{a}e\mathbf{y}$, where a divides f and $e \in k[\mathcal{A}]$. By Lemma 7.4.4 we conclude that $\left(T_{\frac{f}{a}}\right)_{r+1} = \left(N_{\frac{f}{a}}\right)_{r+1}$ is squarefree lex. Hence, $u \in N$ implies that $\frac{f}{a}ew\mathbf{z} \in N$. As N is squarefree Borel, we conclude that $\left(\frac{fe}{a}\right)\left(\frac{a}{e}\right)w\mathbf{z} = fw\mathbf{z} \in N$.

Suppose $u = f\frac{\mathbf{y}}{e}y_j$, where $e \in k[\mathcal{A}]$. Then $e = z_i$. Since $\frac{\mathbf{y}e}{y_j} \in T_f$, by Lemma 7.4.4 we conclude that $(T_f)_r = (N_f)_r$ is squarefree lex, and so $w\mathbf{z} \in N_f$.

Suppose $u = f\frac{\mathbf{y}c}{y_j}$, where $c \in k[\mathcal{A}^c]$. By Lemma 7.4.4 we conclude that $\left(T_{fc}\right)_{r-1} = \left(N_{fc}\right)_{r-1}$, so $u \in N$. By Construction 7.4.2, the ideal N is $(\{c\}\cup\mathcal{A}\setminus y_j)$-compressed since $c < y_j$. Hence, $u \in N$ implies that $fw\mathbf{z} \in N$ as $w\mathbf{z} > \frac{\mathbf{y}c}{y_j}$. ☐

Construction 7.4.7. Each of the colon ideals $(N : \sigma)$ can be decomposed in the notation of Definition 7.3.5 as follows:

$$(N : \sigma) = \bigoplus_f f\,(N : \sigma)_f\,.$$

Each $(N : \sigma)_f$ is an ideal in $k[x_i \in \mathcal{A} \,|\, i \notin \sigma]/(x_i^2 \,|\, i \notin \sigma)$. Similarly, we have

$$(T : \sigma) = \bigoplus_f f\,(T : \sigma)_f\,.$$

Lemma 7.4.8. *Let f be a squarefree monomial in $k[\mathcal{A}^c]$.*

(1) For every $\sigma \subseteq \mathcal{A}$ we have $(N : \sigma)_f = (N_f : \sigma)$.

(2) If $\sigma, \gamma \subseteq \mathcal{A}$ and $\sigma \cap \gamma = \emptyset$, then $(N : \sigma\gamma)_f = \big((N_f : \sigma) : \gamma\big)$.

(3) If $\tau \subseteq \mathcal{A}^c$, then $(N : \tau)_f = N_{f\tau}$.

Proof. First, we prove (1). Let $m \in k[\mathcal{A} \setminus \sigma]$ be a monomial. We have that

$$m \in (N : \sigma)_f \Leftrightarrow fm \in (N : \sigma) \Leftrightarrow fm\sigma \in N \Leftrightarrow m\sigma \in N_f \Leftrightarrow m \in (N_f : \sigma)\,.$$

Applying (1), we prove (2) as follows:

$$\big((N_f : \sigma) : \gamma\big) = \big((N : \sigma)_f : \gamma\big) = \big((N : \sigma) : \gamma\big)_f = (N : \sigma\gamma)_f\,.$$

(3) For a monomial ideal U, we denote by $\{U\}$ the set of (squarefree) monomials in U. We have that

$$\begin{aligned}\{(N:\tau)_f\} &= \{\text{monomials } m \in k[A] \mid mf \in (N:\tau)\} \\ &= \{\text{monomials } m \in k[A] \mid mf\tau \in N\} \\ &= \{N_{f\tau}\}.\end{aligned}$$

□

Lemma 7.4.9. *Let* $\mathcal{N} = (w\dfrac{\mathbf{y}}{y_r}, \cdots, wy_r\dfrac{\mathbf{z}}{z_2}, \mathbf{y})$ *be an ideal in* $k[\mathcal{A}]$, *(where "* $\cdots$ *" means that we take all the squarefree monomials that are lex-between* $w\frac{\mathbf{y}}{y_r}$ *and* $wy_r\frac{\mathbf{z}}{z_2}$*). Then*

- *(1)* $\mathcal{T} = (w\dfrac{\mathbf{y}}{y_r}, \cdots, w\mathbf{z}, \mathbf{y}z_2, \cdots, \mathbf{y}z_r)$ *is the squarefree lex ideal with the same Hilbert function, and* $\mathcal{N}_{r+1} = \mathcal{T}_{r+1}$.

- *(2) If* $\mu \subset \mathcal{A}$ *is not a subset of supp*$(\mathbf{y})$ *or of supp*$(w\mathbf{z})$*, then* $(\mathcal{N}:\mu) = (\mathcal{T}:\mu)$*. All other possibilities for* μ*, and the corresponding ideals* $(\mathcal{N}:\mu)$ *and* $(\mathcal{T}:\mu)$*, are listed in the two tables below. The first table lists the cases when* $\mathrm{gens}(\mathcal{N}:\mu) \subset \mathrm{gens}(\mathcal{T}:\mu)$*:*

μ	$(\mathcal{N}:\mu)$	$(\mathcal{T}:\mu)$
$\emptyset \subset \zeta \subset supp(\mathbf{z})$	$(((w(y_1,\cdots,y_r))_{r-\lvert\zeta\rvert}), \mathbf{y})$	$(((w(y_1,\cdots,y_r))_{r-\lvert\zeta\rvert}), w\frac{\mathbf{z}}{\zeta}, \mathbf{y})$
$supp(w)$	$((y_1,\cdots,y_r)_{r-1})$	$(((y_1,\cdots,y_r)_{r-1}), \mathbf{z})$
$supp(w)\zeta$ *with* $\emptyset \subset \zeta \subset supp(\mathbf{z})$	$((y_1\cdots,y_r)_{r-1-\lvert\zeta\rvert})$	$(((y_1\cdots,y_r)_{r-1-\lvert\zeta\rvert}\frac{\mathbf{z}}{\zeta})$

The second table lists the remaining cases:

μ	$(\mathcal{N} : \mu)$	$(\mathcal{T} : \mu)$
$\emptyset$	$(((w(y_1, \cdots, y_r))_r), \mathbf{y})$	$(((w(y_1, \cdots, y_r))_r), w\mathbf{z}, \mathbf{y}z_2, \cdots, \mathbf{y}z_r)$
$\emptyset \subset \rho \subset supp(\mathbf{y})$	$(((w)_{r-\|\rho\|}), \frac{\mathbf{y}}{\rho})$	$(((w)_{r-\|\rho\|}), \frac{\mathbf{y}}{\rho}z_2, \cdots, \frac{\mathbf{y}}{\rho}z_r)$
$supp(\mathbf{y})$	(1)	$(w, z_2, \cdots, z_r)$
$supp(\mathbf{z})$	$(wy_1, \cdots, wy_r, \mathbf{y})$	$(w, \mathbf{y})$
$supp(w\mathbf{z})$	$(y_1, \cdots, y_r)$	(1)

- *(3) Let Y be a $(2r-2)$-compressed ideal in the polynomial ring $k[\mathcal{A}]/(P \cap k[\mathcal{A}])$ such that $Y_r = \mathcal{N}_r$, and let Z be the lex ideal of $k[\mathcal{A}]/(P \cap k[\mathcal{A}])$ with the same Hilbert function as Y. For every subset μ of $\mathcal{A}$, we have:*

$$\text{gens}(\mathcal{N} : \mu) \setminus \text{gens}(\mathcal{T} : \mu) = \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)$$

$$\text{gens}(\mathcal{T} : \mu) \setminus \text{gens}(\mathcal{N} : \mu) = \text{gens}(Z : \mu) \setminus \text{gens}(Y : \mu)\,.$$

Proof. (1) The ideal $\mathcal{T}$ is clearly squarefree lex. We need to show that $\mathcal{T}$ has the same Hilbert function as $\mathcal{N}$. Obviously, $\dim_k((\mathcal{N})_r) = \dim_k((\mathcal{T})_r)$. Let $\mathcal{T}' = (w\dfrac{\mathbf{y}}{y_r}, \cdots, w\mathbf{z})$. It is straightforward to verify that $\mathcal{N}_{r+1} \supset \mathcal{T}'_{r+1}$ and$\{\mathcal{N}_{r+1}\} \setminus \{\mathcal{T}'_{r+1}\}$ $= \{\mathbf{y}z_2, \cdots, \mathbf{y}z_r\}$. Therefore, $\mathcal{N}_{r+1} = \mathcal{T}_{r+1}$.

(2) Recall that $\mu \subseteq \mathcal{A}$. A simple computation shows that $(\mathcal{N} : wy_i) = (\mathcal{T} : wy_i)$ and $(\mathcal{N} : y_iz_j) = (\mathcal{T} : y_iz_j)$ for all i, j. If μ is not divisible by $\mathbf{y}$ or $w\mathbf{z}$, it follows that μ contains either $\sup(wy_i)$ or $\sup(y_iz_j)$, for some i, j, so $(\mathcal{N} : \mu) = (\mathcal{T} : \mu)$. For all other μ, straightforward computation yields the ideals $(\mathcal{N} : \mu)$ and $(\mathcal{T} : \mu)$ listed in the tables.

(3) For a monomial m in $k[\mathcal{A}]$, we will use the notation $m : \mu$ for $\dfrac{m}{\gcd(m, \mu)}$.

Observe first that $Y_j = Z_j$ for all $j \neq r$ by Lemma 7.4.4(3). Hence, $\{Z\} \setminus \{Y\} = w\mathbf{z}$ and $\{Y\} \setminus \{Z\} = \mathbf{y}$.

Suppose that $m : \mu$ is a minimal monomial generator for $(Y : \mu)$ but not $(Z : \mu)$. We assume that $m : \mu \notin (\mathcal{N} : \mu)$ and will derive a contradiction. Note that $m \notin \mathcal{N}$. Then $\deg(m) \neq r$, because $m \in Y$ and $Y_r = \mathcal{N}_r$. Since $m \in Y$ and $Y_{\deg(m)} = Z_{\deg(m)}$ we conclude that $m \in Z$, and so $m : \mu \in (Z : \mu)$. Since $m : \mu$ is not a minimal monomial generator for $(Z : \mu)$, there must be a monomial $u \in Z \setminus Y$ such that $u : \mu$ properly divides $m : \mu$. The only monomial of $Z \setminus Y$ is $w\mathbf{z}$, so $u = w\mathbf{z}$. Then $\mu(m : \mu)$ is a proper multiple of $w\mathbf{z}$, and so must be in $\mathcal{N}$ by (1). On the other hand, since $m : \mu \notin (\mathcal{N} : \mu)$, it follows that $\mu(m : \mu) \notin \mathcal{N}$, a contradiction. Thus we must have $m : \mu \in (\mathcal{N} : \mu)$. Note that $Y \supset \mathcal{N}$. We conclude that $m : \mu$ is a minimal monomial generator of $(\mathcal{N} : \mu)$. We have proved that

$$\operatorname{gens}(Y : \mu) \setminus \operatorname{gens}(Z : \mu) \subseteq \operatorname{gens}(\mathcal{N} : \mu)\,.$$

Now, suppose further that $m : \mu$ is a minimal monomial generator of $(\mathcal{T} : \mu)$. We will derive a contradiction. Note that m may be chosen to be a minimal monomial generator of $\mathcal{N}$, because $m : \mu$ is a minimal monomial generator of $(\mathcal{N} : \mu)$.

Since $m : \mu$ is not a minimal monomial generator of $(Z : \mu)$, there must be a monomial $u \in Z \setminus \mathcal{T}$ such that $u : \mu$ properly divides $m : \mu$. As $Y_j = Z_j$ and $\mathcal{N}_j = \mathcal{T}_j$ for $j \neq r$, and $Y_r = \mathcal{N}_r$, and $Z_r = \mathcal{T}_r$, we get $\{Z\} \setminus \{\mathcal{T}\} = \{Y\} \setminus \{\mathcal{N}\}$. Hence $u \in Y$, so that $m : \mu$ is not a minimal monomial generator for $(Y : \mu)$, a contradiction. This shows that

$$\operatorname{gens}(Y : \mu) \setminus \operatorname{gens}(Z : \mu) \subseteq \operatorname{gens}(\mathcal{N} : \mu) \setminus \operatorname{gens}(\mathcal{T} : \mu).$$

In order to prove the equality in the first formula in (3), we need to show that the opposite inclusion holds. To this end, suppose that $m : \mu$ is a minimal monomial generator of $(\mathcal{N} : \mu)$ but not $(\mathcal{T} : \mu)$. Then, by (2), either $m = \mathbf{y}$ and μ divides $\mathbf{y}$, or $\mu = \mathbf{z}$ or $\mu = w\mathbf{z}$.

Suppose first that $m = \mathbf{y}$. As $Y \supseteq \mathcal{N}$, we have that $\mathbf{y} : \mu \in (Y : \mu)$. If $\mathbf{y} : \mu$ were not a minimal monomial generator of $(Y : \mu)$, we would have a monomial $u \in Y \setminus \mathcal{N} = Z \setminus \mathcal{T}$, such that $u : \mu$ properly divides $\mathbf{y} : \mu$. But then u must be a proper divisor of $\mathbf{y}$, and $u \in Z$ implies the contradiction $\mathbf{y} \in Z$. Hence $m : \mu \in \operatorname{gens}(Y)$. If $m : \mu \in \operatorname{gens}(Z)$, we would have $\mathbf{y} \in Z$, a contradiction. Thus in this case we have $m : \mu \in \operatorname{gens}(Y : \mu) \setminus \operatorname{gens}(Z : \mu)$.

Suppose now that $\mu = \mathbf{z}$ or $\mu = w\mathbf{z}$. Straightforward computation using (2) shows that one of the following two cases holds:

μ	$m : \mu$	$w\mathbf{z} : \mu$
$\mathbf{z}$	wy_j	w
$w\mathbf{z}$	y_j	1

where $1 \leq j \leq n$. In particular, $w\mathbf{z} : \mu$ is a proper divisor of $m : \mu$ in either case. As $Z_r = \mathcal{T}_r$, we get $w\mathbf{z} : \mu \in (Z : \mu)$, so $m : \mu$ is not a minimal monomial generator for $(Z : \mu)$. If $m : \mu$ were not a minimal monomial generator for $Y : \mu$, there would be a monomial $u \in Y \setminus \mathcal{N}$ such that $u : \mu$ is a proper divisor of $m : \mu$. The table above implies that one of the following two cases holds:

μ	$u : \mu$	$\mu(u : \mu) \in Y$
$\mathbf{z}$	$w,$ or $y_j,$ or 1	$w\mathbf{z},$ or $y_j\mathbf{z},$ or $\mathbf{z}$
$w\mathbf{z}$	1	$w\mathbf{z}$

where $1 \leq j \leq n$. As $Y_r = \mathcal{N}_r$, none of the monomials in the third column are in Y. This is a contradiction. Thus $m : \mu$ is a minimal monomial generator for $(Y : \mu)$. Therefore, $m : \mu \in \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)$ in this case.

We have shown that the first formula in (3) holds:

$$\text{gens}(\mathcal{N} : \mu) \setminus \text{gens}(\mathcal{T} : \mu) = \text{gens}(Y : \mu) \setminus \text{gens}(Z : \mu)\,.$$

A very similar argument yields the second formula:

$$\text{gens}(\mathcal{T} : \mu) \setminus \text{gens}(\mathcal{N} : \mu) = \text{gens}(Z : \mu) \setminus \text{gens}(Y : \mu). \qquad \square$$

Notation 7.4.10. We write an EK-triple in the form $(\tau\mu, gq, \alpha)$ so that $\tau \in k[\mathcal{A}^c]$, $\mu \in k[\mathcal{A}]$, $g \in k[\mathcal{A}^c]$, $q \in k[\mathcal{A}]$. By $\tau\mu$ we mean the union of τ and μ.

For $\mu \subset \mathcal{A}$, we set $\mathbf{n}$ to be the homogeneous maximal ideal of the ring $k[\mathcal{A}\backslash\mu]/P$.

The next lemma provides a list of the possible EK-triples for N:

Lemma 7.4.11.

(1) There are three types of EK-triples $(\tau\mu, gq, \alpha)$ for N:

Type 1: $(\tau\mu, gq, \alpha)$ is an EK-triple for T.

Type 2: $(\tau\mu, gq, \alpha)$ is not an EK-triple for T and q is a minimal monomial generator of both $(N : \tau\mu)_g$ and $(T : \tau\mu)_g$.

Type 3: $(\tau\mu, gq, \alpha)$ is not an EK-triple for T and q is a minimal monomial generator of $(N : \tau\mu)_g$ but not of $(T : \tau\mu)_g$.

(2) The EK-triples of Type 2 satisfy $q\mu = w\mathbf{z}$ and $\max(g) > \max(q)$. In particular, $\mu \neq 1$, because $x_n = z_r$ divides μ.

(3) Let $\mathcal{N}$ and $\mathcal{T}$ be as in Lemma 7.4.9. If $(\tau\mu, gq, \alpha)$ is an EK-triple of Type 3 for N, then q is a minimal monomial generator for $(\mathcal{N} : \mu)$ but not for $(\mathcal{T} : \mu)$.

(4) *Suppose that $(\tau\mu, gq, \alpha)$ is an EK-triple of Type 3 for N. All possibilities for μ, and the corresponding ideals $(\mathcal{N} : \mu)$ and $(\mathcal{T} : \mu)$, are listed in the second table in Lemma 7.4.9(2).*

Proof. (1) It suffices to show that q is a minimal monomial generator of the ideal $(N : \tau\mu)_g$. Since $gq \in (N : \tau\mu)$, we have that $q \in (N : \tau\mu)_g$. If we had $\frac{q}{a} \in (N : \tau\mu)_g$, it would follow that $\frac{gq}{a} \in (N : \tau\mu) = (N : \tau\mu)$, so that gq would not be a minimal monomial generator of $(N : \tau\mu)$. Thus, q is a minimal monomial generator of $(N : \tau\mu)_g$.

(2) Since $gq \in (T : \tau\mu)$ is not a minimal monomial generator, there exists a variable c dividing gq such that $\frac{gq}{c} \in (T : \tau\mu)$. Since T is Borel, we may take $c = x_{\max(gq)}$. If c divides q, we have $\frac{q}{c} \in (T : \tau\mu)_g$, so that q is not a minimal monomial generator of $(T : \tau\mu)_g$. Therefore c divides g. Since $c = x_{\max(gq)}$, we have $\max(g) > \max(q)$. As gq is a minimal monomial generator of $(N : \tau\mu)$, we have $\frac{g}{c}q \notin (N : \tau\mu)$, so $\frac{g}{c}q\tau\mu \in T \setminus N$. By Lemma 7.4.6(2) it follows that $q\mu = w\mathbf{z}$.

(3) By Lemma 7.4.8, we have that

$$(N : \tau\mu)_g = ((N : \tau)_g : \mu) \quad \text{and} \quad (T : \tau\mu)_g = ((T : \tau)_g : \mu)\,.$$

We are going to apply Lemma 7.4.9(3) to the ideals $Y = (N : \tau)_g$ and $Z = (T : \tau)_g$. By Lemma 7.4.8(3), we have that $Y = N_{g\tau}$ and $Z = T_{g\tau}$. By Lemma 7.4.6 we get that $Y_r = \mathcal{N}_r$. Clearly, Z is the squarefree lex ideal with the same Hilbert function as Y. Note that Y is $(2r-2)$-compressed since N is. Therefore, Y and Z satisfy the conditions of Lemma 7.4.9(3) and we can apply it.

We have that

$$q \in \operatorname{gens}(Y : \mu) \setminus \operatorname{gens}(Z : \mu)$$

since we consider EK-triples of Type 3. Lemma 7.4.9(3) yields

$$q \in \operatorname{gens}(\mathcal{N} : \mu) \setminus \operatorname{gens}(\mathcal{T} : \mu).$$

(4) follows from (3) and Lemma 7.4.9(2). $\square$

Next, we construct a map from the set of EK-triples for N to the set of EK-triples for T. We will use this map to prove the Main Lemma 7.3.9.

Construction 7.4.12. We will define a map ϕ from the set of EK-triples for N to the EK-triples for T. First, we introduce notation.

If $\alpha = v \prod_{i \in \mathcal{I}} y_i \prod_{j \in \mathcal{J}} z_j$, where v is a squarefree monomial in $k[\mathcal{A}^c]$ and we use the convention $w = z_1$, then let $\hat{\alpha}$ be the monomial

$$\hat{\alpha} = v \prod_{i \in \mathcal{I}} z_i \prod_{j \in \mathcal{J}} y_j \,.$$

Let $t_1 > \cdots > t_s = z_r = x_n$ be all the variables of S not in $\tau\mathbf{y}$, ordered lexicographically. For a monomial m, such that $m\tau$ is squarefree, set

$$t_m = \begin{cases} x_{\max(m)} & \text{if } x_{\max(m)} \notin \mathbf{y} \\ \\ \text{the lex-last variable among the } t\text{-variables} & \\ \text{that is lex-before } x_{\max(m)} & \text{if } x_{\max(m)} \in \mathbf{y}\,, \end{cases}$$

and furthermore, for a monomial m and an integer j, set

$$t_{m+j} = t_{p+j} \text{ where the integer } p \text{ is defined by } t_p = t_m \,.$$

In cases 2 and 3 below, we will set the integer d such that $\max(g) = \max(t_d)$. In case 2, e will be the integer such that t_d is between z_{r-e} and z_{r-e+1}. Thus, $r - e = \#\{z_j : \max(z_j) < \max(t_d)\}$ (recall the convention $z_1 = w$). In case 3 we will set i such that t_d is between y_{i-1} and y_i.

Denote by $\tilde{\alpha}$ the monomial

$$\tilde{\alpha} = \alpha \prod_{\substack{y_j \text{ divides } \alpha \\ j \neq 1}} \frac{t_{gq+j-1}}{y_j} \prod_{y_1 \text{ divides } \alpha} \frac{w}{y_1}.$$

and denote by $\bar{\alpha}$ the monomial

$$\bar{\alpha} = \alpha \Bigg(\prod_{\substack{y_j \text{ divides } \alpha, \\ \max(z_j) < \max(g)}} \frac{z_j}{y_j} \Bigg) \Bigg(\prod_{\substack{y_j \text{ divides } \alpha, \\ \max(z_j) > \max(g)}} \frac{t_{d+j-(r-e)}}{y_j} \Bigg).$$

The map ϕ is defined as follows: If Γ is an EK-triple for N of the form given in the third column in the table below, then $\phi(\Gamma)$ is given in the fourth column.

Case	Type of Γ	EK-triple Γ	$\phi(\Gamma)$
1)	Type 1	$\Gamma = (\tau\mu, gq, \alpha)$	$\Gamma = (\tau\mu, gq, \alpha)$
2)	Type 2	$(\tau\mu, g\frac{w\mathbf{z}}{\mu}, \alpha)$, $\emptyset \subset \mu \subset \sup(w\mathbf{z})$, $\max(g) > \max(y_r)$	$(\tau\hat{\mu}, \frac{g}{x_{\max(g)}} t_{d+e}\frac{\mathbf{y}}{\hat{\mu}}, \bar{\alpha})$,
3)	Type 2	$(\tau\mathbf{z}, gw, \alpha)$, $\max(y_1) < \max(g) < \max(y_r)$ or $(\tau w\mathbf{z}, g, \alpha)$	$(\tau\frac{\mathbf{y}}{y_1}, \frac{g}{x_{\max(g)}} t_{d+i-1}y_1, \tilde{\alpha})$ or $(\tau\mathbf{y}, \frac{g}{x_{\max(g)}} t_{d+i-1}, \tilde{\alpha})$
4)	Type 2	$(\tau\mathbf{z}, gw, \alpha)$, $\max(w) < \max(g) < \max(y_1)$	$(\tau\frac{\mathbf{y}}{y_1 y_2}, \frac{g}{x_{\max(g)}} y_1 y_2 z_2, \alpha x_{\max(g)})$
5)	Type 3	$(\tau, g\mathbf{y}, \alpha)$	$(\tau, gw\mathbf{z}, \hat{\alpha})$
6)	Type 3	$(\tau\rho, g\frac{\mathbf{y}}{\rho}, \alpha), \emptyset \subset \rho \subset \sup(\mathbf{y}),\ \rho \neq \frac{\mathbf{y}}{y_1}$	$(\tau\hat{\rho}, g\frac{w\mathbf{z}}{\hat{\rho}}, \hat{\alpha})$
7)	Type 3	$(\tau\frac{\mathbf{y}}{y_1}, gy_1, \alpha)$	$(\tau\frac{\mathbf{z}z_r}{,} gwz_r, y_r\hat{\alpha})$
8)	Type 3	$(\tau\mathbf{y}, g, \alpha)$	$(\tau\frac{w\mathbf{z}}{z_r}, gz_r, y_r\hat{\alpha})$
9)	Type 3	$(\tau\mathbf{z}, gwy_j, \alpha)$	$(\tau\frac{\mathbf{y}}{y_j}, gy_j t_{y_j+j}, \tilde{\alpha})$
10)	Type 3	$(\tau w\mathbf{z}, gy_j, \alpha),\ j \neq 1$	$(\tau\mathbf{y}, gt_{y_j+j}, \tilde{\alpha})$
11)	Type 3	$(\tau w\mathbf{z}, gy_1, \alpha)$	$(\tau\frac{\mathbf{y}}{y_1}, gy_1 t_{y_1+1}, w\tilde{\alpha})$

Lemma 7.4.13. *The third column in the table in Construction 7.4.12 lists all the possibilities for EK-triples for N.*

Proof. Apply Lemma 7.4.11. For EK-triples of Type 3, straightforward computation yields q since we know all possibilities for$(\mathcal{N} : \mu)$ and $(\mathcal{T} : \mu)$ (see the second table in Lemma 7.4.9(3)) and in view of Lemma 7.4.11(3). $\square$

Lemma 7.4.14.

(1) The map ϕ is well-defined (i.e., $\phi(\Gamma)$ is an EK-triple for T, for all Γ).

(2) The map ϕ preserves bidegree.

(3) The map ϕ is an injection.

Proof. (2) Straightforward verification shows that ϕ preserves bidegrees.

(3) Let $\Gamma = (\tau\mu, gq, \alpha)$, $\Gamma' = (\tau'\mu', g'q', \alpha')$, $\phi(\Gamma) = (\theta, \ell, \beta)$, $\phi(\Gamma') = (\theta', \ell', \beta')$. Write $\theta = \tau\lambda$, $\theta' = \tau'\lambda'$, $\ell = uv$ and $\ell' = u'v'$ with $\lambda, \lambda', u, u' \in k[\mathcal{A}]$, $v, v' \in k[\mathcal{A}^c]$. Suppose that $\Gamma \neq \Gamma'$.

If $\tau \neq \tau'$, then $\phi(\Gamma) \neq \phi(\Gamma')$ and we are done. For the rest of the proof, we assume that $\tau = \tau'$. In this case, note that we have the same choice for the variables t_j in Construction 7.4.12 for Γ and Γ'. Also, note that if $\alpha \neq \alpha'$, then $\hat{\alpha} \neq \hat{\alpha}'$, $\bar{\alpha} \neq \bar{\alpha}'$, and $\tilde{\alpha} \neq \tilde{\alpha}'$ by Construction 7.4.12.

If Γ and Γ' fall under the same Case in Construction 7.4.12, it is immediate that $\phi(\Gamma) \neq \phi(\Gamma')$, except in Cases 2 and 3. In Cases 2 and 3, we have to consider the situation when $\dfrac{g}{x_{\max(g)}} = \dfrac{g'}{\max(g')}$, and $\tau = \tau'$, $\mu = \mu'$, $\alpha = \alpha'$. Let d', e', i' be defined analogously to d, e, i. Without loss of generality we can assume that $\max(g) > \max(g')$. Hence, $\max(t_d) > \max(t_{d'})$, and so $d > d'$. Therefore, we have the inequalities $i \geq i'$ and

$$e' - e = \#\{z_j : z_j \text{ between } t_d \text{ and } t_{d'}\} \leq 1 + \#\{t_j : t_j \text{ between } t_d \text{ and } t_{d'}\} = d - d'\,.$$

It follows that $d + e > d' + e'$ and $d + i - 1 > d' + i' - 1$. Therefore, $\ell \neq \ell'$.

Thus, we may assume that Γ and Γ' belong to different Cases.

Suppose first that Γ' falls under Case 1. We will show that $\phi(\Gamma)$ is not an EK-triple for N, for Γ in each of the Cases 2 through 11. In Cases 2, 3, and 4, we have that $\ell\theta$ is properly divisible by the monomial $\tau \dfrac{g}{x_{\max(g)}}\mathbf{y}$, which is in N by Lemma 7.4.16; hence $\ell\theta$ is not a minimal monomial generator of N. In Cases 9, 10, and 11, we have that $\ell\theta$ is properly divisible by the monomial $\tau g \mathbf{y}$, which is in N because $(N : \tau)_g \neq (T : \tau)_g$ implies $\mathbf{y} \in (N : \tau)_g$; hence $\ell\theta$ is not a minimal monomial generator of N. In Cases 5 through 8, $\lambda = \mu$ and $u = q$ have concrete values, and the second table in Lemma 7.4.9(3) shows that q is not a minimal monomial generator for $(\mathcal{N} : \mu)$; hence Lemma 7.4.11(3) implies that $\phi(\Gamma)$ is not an EK-triple for N.

For the rest of the proof, we assume that that neither Γ nor Γ' is in Case 1.

In many cases, it is clear that $\lambda \neq \lambda'$. These cases are listed in the following table.

Case of Γ	Case of Γ'	Difference between λ and λ'
2	5	$\mu \neq \emptyset$, so $\lambda = \hat{\mu} \neq \emptyset$. But $\lambda' = \emptyset$.
2	6,7,8	$\mu \subset \sup(w\mathbf{z})$ by Lemma 7.4.11(2), so $\emptyset \neq \lambda = \hat{\mu} \subset \sup(\mathbf{y})$. But $\emptyset \neq \lambda' \subset \sup(w\mathbf{z})$.
3	4	If $\lambda = \lambda' = \dfrac{\mathbf{y}}{y_1 y_2}$ then $\mu \dfrac{\mathbf{z}}{z_2}$, but μ is $\mathbf{z}$ or $w\mathbf{z}$.
3	5	$\mu = \mathbf{z}$ or $\mu = w\mathbf{z}$, so $\lambda = \hat{\mu}$ is $\frac{\mathbf{y}}{y_1}$ or $\mathbf{y}$, but $\lambda' = \emptyset$.
3	6,7,8	$\mu = \mathbf{z}$ or $\mu = w\mathbf{z}$, so $\lambda = \hat{\mu}$ is $\frac{\mathbf{y}}{y_1}$ or $\mathbf{y}$. But $\emptyset \neq \lambda' \subset \sup(w\mathbf{z})$.
4	6	$\lambda = \frac{\mathbf{y}}{y_1 y_2}$, but $\emptyset \neq \lambda' = \hat{\rho} \subset \sup(w\mathbf{z})$.
4	8,9,10,11	$\deg(\lambda) = r - 2$, but $\deg(\lambda') \geq r - 1$.
5	6	$\lambda = \emptyset$ and $\lambda' \neq \emptyset$.
5	8,9,10,11	$\lambda = \emptyset$ and $\lambda' \neq \emptyset$.
6	9,10,11	$\emptyset \subset \rho \subset \sup(\mathbf{y})$, so $\emptyset \neq \lambda = \hat{\rho} \subset \sup(w\mathbf{z})$. But $\emptyset \neq \lambda' \subseteq \sup(\mathbf{y})$.
7	8,9,10,11	$\lambda = \frac{\mathbf{z}}{z_r}$, but λ' has a different value.
8	9,10,11	$\lambda = \frac{w\mathbf{z}}{z_r}$, but λ' has a different value.
9	10	$\lambda = \frac{\mathbf{y}}{y_j}$, but $\lambda' = \mathbf{y}$.
10	11	$\lambda = \mathbf{y}$, but $\lambda' = \frac{\mathbf{y}}{y_1}$.

If $\lambda = \lambda'$, then Γ and Γ' must belong to one of the pairs of Cases listed in the following table. We assume that $\lambda = \lambda'$ and give the differences between Γ and Γ' in the last column of the table.

Case of Γ	Case of Γ'	Difference
2	3	$\lambda = \lambda' = \frac{\mathbf{y}}{y_1}$, so $u = t_{d+e}y_1$ and $u' = t_{d'+i'-1}y_1$. By Lemma 7.4.17 (1,2), $u \neq u'$.
2	4	If $\lambda = \lambda' = \dfrac{\mathbf{y}}{y_1 y_2}$, then $\mu = \dfrac{\mathbf{z}}{z_2}$ so $\max(g) > \max(z_2)$ by Lemma 7.4.10(2), but $\max(g') < \max(z_2)$.
2	9,10,11	$\max(t_d) = \max(g) > \max(y_r)$, so $\max(\ell) = \max(t_{d+e}) > \max(t_{y_r+r})$ by Lemma 7.4.17(1). But $\max(\ell') = \max(\,(\max(g'), \max(t_{y_j+j})\,) \leq \max(t_{y_r+r})$, because $\max(t_{y_j+j}) \leq \max(t_{y_r+j}) \leq \max(t_{y_r+r})$ and $\max(g') < \max(y_r) < \max(t_{y_r+r})$ by Lemmas 7.4.15 and 7.4.17(4).
3	9,10,11	$\max(\ell) = \max(t_{d+i-1})$. In case 11, set $j' = 1$. Then $\max(g') < \max(y_{j'})$. So $\max(\ell') = \max(t_{y_{j'}+j'})$ by Lemma 7.4.15. By Lemma 7.4.17(3), $\max(\ell) \neq \max(\ell')$.
4	5,7	$u = y_1 y_2 z_2$, but wz_r divides u'.
5	7	$y_r \notin \beta$, but $y_r \in \beta'$
6	7,8	$y_r \in \operatorname{supp}(\beta')$. But $z_r = x_n \notin \alpha$ so $y_r \notin \operatorname{supp}(\beta)$.
9	11	$w \in \operatorname{supp}(\beta')$. If $w \in \operatorname{supp}(\beta)$, then $y_1 \in \alpha$, so $j \neq 1$, and then $y_1 \notin \ell$ but $y_1 \in \ell'$.

It remains to prove (1). In each of the Cases in Construction 7.4.12, we will show that $\phi(\Gamma)$ is an EK-triple for T. Set $\phi(\Gamma) = (\theta, \ell, \beta)$. In all cases it is immediate that $\theta\ell\beta$ is squarefree and that $\max(\beta) < \max(\ell)$. Thus we need only verify that ℓ is a minimal monomial generator for $(T : \theta)$.

Case 1 is clear.

Case 2: We have that $\tau\mu\frac{gq}{x_{\max(g)}} = \tau\frac{g}{x_{\max(g)}}w\mathbf{z} \in T \setminus N$. Hence $\tau\frac{g}{x_{\max(g)}}\mathbf{y} \in N$, so $\tau\frac{g}{x_{\max(g)}}\mathbf{y}z_r \in N$. Thus $\tau\frac{g}{x_{\max(g)}}\mathbf{y}z_r \in T$. Since T is squarefree Borel, we have $\tau\frac{g}{x_{\max(g)}}\mathbf{y}t_{d+e} \in T$, and hence $\frac{g}{x_{\max(g)}}t_{d+e}\frac{\mathbf{y}}{\hat{\mu}} \in (T : \tau\hat{\mu})$. Suppose that this is not a minimal monomial generator. Then $\tau\hat{\mu}\frac{g\hat{q}}{x_{\max(g)}c}t_{d+e} \in T$. Since T is squarefree Borel and t_{d+e} is lex-after $x_{\max(g)}y_n$, it follows that $\tau\hat{\mu}\frac{g}{x_{\max(g)}}\hat{q} = \tau\frac{g}{x_{\max(g)}}\mathbf{y} \in T$. Hence $\tau\frac{g}{x_{\max(g)}}w\mathbf{z} \in N$. So we get the contradiction $\frac{g}{x_{\max(g)}q} \in (N : \tau\mu)$.

Case 3: Note that $\max(q) < \max(g) < \max(y_r) < \max(z_2)$ implies that either $q = w$ or $q = 1$. If $\max(g) > y_1$ or $q = 1$, we have that t_{d+i-1} is lex-after $x_{\max(g\hat{q})}$ and the proof of Case 2 holds, mutatis mutandis. If not, we are in Case 4.

Case 4: We have $\tau\frac{g}{x_{\max(g)}}w\mathbf{z} \in T\setminus N$, so $\tau\frac{g}{x_{\max(g)}}\mathbf{y} \in N\setminus T$. Thus $\tau\frac{g}{x_{\max(g)}}\mathbf{y}z_2 \in N$ and also T. This yields $\frac{g}{x_{\max(g)}}y_1y_2z_2 \in (T : \tau\frac{\mathbf{y}}{y_1y_2})$. If this were not a minimal monomial generator, we would have $\frac{g}{x_{\max(g)}}y_1y_2 \in \left(T : \tau\frac{\mathbf{y}}{y_1y_2}\right)$, so $\tau\frac{g}{x_{\max(g)}}\mathbf{y} \in T$.

In all of the remaining cases, Γ is of Type 3, so it is immediate from the table in Lemma 7.4.9(2) that $\ell \in (T : \theta)$.

Case 5: If $gw\mathbf{z}$ were not a minimal monomial generator for $(T : \tau)$, we would have $\frac{g}{c}w\mathbf{z} \in (T : \tau)$ which implies $\frac{g}{c}\mathbf{y} \in (N : \tau)$, contradicting the assumption that $g\mathbf{y}$ is a minimal monomial generator of $(N : \tau)$.

Case 6: If $g\frac{w\mathbf{z}}{\hat{\rho}}$ were not a minimal monomial generator for $(T : \tau\hat{\rho})$, we would have $\frac{g}{c}w\mathbf{z} \in (T : \tau)$.

Case 7: If gwz_r were not a minimal monomial generator for $\left(T : \tau\frac{\mathbf{z}}{z_r}\right)$, we would have $\frac{g}{c}w\mathbf{z} \in (T : \tau)$.

Case 8: If gz_r were not a minimal monomial generator for $\left(T : \frac{w\mathbf{z}}{z_r}\right)$, we would have $\frac{g}{c}w\mathbf{z} \in (T : \tau)$.

Case 9: If $gy_it_{y_i+i}$ were not a minimal monomial generator for $\left(T : \tau\frac{\mathbf{y}}{y_j}\right)$, we would have $\frac{g}{c}\mathbf{y}t_{y_j+j} \in (T : \tau)$. Since T is squarefree Borel and $x_{\max(g)}$ is lex-before

y_j and hence lex-before t_{y_j+j} by Lemma 7.4.15, it would follow that $g\mathbf{y} \in (T : \tau)$.

Case 10: If gt_{y_j+j} were not a minimal monomial generator for $(T : \tau\mathbf{y})$, we would have $\frac{g}{c}t_{y_j+j} \in (T : \tau)$. Since T is squarefree Borel and $x_{\max(g)}$ is lex-before y_j and hence lex-before t_{y_j+j} by Lemma 7.4.15, it would follow that $g\mathbf{y} \in (T : \tau)$.

Case 11: If $gy_1t_{y_1+1}$ were not a minimal monomial generator for $(T : \tau\mathbf{y})$, we would have $\frac{g}{c}y_1t_{y_1+1} \in (T : \tau)$. Since T is squarefree Borel and $x_{\max(g)}$ is lex-before y_i and hence lex-before t_{y_i+i} by Lemma 7.4.15, it would follow that $g\mathbf{y} \in (T : \tau)$. ☐

In the proof of the above Lemma 7.4.14 we used the following supplementary lemmas:

Lemma 7.4.15. *Let $\tau \in k[\mathcal{A}^c]$ and g be a squarefree monomial in $k[\mathcal{A}^c]$ such that $g\tau\mathbf{y} \in N$. Suppose that either gy_j is a minimal monomial generator of $(N : \tau w\mathbf{z})$ or that gwy_j is a minimal monomial generator of $(N : \tau\mathbf{z})$. Then* $\max(g) < \max(y_j)$.

Proof. Suppose the opposite. Let $c = x_{\max(g)} < y_j$, and $\frac{g}{c}c\tau\mathbf{y} \in N$. By Construction 7.4.2, it follows that the ideal N is $(\{c\} \cup \mathcal{A} \setminus \{y_j\})$-compressed. Therefore, $\frac{g}{c}y_j(\tau w\mathbf{z}) \in N$. Hence, we have that $\frac{g}{c}y_j \in (N : \tau w\mathbf{z})$ and $\frac{g}{c}y_jw \in (N : \tau\mathbf{z})$. This is a contradiction. ☐

Lemma 7.4.16. *Let $(\tau\mu, gq, \alpha)$ be an EK-triple of Type 2 for N. Then $\tau\frac{g}{x_{\max(g)}}\mathbf{y} \in N$.*

Proof. Applying Lemma 7.4.11(2), we have that $\mu q = w\mathbf{z}$. Thus $gq \in T_{\tau\mu}$, so, since $(\tau\mu, gq, \alpha)$ is an EK-triple of Type 2, it must be the case that $\frac{gq}{x_{\max(gq)}} = \frac{g}{x_{\max(g)}}q \in T_{\tau\mu} \setminus N_{\tau\mu}$. Hence, $\frac{g}{x_{\max(g)}}\tau q\mu = \frac{g}{x_{\max(g)}}\tau w\mathbf{z} \in T \setminus N$, so $\frac{g}{x_{\max(g)}}\tau\mathbf{y} \in N \setminus T$. ☐

Lemma 7.4.17. *Let everything be as in the proof of Lemma 7.4.14(3). Then:*

(1) If $\max(g) > \max(y_r)$*, then* $\max(t_{d+e}) > \max(t_{y_r+r})$*.*

(2) If $\max(g) < \max(y_r)$*, then* $\max(t_{d+i-1}) < \max(t_{y_r+r})$*.*

(3) $t_{d+i-1} \neq t_{y_j+j}$ *for any* $j \geq 1$*.*

(4) $\max(y_j) < \max(t_{y_j+j})$ *for any* $j \geq 1$*.*

Proof. (1) We have $\max(t_d) > \max(t_{y_r+r-e})$, as $r-e = \#\{z_j : \max(z_j) < \max(t_d)\}$.

(2) We have $\max(t_d) < \max(y_r)$ and $i - 1 < r$.

(3) If $\max(t_d) \leq \max(t_{y_j})$, then $i < j$. If $\max(t_d) > \max(t_{y_j})$, then $i \geq j$.

(4) t_{y_j} is the lex-last t-variable that is lex-before y_j. Hence, t_{y_j+1} is lex-after y_j. The variable t_{y_j+j} comes lex-later still. Thus, $y_j > t_{y_j+j}$. □

We are ready for the proof of the Main Lemma 7.3.9.

Proof of the Main Lemma 7.3.9. Let T be the ideal constructed in Construction 4.5. By Lemma 7.4.6, T is a squarefree Borel ideal lexicographically greater than N, and it has the same Hilbert function as N.

By Theorem 7.3.3, the graded Betti numbers of $S/(N + P)$ and of $S/(T + P)$ can be counted using EK-triples. By Lemma 7.4.14, there exists an injection ϕ from the set of EK-triples for N to the EK-triples for T which preserves bidegree. Therefore, there are at least as many EK-triples for T as for N in every bidegree. It follows that for all p, s, the graded Betti numbers satisfy

$$b_{p,s}(S/(T + P)) \geq b_{p,s}(S/(N + P))\,.$$

□

7.5 Ideals plus squares

Let F be a graded ideal containing $P = (x_1^2, \dots, x_n^2)$; we say that F is an *ideal-plus-squares.* If $F = I + P$ for some ideal I which is squarefree Borel or squarefree lex, we say that F is *Borel-plus-squares* or *lex-plus-squares* respectively. By Kruskal-Katona's Theorem [Kr,Ka], there exists a squarefree lex ideal L such that F and the lex-plus-squares ideal $L+P$ have the same Hilbert function.

Theorem 7.5.1. *Suppose that* $\mathrm{char}\,(k)=0$*. Let F be a graded ideal containing $P=(x_1^2,\ldots,x_n^2)$. Let L be the squarefree lex ideal such that F and the lex-plus-squares ideal $L+P$ have the same Hilbert function. The graded Betti numbers of $L+P$ are greater than or equal to those of F.*

Proof. The proof has 5 steps. In each of the first four steps, we replace the original (non-lex) ideal by an ideal with the same Hilbert function and greater graded Betti numbers.

Step 1: Let F' be the initial ideal of F (with respect to any fixed monomial order). It has the following properties:

∘ $F'\supseteq P$.

∘ F' is a monomial ideal with the same Hilbert function as F.

∘ The graded Betti numbers of F' are greater than or equal to those of F.

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal $L+P$ are greater than or equal to those of F'.

Step 2: Now, we change the ground field k to an infinite field $\tilde{k}$ of characteristic 2. We denote by $\tilde{F}\subset\tilde{k}[x_1,\cdots,x_n]$ the monomial ideal generated by the monomials in F'. It has the following properties:

∘ $\tilde{F}\supseteq P$.

∘ $\tilde{F}$ is a monomial ideal with the same Hilbert function as F'.

∘ The graded Betti numbers of $\tilde{F}$ are greater than or equal to those of F'.

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal $L + P$ are greater than or equal to those of $\tilde{F}$.

Step 3: Now, let $\tilde{M}$ be a generic initial ideal of $\tilde{F}$ (with respect to any monomial order). It has the following properties:

∘ $\tilde{M} \supseteq P$ because the characteristic of $\tilde{k}$ is 2.

∘ $\tilde{M}$ is a Borel-plus-squares ideal with the same Hilbert function as $\tilde{F}$.

∘ The graded Betti numbers of $\tilde{M}$ are greater than or equal to those of $\tilde{F}$.

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal $L + P$ are greater than or equal to those of $\tilde{M}$.

Step 4: The Eliahou-Kervaire resolution [EK] shows that the graded Betti numbers of a squarefree Borel ideal do not depend on the characteristic. By Theorem 7.2.1(4) and Lemma 7.3.1(1), it follows that the graded Betti numbers of a Borel-plus-squares ideal do not depend on the characteristic. So now, we return to the ground field k. We denote by $M \subset k[x_1, \cdots, x_n]$ the monomial ideal generated by the monomials in $\tilde{M}$. It has the following properties:

∘ $M \supseteq P$.

∘ M is a Borel-plus-squares ideal with the same Hilbert function as $\tilde{M}$.

∘ The graded Betti numbers of M are equal to those of $\tilde{M}$.

We will prove the theorem by showing that the graded Betti numbers of the lex-plus-squares ideal $L + P$ are greater than or equal to those of M.

Step 5: Let N be the squarefree Borel ideal such that $M = N + P$. Since $N + P$ and $L + P$ have the same Hilbert function, we can apply Theorem 7.3.4. It yields that the graded Betti numbers of the lex-plus-squares ideal $L + P$ are greater than or equal to those of $M = N + P$. $\square$

7.6 Ideals plus powers

Let $\mathbf{a} = \{a_1 \leq a_2 \leq \cdots \leq a_n\}$ be a sequence of integers or ∞. Set $U = (x_1^{a_1}, \ldots, x_n^{a_n})$, where $x_i^{\infty} = 0$. We say that a monomial $m \in S$ is an $\mathbf{a}$-*monomial* if the image of m in S/U is non-zero. Following [GHP], an ideal in S is called an $\mathbf{a}$-*ideal* if it is generated by $\mathbf{a}$-monomials.

Set $\sigma^{\mathbf{a}} = \prod_{i \in \sigma} x_i^{a_i}$, and for a monomial $\mathbf{a}$-ideal I set $F_\sigma = S/(I : \sigma^{\mathbf{a}})(-2\sigma^{\mathbf{a}})$. Note that $F_\sigma = 0$ if, for any $i \in \sigma$, $a_i = \infty$. Note also that $(I : \sigma^{\mathbf{a}}) = (I : \prod_{i \in \sigma} x_i^{a_i - 1})$ is the ideal formed by "erasing" all the variables in σ from a generating set for I. The argument in the proof of Theorem 7.2.1 yields:

Theorem 7.6.1. *Let I be a monomial $\mathbf{a}$-ideal.*

(1) We have the long exact sequence

$$0 \to \bigoplus_{|\sigma|=n} F_\sigma \xrightarrow{\varphi_n} \ldots \to \bigoplus_{|\sigma|=i} F_\sigma \xrightarrow{\varphi_i} \bigoplus_{|\sigma|=i-1} F_\sigma \to$$

$$\cdots \to \bigoplus_{|\sigma|=1} F_\sigma \xrightarrow{\varphi_1} \bigoplus_{|\sigma|=0} F_\sigma = S/I \to S/(I+U) \to 0 \tag{7.6.2}$$

with maps φ_i the Koszul maps for the sequence $x_1^{a_1}, \ldots, x_n^{a_n}$.

(2) Each of the ideals $(I : \sigma^{\mathbf{a}})$ in (1) is an $\mathbf{a}$-ideal monomial ideal.

(3) $S/(I+U)$ is minimally resolved by the iterated mapping cones from (7.6.2).

Remark 7.6.3. The other results in the previous sections cannot be generalized to this situation. The first problem is that if I and J are $\mathbf{a}$-ideals, then it is not true that I and J have the same Hilbert function if and only if $I+U$ and $J+U$ have the same Hilbert function. The following example from [GHP] illustrates this: the ideals $I = (x^2, y^2)$ and $J = (x^2, xy)$ have different Hilbert functions, but the ideals $I + (x^3, y^3)$ and $J + (x^3, y^3)$ have the same Hilbert function.

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