THE LEX-PLUS-POWERS CONJECTURE HOLDS FOR PURE POWERS

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Abstract: We prove Evans' Lex-Plus-Powers Conjecture for ideals containing a monomial regular sequence.

1. Introduction

Let $S = \mathbb{k}[x_1, \cdots, x_n]$ be the polynomial ring in n variables over an arbitrary field. Fix $r \leq n$ and a nondecreasing sequence of positive integers, $2 \leq e_1 \leq e_2 \leq \cdots \leq e_r$, and let $P = (x_1^{e_1}, \cdots, x_r^{e_r})$ be the ideal generated by those powers of the variables. (If $r \neq n$, it is sometimes convenient to set $e_{r+1} = \cdots = e_n = \infty$ and $x_i^\infty = 0$.)

The Hilbert function of a homogeneous ideal of S is a well-studied and important invariant with applications in many areas, including Algebraic Geometry, Commutative Algebra, and Combinatorics. One of the basic tools in the study of Hilbert functions was provided by Macaulay [Ma] in 1927: every Hilbert function of a homogeneous ideal of S is attained by a lexicographic ideal. Macaulay's insight was that the lex ideals, which are defined combinatorially, are a useful tool in studying the combinatorial invariants of the polynomial ring. Later, Macaulay's theorem was extended to many other rings, including the quotient ring S/P (due to Clements and Lindström [CL]).

Motivated by Macaulay's theorem and applications in Algebraic Geometry, Eisenbud, Green, and Harris made the following conjecture about Hilbert functions [EGH1, EGH2]:

Conjecture 1.1 (Eisenbud, Green, Harris). *Let $F = (f_1, \cdots, f_r)$ be a homogeneous regular sequence, such that $\deg f_i = e_i$ for all i, and let I be any homogeneous ideal containing F. Then there is a lex ideal L such that $L + P$ and I have the same Hilbert function.*

The conjecture is well-studied but remains wide open. The largest class of ideals where it is known is due to recent work of Caviglia and Maclagan [CM], who prove the conjecture whenever the degrees of the regular sequence increase quickly enough (i.e., if $e_k > \sum_{\ell=1}^{k-1} e_\ell$ for all k).

In recent decades, graded Betti numbers have become an important topic in Commutative Algebra. One influential result is due to Bigatti [Bi], Hulett [Hu], and Pardue [Pa] in the 1990s. They showed that the lex ideals of S have maximal graded

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Betti numbers among all ideals with a fixed Hilbert function, providing a sharp upper bound on the graded Betti numbers of a homogeneous ideal with a given Hilbert function. Because of the importance of Bigatti, Hulett, and Pardue's results, similar statements are known or conjectured in many settings where Macaulay-type theorems hold, including the exterior algebra and the ring S/P (see for example Aramova-Herzog-Hibi [AHH1, AHH2] and Gasharov-Hibi-Peeva [GHP]).

Inspired by Bigatti, Hulett, and Pardue's results, Evans [FR] extended the Eisenbud-Green-Harris conjecture to include a statement about Betti numbers:

Conjecture 1.2 (Evans, The Lex-Plus-Powers Conjecture). *Let F, I, and L be as in Conjecture 1.1. Then for all i and j the graded Betti numbers of I and $L+P$ satisfy $b_{i,j}(L+P) \geq b_{i,j}(I)$.*

Both conjectures are open. In particular, the Lex-Plus-Powers Conjecture has been open even if F consists of pure powers of the variables (i.e., $F = P$). The main result of this paper is that the Lex-Plus-Powers Conjecture holds if F consists of monomials, a case in which the Eisenbud-Green-Harris Conjecture is a straightforward consequence of Clements and Lindström's Theorem.

For a subset τ of the variables, put $\mathbf{x}_\tau = \prod_{x_i \in \tau} x_i^{e_i}$. In [MPS], Mermin, Peeva, and Stillman use mapping cones to give a formula for the Betti numbers of a monomial-plus-P ideal in terms of its colon ideals: If M is a monomial ideal not containing any $x_i^{e_i}$, then we have

$$b_{i,j}(M+P) = \sum_{\tau} b_{i-|\tau|,j-\deg \mathbf{x}_\tau}(M : \mathbf{x}_\tau). \tag{1.3}$$

Using this formula and the Eliahou-Kervaire resolution [EK], Murai shows in [Mu] that the Lex-Plus-Powers Conjecture holds for Borel-plus-P ideals:

Theorem 1.4 ([Mu]). *Suppose that B is Borel, and let L be a lex ideal such that $L+P$ has the same Hilbert function as $B+P$. Then for all i, j we have $b_{i,j}(L+P) \geq b_{i,j}(B+P)$.*

Thus, the Lex-Plus-Powers conjecture would be proved by reduction to the Borel case:

Question 1.5. *Let I and F be as in Conjecture 1.1. Does there exist a Borel ideal B such that $B+P$ has the same Hilbert function as, and larger Betti numbers than, I?*

In Theorems 3.1 and 8.1, we give a positive answer to Question 1.5 in the case that F consists of monomials.

In section 2, we introduce notation which will be used throughout the paper. In section 3, using a walk on the Hilbert scheme, we prove the Lex-Plus-Powers Conjecture for ideals containing powers of the variables in characteristic zero. This approach yields a short proof, but does not work in positive characteristic.

In sections 4 through 8, we give a characteristic-free proof of the same result. While the proof is long, we introduce some new techniques to study Hilbert functions and Betti numbers of monomial ideals, including Theorem 4.5, a formula for the multigraded Betti numbers of any monomial ideal. Our main tool is a generalization of the combinatorial "shifting" operation of Erdös, Ko, and Rado [EKR].

Shifting is an operation which associates to every simplicial complex another complex with the same face vector and certain special properties, called "shifted". (See [AHH2, MH].) We generalize combinatorial shifting to monomial ideals, and show that Betti numbers are nondecreasing under this operation. We use shifting and compression (defined in [Me2]) to compare the Betti numbers of an ideal I containing P with those of a Borel-plus-P ideal.

We also consider some related problems. In section 9, we show that the Betti numbers of I are obtained from those of the lex-plus-powers ideal $L + P$ by consecutive cancellations. In section 10, we briefly discuss some open problems.

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2. Background and Notation

We recall some notation and results that will be used throughout the paper.

Metadefinition 2.1. For a property $(*)$ of ideals, and an ideal I containing P, we say that I is *$(*)$-plus-P* if there exists an ideal $\hat{I}$ satisfying $(*)$ such that $I = \hat{I} + P$. In this paper, we will consider homogeneous-plus-P, lex-plus-P, compressed-plus-P, Borel-plus-P, and shifted-plus-P ideals.

Notation 2.2. The ring S is graded by setting $\deg x_i = 1$ for all i. All the S-modules we consider will inherit a natural grading from S; if M is a graded module we write M_d for the $\Bbbk$-vector subspace spanned by the homogeneous forms of degree d in M. We denote shifts in the grading in the usual way; that is, $M(-d)$ is the module isomorphic to M but with all degrees increased by d, so that, as vector spaces, $M(-d)_e = M_{e-d}$.

Definition 2.3. We will use both the graded *lexicographic* and *reverse lexicographic* monomial orderings. Let u and v be monomials of the same degree, and write $u = x_1^{e_1}x_2^{e_2}\cdots x_n^{e_n}$ and $v = x_1^{f_1}x_2^{f_2}\cdots x_n^{f_n}$. We say that u is *greater than v with respect to the lexicographic order*, or $u >_{\text{lex}} v$, if there exists an i such that $e_i > f_i$ and $e_j = f_j$ for all $j < i$. We say that u is *greater than v with respect to the reverse lexicographic order*, or $u >_{\text{rev}} v$, if there exists an i such that $e_i < f_i$ and $e_j = f_j$ for all $j > i$.

Definition 2.4. We say that a monomial ideal $L \subset S$ is *lex* or *lexicographic* if, for all degrees d, the vector space L_d is generated by an initial segment in the lexicographic order. That is, if u and v are monomials of the same degree such that $u <_{\text{lex}} v$ and $u \in L$, then we must have $v \in L$ as well.

Definition 2.5. We can use these orderings to compare monomial ideals as well. Let $\mathcal{I} = \{u_1, \cdots, u_s\}$ and $\mathcal{J} = \{v_1, \cdots, v_s\}$ be sets of degree d monomials, each ordered reverse lexicographically (so $u_i >_{\text{rev}} u_j$ and $v_i >_{\text{rev}} v_j$ whenever $i < j$). Then we say that $\mathcal{I}$ *is reverse lexicographically greater than* $\mathcal{J}$, $\mathcal{I} >_{\text{rev}} \mathcal{J}$, if there exists an i such that $u_i >_{\text{rev}} v_i$ and $u_j = v_j$ for all $j < i$. For monomial ideals $I \neq J$ having the same Hilbert function, and for a degree d, let $\{I_d\}$ and $\{J_d\}$ be the sets of degree d monomials in I and J, respectively. We say that I *is reverse lexicographically greater than* J if, for all d, $I_d = J_d$ or $\{I_d\}$ is reverse lexicographically greater than $\{J_d\}$.

Lemma 2.6. *Let $\mathcal{I} = \{u_1, \dots, u_t\}$ and $\mathcal{J} = \{v_1, \dots, v_t\}$ be sets of monomials, all with the same degree. If $u_k \geq_{\text{rev}} v_k$ for all k, then $\mathcal{I}$ is reverse lexicographically greater than or equal to $\mathcal{J}$.*

Proof. We use induction on t. If $t = 1$, the statement is immediate. Otherwise, let u_p and v_q be the smallest elements of $\mathcal{I}$ and $\mathcal{J}$, respectively, with respect to the reverse lex order. Then, by assumption, we have $u_q \geq_{\text{rev}} u_p \geq_{\text{rev}} v_p \geq_{\text{rev}} v_q$, so we can apply the inductive hypothesis to get that $\mathcal{I} \setminus \{u_p\}$ is reverse lexicographically greater than or equal to $\mathcal{J} \setminus \{v_q\}$. Since u_p and v_q are the smallest elements of $\mathcal{I}$ and $\mathcal{J}$, it follows that $\mathcal{I}$ is reverse lexicographically greater than or equal to $\mathcal{J}$ as desired. $\square$

Term orders allow us to associate to any ideal of S a monomial ideal, called its initial ideal. In this paper we consider only reverse lexicographic initial ideals, but the definition below works with any term order.

Definition 2.7. For a (homogeneous) polynomial g, write $g = \sum a_m m$ with $a_m \in \Bbbk$ and m ranging over the monomials. The *initial monomial* of g, $\text{in}_{\text{rev}}(g)$, is the maximal m in the reverse lexicographic order such that a_m is nonzero. For an (homogeneous) ideal I, the *initial ideal* of I is the monomial ideal generated by the initial monomials of every form in I, $\text{in}_{\text{rev}}(I) = (\text{in}_{\text{rev}}(g) : g \in I)$. It is well-known that $\text{in}_{\text{rev}}(I)$ has the same Hilbert function as I and larger graded Betti numbers.

Definition 2.8. For a graded module M, the *Hilbert function* of M assigns to each degree d the dimension of the vector space M_d. We write $\text{Hilb}(M)(d) = \dim_{\Bbbk} M_d$.

Definition 2.9. A *free resolution* of the graded module M is an exact sequence

$$\mathbb{F} : \cdots \to F_1 \to F_0 \to M \to 0$$

such that each F_i is a free S-module. We say that $\mathbb{F}$ is the *minimal free resolution* of M if each F_i has minimum possible rank. Equivalently, $\mathbb{F}$ is minimal if, for all i, the nonzero entries of the matrix associated to the map $d_i : F_i \to F_{i-1}$ are contained in the homogeneous maximal ideal, $(x_1, \cdots, x_n)$. Up to an isomorphism of complices, every finitely generated module has a unique minimal free resolution.

Definition 2.10. If $\mathbb{F}$ is the minimal free resolution of M, the *Betti numbers* of M are given by $b_i(M) = \text{rk}\, F_i$. If we decompose the F_i as graded free modules, $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{b_{i,j}}$, then the $b_{i,j}$ are the *graded Betti numbers* of M.

Definition 2.11. A monomial ideal I is *Borel* or 0-*Borel-fixed* if it satisfies the property:

$$\text{If } fx_j \in I \text{ and } i < j, \text{ then } fx_i \in I.$$

Borel ideals are important because they occur (in characteristic zero) as generic initial ideals [BS, Ga]. They are combinatorially useful because they are minimally resolved by the Eliahou-Kervaire resolution [EK], which gives explicit formulas for their Betti numbers. Borel ideals can also be attained via a characteristic-free technique called compression.

Definition 2.12. Fix a subset $\mathcal{A} \subset \{x_1, \cdots, x_n\}$. Any monomial ideal I decomposes (as a $\Bbbk$-vector space) into a direct sum over monomials $f \in \Bbbk[\{x_1, \cdots, x_n\} \smallsetminus \mathcal{A}]$, $I = \bigoplus_f fV_f$. Each V_f is an ideal of $\Bbbk[\mathcal{A}]$. If the V_f are all lex ideals in $\Bbbk[\mathcal{A}]$, we say that I is $\mathcal{A}$-*compressed*.

Set W_f equal to the lex ideal of $\Bbbk[\mathcal{A}]$ having the same Hilbert function as V_f. We say that $J = \bigoplus fW_f$ is the $\mathcal{A}$-*compression* of I.

Compression and compressed ideals have been used by Macaulay and others [CL, Ma, Me1, Me2, MP1, MP2, MPS, MH] to study Hilbert functions and Betti numbers. In [Me2], Mermin proves the following:

Theorem 2.13 ([Me2])**.** *Let N be a monomial ideal, and let T be the $\mathcal{A}$-compression of N. Then:*

(i) *T is an ideal.*
(ii) *N and T have the same Hilbert function.*
(iii) *$b_{i,j}(T) \geq b_{i,j}(N)$ for all i and j.*
(iv) *N is Borel if and only if N is $\{x_i, x_j\}$-compressed for all i and j.*

Definition 2.14 (**Polarization**)**.** For ease of notation, we define a simplified version of polarization. For a fuller version of the theory, see e.g. [Ei2, Exercise 3.24]. Fix a variable $b = x_k$. For a monomial $u = \prod x_i^{f_i}$, the b^{th} *polarization* of u is

$$\operatorname{pol}_b(u) = \left(\prod_{x_i \neq b} x_i^{f_i} \right) (bc_1 \cdots c_{f_k - 1}),$$

where the c_i are new variables (and $\operatorname{pol}_b(u) = u$ if b does not divide u).

Let s be sufficiently large (e.g., the largest power of b occuring in any generator of any ideal under consideration), and set $S^{\text{po}} = \Bbbk[x_1, \cdots, x_n, c_1, \cdots, c_{s-1}]$. (We order the variables so that $x_n >_{\text{rev}} c_k$ for all k.) For a monomial ideal I, set $\operatorname{gens}(I)$ equal to the (unique) set of minimal monomial generators of I. Then for $u \in \operatorname{gens}(I)$, we have $\operatorname{pol}_b(u) \in S^{\text{po}}$. The *polarization* of I is the ideal I^{po} of S^{po} generated by these monomials,

$$I^{\text{po}} = (\operatorname{pol}_b(u) : u \in \operatorname{gens}(I)) \, .$$

A monomial ideal $I \in S$ is naturally associated to two ideals of S^{po}, namely I^{po} and IS^{po}. We have the following:

Proposition 2.15.

(i) *For all i and j, $b_{i,j}(I) = b_{i,j}(I^{\mathrm{po}}) = b_{i,j}(IS^{\mathrm{po}})$. (Here, $b_{i,j}(I^{\mathrm{po}})$ and $b_{i,j}(IS^{\mathrm{po}})$ are taken over S^{po}.)*

(ii) *Let I and J be monomial ideals of S. Then I^{po} and IS^{po} have the same Hilbert function, and IS^{po} and JS^{po} have the same Hilbert function if and only if I and J have the same Hilbert function.*

(iii) *Let $u \in I$ be a monomial of S such that $\mathrm{pol}_b(u) \in S^{\mathrm{po}}$. Then $\mathrm{pol}_b(u) \in I^{\mathrm{po}}$.*

Proof. (i) is [BH, Lemma 4.2.16], and (ii) is immediate from (i) and the formula

$$\mathrm{Hilb}(I)(d) = \sum_{i,j} \left((-1)^i b_{i,j}(I)\,\mathrm{Hilb}(S)(d-j)\right).$$

For (iii), observe that $\mathrm{pol}_b(v)$ divides $\mathrm{pol}_b(u)$ whenever v divides u. □

3. The proof in characteristic zero

In this section we prove the following:

Theorem 3.1. *Let $\Bbbk$ have characteristic zero, and let $F = (f_1, \cdots, f_r)$ be a regular sequence of monomials, in degrees $e_1 \leq \cdots \leq e_r$. Set $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. If I is any ideal containing F, then there exists a lex-plus-P ideal L such that I and L have the same Hilbert function and $b_{i,j}(L) \geq b_{i,j}(I)$ for all i and j.*

Throughout the section, $F = (f_1, \cdots, f_r)$ will be a regular sequence of monomials in degrees $e_1, \cdots, e_r$, and P will be the pure powers in these degrees, $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. First, we reduce to the case that I is monomial-plus-P.

Lemma 3.2. *Let I be a homogeneous ideal containing F. Then there exists a monomial ideal J containing P such that I and J have the same Hilbert function and $b_{i,j}(J) \geq b_{i,j}(I)$ for all i and j.*

Proof. For any monomial u of S, we set $\mathrm{supp}(u) = \{x_k : x_k \text{ divides } u\}$. Since $f_1, \cdots, f_r$ is a regular sequence, we have $\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) = \varnothing$ for all $i \neq j$. After reordering the variables if necessary, we may assume $x_i \in \mathrm{supp}(f_i)$.

Write $\mathrm{supp}(f_1) = \{x_{i_1}, \ldots, x_{i_t}\}$. We may assume $i_1 = 1$. Consider the automorphism ϕ of S given by $\phi(x_k) = x_k$ for $x_k = x_1$ or $x_k \notin \mathrm{supp}(f_1)$ and $\phi(x_k) = x_1 + x_k$ for $x_k \in \mathrm{supp}(f_1) \setminus \{x_1\}$. We have $\phi(f_k) = f_k$ for $k \neq 1$, and we can write $\phi(f_1) = x_1^{e_1} + g$ for some polynomial g. Set $I' = \mathrm{in}_{\mathrm{rev}}(\phi(I))$. Then I' contains $(x_1^{e_1}, f_2, \cdots, f_r)$, has the same Hilbert function as I, and $b_{i,j}(I') \geq b_{i,j}(I)$. Repeating this procedure for each f_k yields an ideal J with the desired properties. □

We remark that the proof of Lemma 3.2 is characteristic-free. However, for the rest of the section, we will assume that $\Bbbk$ has characteristic zero and that I is a monomial-plus-P ideal. Since the resolution of a monomial ideal depends only on the characteristic of the ground field, we may, without loss of generality, replace $\Bbbk$ with any field of characteristic zero. Thus, we will assume that $\Bbbk = \mathbb{C}$.

The idea of our proof is similar to that of Pardue [Pa]. For a monomial-plus-P ideal I which is not Borel-plus-P, we construct another ideal J satisfying:

- J contains P.
- $\mathrm{Hilb}(J) = \mathrm{Hilb}(I)$.
- $b_{i,j}(J) \geq b_{i,j}(I)$ for all i, j.
- J is reverse lexicographically greater than I.

After applying this construction repeatedly, we will obtain a Borel-plus-P ideal and apply Theorem 1.4.

Definition 3.3. Any homogeneous polynomial $f \in S_d$ may be written $f = \sum \alpha_v v$, where v ranges over the degree d monomials and $\alpha_v \in \mathbb{C}$. The *monomial support* of f is the set of monomials with nonzero coefficients, $\mathrm{Supp}(f) = \{v : \alpha_v \neq 0\}$.

Lemma 3.4. *Let $d \geq 0$ be an integer and let $u_1 >_{\mathrm{rev}} \cdots >_{\mathrm{rev}} u_t$ be monomials of degree d. Suppose that $f_1, \ldots, f_t$ are $\mathbb{C}$-linearly independent polynomials of degree d such that $u_k \in \mathrm{Supp}(f_k)$ for all k and $u_k \notin \mathrm{Supp}(f_\ell)$ whenever $\ell \lneqq k$. Then $\big\{ \mathrm{in}_{\mathrm{rev}}(f) : f \in \mathrm{span}_{\mathbb{C}}\{f_1, \ldots, f_t\}\big\}$ is reverse lexicographically greater than or equal to $\{u_1, \ldots, u_t\}$.*

Proof. We induct on t. If $t = 1$ then the statement is obvious. Otherwise, let $F = \{\mathrm{in}_{\mathrm{rev}}(f) : f \in \mathrm{span}_{\mathbb{C}}\{f_1, \ldots, f_{t-1}\}\}$. By induction, we have $F \geq_{\mathrm{rev}} \{u_1, \ldots, u_{t-1}\}$. Let

$$v \in \{\mathrm{in}_{\mathrm{rev}}(f) : f \in \mathrm{span}_{\mathbb{C}}\{f_1, \ldots, f_t\}\} \smallsetminus F.$$

By Lemma 2.6, it is enough to show that $v \geq_{\mathrm{rev}} u_t$. By the definition of v, there exist $\alpha_1, \ldots, \alpha_t \in \mathbb{C}$ such that $v = \mathrm{in}_{\mathrm{rev}}(\alpha_1 f_1 + \cdots + \alpha_t f_t)$. Since $v \notin F$ we have $\alpha_t \neq 0$, so $u_t \in \mathrm{Supp}(\alpha_1 f_1 + \cdots + \alpha_t f_t)$. Thus, $v = \mathrm{in}_{\mathrm{rev}}(\alpha_1 f_1 + \cdots + \alpha_t f_t) \geq_{\mathrm{rev}} u_t$ as desired. ☐

For the remainder of the section, fix two variables $a >_{\mathrm{rev}} b$.

Proposition 3.5. *Suppose that P contains no power of b, and that I is not $\{a, b\}$-compressed-plus-P. Consider the automorphism ϕ of S given by $\phi(x_k) = x_k$ for $x_k \neq b$ and $\phi(b) = a - b$. Set $J = \mathrm{in}_{\mathrm{rev}}(\phi(I))$. Then:*

(i) *J contains P.*
(ii) *J has the same Hilbert function as I.*
(iii) *$b_{i,j}(J) \geq b_{i,j}(I)$ for all i and j.*
(iv) *$J \neq I$.*
(v) *J is reverse lexicographically greater than I.*

Proof. (i), (ii), (iii), and (iv) are immediate; we prove (v). For any degree d, let $\{I_d\}$ be the set of degree d monomials in I. Write $\{I_d\} = \{u_1, \cdots, u_t\}$, ordered reverse lexicographically. Then $\{\phi(u_1), \cdots, \phi(u_t)\}$ is a $\mathbb{C}$-basis for $\phi(I)_d$. Clearly, $u_k \in \mathrm{Supp}(\phi(u_k))$ for all k and $u_k \notin \mathrm{Supp}(\phi(u_\ell))$ for all $\ell \lneqq k$. Applying Lemma 3.4, $\{J_d\}$ is reverse lexicographically greater than or equal to $\{I_d\}$. Since d was arbitrary, it follows that J is reverse lexicographically greater than I, proving (v). ☐

Next, we consider the case that P contains some power of b.

Definition 3.6. Let e_b be the smallest power of b appearing in P (i.e., b^{e_b} is a generator of P), and let ζ be a primitive ${e_b}^{\text{th}}$ root of unity (e.g., $\zeta = \cos\frac{2\pi}{e_b} + \sqrt{-1}\sin\frac{2\pi}{e_b}$). Let $\widetilde{\phi}$ be the autormorphism of S^{po} given by $\widetilde{\phi}(x_k) = x_k$ for $x_k \neq b$, $\widetilde{\phi}(b) = a - b$, and $\widetilde{\phi}(c_k) = a - \zeta^k b + c_k$ for all c_k. Put $\widetilde{J} = \mathrm{in}_{\mathrm{rev}}(\widetilde{\phi}(I^{\mathrm{po}}))$.

We recall an arithmetic fact about roots of unity:

Lemma 3.7. *Let $f = (a-b)(a-\zeta b)\cdots(a-\zeta^k b)$. Then:*

(i) *If $k = e_b - 1$, then $f = a^k - b^k$.*
(ii) *If $k \lneqq e_b - 1$, then $ab^{k-1} \in \mathrm{Supp}(f)$.*

Lemma 3.8. *Suppose that P contains some power of b, and that I is not $\{a,b\}$-compressed-plus-P. Then:*

(i) *$\widetilde{J}$ contains PS^{po}.*
(ii) *$\widetilde{J}$ has the same Hilbert function as IS^{po}.*
(iii) *$b_{i,j}(\widetilde{J}) \geq b_{i,j}(IS^{\mathrm{po}})$ for all i and j.*
(iv) *$\widetilde{J} \neq IS^{\mathrm{po}}$.*
(v) *$\widetilde{J} \cap S$ is reverse lexicographically greater than I.*
(vi) *$\widetilde{J} = (\widetilde{J} \cap S)S^{\mathrm{po}}$.*

Proof. To prove (i), it suffices to show that $b^{e_b} \in \widetilde{J}$. We have

$$\begin{aligned}\widetilde{\phi}\left(\mathrm{pol}_b(b^{e_b})\right) &= (a-b)\prod_{k=1}^{e_b-1}\Big((a-\zeta^k b) + c_k\Big)\\ &= \prod_{k=1}^{e_b}\Big(a - \zeta^k b\Big) + g\\ &= a^{e_b} - b^{e_b} + g,\end{aligned}$$

where every term of the polynomial g is divisible by some c_k. In particular, since $a^{e_b} \in I^{\mathrm{po}}$, we have $\widetilde{\phi}(a^{e_b} - \mathrm{pol}_b(b^{e_b})) = b^{e_b} + g \in \widetilde{\phi}(I^{\mathrm{po}})$, so $b^{e_b} \in \widetilde{J}$.

(ii) and (iii) are immediate from Proposition 2.15. We will prove (iv), (v), and (vi) simultaneously.

Let $\mathcal{A} = \{a, b, c_1, \cdots, c_{s-1}\}$ be the set consisting of a, b, and all of the c-variables, put $R = \mathbb{C}[\mathcal{A}]$, and consider the decomposition $I = \bigoplus fI_f$, where f ranges over the monomials of S which are not divisible by a or b. Since $\widetilde{\phi}$ restricts to an automorphism of R, we get $\widetilde{J} = \bigoplus f\widetilde{J}_f = \bigoplus f\left(\mathrm{in}_{\mathrm{rev}}(\widetilde{\phi}((I_f)^{\mathrm{po}}))\right)$. It suffices to show that $\{(\widetilde{J}_f \cap \mathbb{C}[a,b])_d\} \geq_{\mathrm{rev}} \{(I_f)_d\}$ for all d (where $\{(I_f)_d\}$ is the set of degree d monomials in I_f, etc.), that $\widetilde{J}_f \neq I_f R$ whenever I_f is not lex-plus-(b^{e_b}) in $\mathbb{C}[a,b]$, and that $\widetilde{J}_f = (\widetilde{J}_f \cap \mathbb{C}[a,b])R$.

Write $\{(I_f)_d\} = \{a^{p_1}b^{q_1}, \cdots, a^{p_t}b^{q_t}\}$ in reverse lex order (so $q_1 < \cdots < q_t$.) We have

$$\widetilde{\phi}(\mathrm{pol}_b(a^{p_k}b^{q_k})) = a^{p_k}\left(\left(\prod_{\ell=0}^{q_k-1}(a-\zeta^\ell b)\right) + g_k\right),$$

where every term of g_k is divisible by some c_t. We have $a^{p_k}b^{q_k} \in \mathrm{Supp}(\widetilde{\phi}(\mathrm{pol}_b(a^{p_k}b^{q_k})))$ for all k, and $a^{p_k}b^{q_k} \notin \mathrm{Supp}(\widetilde{\phi}(\mathrm{pol}_b(a^{p_\ell}b^{q_\ell})))$ for all $\ell \lneqq k$. Set

$$F_d = \{\mathrm{in}_{\mathrm{rev}}(\widetilde{\phi}(\mathrm{pol}_b(a^{p_k}b^{q_k}))) : a^{p_k}b^{q_k} \in \{(I_f)_d\}\}.$$

Then, by Lemma 3.4, it follows that $F_d \geq_{\mathrm{rev}} \{(I_f)_d\}$. In particular, $F_d \subset \mathbb{C}[a,b]$.

Let J_f be the ideal of $\mathbb{C}[a,b]$ generated by all the F_d, $d \geq 0$. Immediately we have $\mathrm{Hilb}(J_f)(d) \geq \mathrm{Hilb}(I_f)(d)$ for all d, so it follows that $\mathrm{Hilb}(J_f R)(d) \geq \mathrm{Hilb}(I_f R)(d) = \mathrm{Hilb}(\widetilde{J}_f)(d)$ for all d. But $J_f R \subset \widetilde{J}_f$, so it must be the case that $J_f R = \widetilde{J}_f$ and $J_f = \bigoplus \mathrm{span}_{\mathbb{C}}(F_d)$. This proves $\widetilde{J}_f \cap \mathbb{C}[a,b] = J_f$, so (vi) holds. Also, since $F_d \geq_{\mathrm{rev}} \{I_f\}_d$, it follows that $\widetilde{J}_f \cap \mathbb{C}[a,b] = \bigoplus \mathrm{span}_{\mathbb{C}}(F_d)$ is reverse lexicographically greater than or equal to I_f.

For (iv) and (v), it remains to show that, if I_f is not lex-plus-(b^{e_b}), then $\widetilde{J}_f \neq I_f R$. In this case, there exists a degree d and an index k such that $q_k \gneqq e_b$, and $v = a^{p_k}b^{q_k} \in (I_f)_d$ but $u = a^{p_k+1}b^{q_k-1} \notin (I_f)_d$. It follows from Lemma 3.7 that $u \in \mathrm{Supp}(\widetilde{\phi}(\mathrm{pol}_b(v)))$, but $u \notin \mathrm{Supp}(\widetilde{\phi}(\mathrm{pol}_b(a^{p_\ell}b^{q_\ell})))$ for any $\ell \lneqq k$. Thus, by Lemma 3.4, F_d is (strictly) reverse lexicographically greater than $\{(I_f)_d\}$, and in particular $J_f \neq I_f$ and $\widetilde{J}_f \neq I_f R$. $\square$

Corollary 3.9. *Suppose that P contains some power of b, and that I is not $\{a,b\}$-compressed-plus-P. Set $J = \widetilde{J} \cap S$. Then:*

(i) *J contains P.*
(ii) *J has the same Hilbert function as I.*
(iii) *$b_{i,j}(J) \geq b_{i,j}(I)$ for all i and j.*
(iv) *$J \neq I$.*
(v) *J is reverse lexicographically greater than I.*

Proposition 3.10. *Let I be a monomial ideal containing P. Then there exists a Borel-plus-P ideal B such that B has the same Hilbert function as I, and $b_{i,j}(B) \geq b_{i,j}(I)$ for all i,j.*

Proof. If I is not already Borel-plus-P, there exist pairs of variables a,b such that I is not $\{a,b\}$-compressed-plus-P. Choose any such pair. Define J as in Corollary 3.9 if P contains some power of b, and as in Proposition 3.5 otherwise. By Corollary 3.9 or Proposition 3.5, J has the same Hilbert function as I and larger Betti numbers. Replace I with J and repeat this procedure. The process must terminate since there are finitely many monomial ideals with the same Hilbert function, and at each step we are replacing the ideal with a reverse lexicographically greater one. Let B be the resulting ideal. $\square$

Theorem 1.4 completes the proof of Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, we may assume without loss of generality that $F = P$, that I is a monomial ideal, and that $\Bbbk = \mathbb{C}$. By Proposition 3.10, we may assume that I is Borel-plus-P. Thus, the desired inequality holds by Theorem 1.4. $\square$

4. Further notation

For the duration of the paper, $\Bbbk$ will be an arbitrary field. Frequently it will be necessary to slice modules more finely than is possible with the standard grading. To this end, we use the multigraded structure of S:

Notation 4.1. We write multidegrees multiplicatively. That is, we set $\operatorname{mdeg} x_i = x_i$ for all i, so that the multidegrees are indexed by the monomials of S. We have $S = \bigoplus S_m$, where m ranges over all the monomials, and S_m is the one-dimensional $\Bbbk$-vector space spanned by $\{m\}$. The modules we consider will all inherit a multigraded structure from S, and shifts in the grading will be written multiplicatively, so, for monomials u and v, we will have $M(u^{-1})_v = M_{u^{-1}v}$ as vector spaces.

Remark. Whenever we have a map $\phi : M \to N$ of graded (respectively, multigraded) modules, ϕ will be *homogeneous of degree* 0 (resp., *multihomogeneous of degree* 1); that is, ϕ will satisfy $\phi(M_d) \subset N_d$ for all d (resp., $\phi(M_m) \subset N_m$ for all m). Verification of this property for each of the maps defined in the paper is straightforward, and so will be omitted.

Definition 4.2. If $\mathbb{F}$ is the minimal free resolution of M, and we decompose the free modules F_i as multigraded modules, $F_i = \bigoplus S(m^{-1})^{b_{i,m}}$, we say that the $b_{i,m}$ are the *multigraded Betti numbers* of M.

Tensoring the resolution $\mathbb{F}$ by $\Bbbk$, we get $b_i(M) = \dim_\Bbbk \operatorname{Tor}_i(\Bbbk, M)$, $b_{i,j}(M) = \dim_\Bbbk \operatorname{Tor}_i(\Bbbk, M)_j$, and $b_{i,m}(M) = \dim_\Bbbk \operatorname{Tor}_i(\Bbbk, M)_m$.

Construction 4.3. Since Tor is balanced, we can compute Betti numbers via a resolution of $\Bbbk$, thus avoiding the more difficult problem of computing a resolution of M. The minimal resolution of $\Bbbk$ is given by the Koszul complex

$$\mathbb{K} : K_n \to K_{n-1} \to \cdots \to K_1 \to S \to \Bbbk \to 0.$$

Each K_i is the i^{th} exterior power of K_1; it has a free basis given by the symbols e_μ, where μ ranges over the squarefree monomials of degree i. The symbol e_μ has degree i and multidegree μ. If $\mu = x_{j_1} \cdots x_{j_i}$ with $j_1 < \cdots < j_i$, we write $e_\mu = e_{x_{j_1}} \wedge \cdots \wedge e_{x_{j_i}}$. The differential is given on this basis by $D(e_{x_{j_1}} \wedge \cdots \wedge e_{x_{j_i}}) = \sum_{c=1}^{i} (-1)^{c+1} x_{j_c} e_{\frac{\mu}{x_{j_c}}}$.

Thus, the Betti numbers of M can be computed from the homology of the complex

$$\cdots \to M \otimes K_{i+1} \to M \otimes K_i \to \cdots .$$

If $M = I$ is a monomial ideal of S, the module $M \otimes K_i$ is the subcomplex of $\mathbb{K}$ generated (as a $\Bbbk$-vector space) by terms of the form fe_μ, where $f \in I$ is a monomial and μ is a squarefree monomial of degree i. The term fe_μ has degree $\deg(f\mu)$ and multidegree $f\mu$. Its differential is $D(fe_\mu) = fD(e_\mu)$.

If $M = S/I$ is the quotient by a monomial ideal, then $\mathrm{Tor}_i(S/I, \Bbbk) = \mathrm{Tor}_{i-1}(I, \Bbbk)$ from the resolutions of I and S/I. We will, without comment, use the homology of $\mathbb{K} \otimes I$ rather than that of $\mathbb{K} \otimes S/I$ in our computations.

This approach yields a formula for the multigraded Betti numbers of any monomial ideal.

Definition 4.4. Let I be a monomial ideal, and let $m = \prod x_\ell^{e_\ell}$ be a monomial. Put $\mathrm{supp}(m) = \{x_\ell : e_\ell \neq 0\}$ and $\sqrt{m} = \prod_{e_\ell \neq 0} x_\ell$. The *shadow of m in I* is the squarefree monomial ideal of $\Bbbk[\mathrm{supp}(m)]$ given by

$$\mathrm{Shadow}_m(I) = \mathrm{sqfree}\left((I : \frac{m}{\sqrt{m}}) \cap \Bbbk[\mathrm{supp}(m)] \right).$$

(For a monomial ideal J, $\mathrm{sqfree}(J)$ is the ideal generated by the squarefree monomials in J.)

Theorem 4.5. *Let I be a monomial ideal, and fix a multidegree m. Then, for all integers i, the following numbers are equal:*

(i) $b_{i,m}(I)$.
(ii) $b_{i,m}(I \cap (\frac{m}{\sqrt{m}}))$.
(iii) $b_{i,\sqrt{m}}(I : \frac{m}{\sqrt{m}})$.
(iv) $b_{i,\sqrt{m}}(\mathrm{Shadow}_m(I))$.

Note that (i), (ii), and (iii) are Betti numbers of ideals of S, while (iv) is a Betti number of an ideal of $\Bbbk[\mathrm{supp}(m)]$. This ideal can, however, be treated as an ideal of S without altering its Betti numbers. Note also that (iv) is a Betti number of a squarefree ideal, and can be computed with Hochster's formula [Ho].

Proof. For a monomial m, the multigraded Betti number $b_{i,m}(I)$ is the i^{th} homology of the complex of vector spaces $(\mathbb{K} \otimes I)_m$, which has a $\Bbbk$-basis given by

$$\{f e_\mu : f \in I, f\mu = m, f \text{ and } \mu \text{ are monomials}, \mu \text{ is squarefree}\}.$$

Since any f appearing in this basis is contained in $I \cap (\frac{m}{\sqrt{m}})$, it follows that $(\mathbb{K} \otimes I)_m = (\mathbb{K} \otimes (I \cap (\frac{m}{\sqrt{m}})))_m$, so (i) is equal to (ii).

On the other hand, if m is squarefree, then any f appearing in this basis is a squarefree monomial of $\Bbbk[\mathrm{supp}(m)]$. Thus, the complices $(\mathbb{K} \otimes (I : \frac{m}{\sqrt{m}}))_{\sqrt{m}}$ and $(\mathbb{K} \otimes \mathrm{Shadow}_m(I))_{\sqrt{m}}$ are the same. Hence (iii)=(iv).

The isomorphism $\frac{m}{\sqrt{m}} \cdot (I : \frac{m}{\sqrt{m}}) = I \cap (\frac{m}{\sqrt{m}})$ gives us (ii)=(iii), completing the proof. □

Finally, we recall "combinatorial shifting" of squarefree ideals.

Definition 4.6. Let I be a squarefree ideal (i.e., I is generated by squarefree monomials). We say that I is *squarefree Borel* or *shifted* if it satisfies the following property:

> Let f be a monomial such that fx_i and fx_j are squarefree, and suppose $i < j$. Then $fx_j \in I \Rightarrow fx_i \in I$.

Shifted ideals arise as the Stanley-Reisner ideals of shifted simplicial complices, and are well-studied in combinatorics.

Definition 4.7. Fix two variables $a >_{\text{lex}} b$. The *combinatorial shift* of a squarefree ideal I is the ideal $\text{Shift}_{a,b}(I)$ generated by:

$$\text{Shift}_{a,b}(I) = \left(\begin{array}{ccc} f & : & f \in I \\ fa & : & fa \in I \text{ or } fb \in I \\ fb & : & fa \in I \text{ and } fb \in I \\ fab & : & fab \in I \end{array} \right),$$

where f runs over all the squarefree monomials not divisible by a or b.

Combinatorial shifting was introduced by Erdös, Ko, and Rado [EKR] for simplicial complexes. Their definition is equivalent to the one given above under the Stanley-Reisner correspondence. The ideal $\text{Shift}_{a,b}(I)$ may readily be shown to be a squarefree ideal having the same Hilbert function as I, and any squarefree ideal can be transformed into a shifted ideal by a sequence of combinatorial shifts. A generalization of this construction to (not necessarily squarefree) monomial ideals is a major element in our proof of Theorem 8.1. In [MH], Murai and Hibi show that Betti numbers increase under combinatorial shifting:

Theorem 4.8 ([MH])**.** *Let I be a squarefree ideal, and put $J = \text{Shift}_{a,b}(I)$. Then $b_{i,j}(S/J) \geq b_{i,j}(S/I)$ for all i and j.*

The proof given in [MH] (which is the inspiration for section 5 of this paper) is involved. For the convenience of the reader, and in the spirit of our proof of Theorem 3.1, we give a shorter proof here:

Proof. Let ϕ be the automorphism of S given by $\phi(b) = a - b$ and $\phi(x_k) = x_k$ for $x_k \neq b$. Put $I' = \text{in}_{\text{rev}}(\phi(I))$. A straightforward computation shows

$$I' \supseteq \left(\begin{array}{ccc} f & : & f \in \text{Shift}_{a,b}(I) \\ fa & : & fa \in \text{Shift}_{a,b}(I) \\ fb & : & fb \in \text{Shift}_{a,b}(I) \\ fa^2 & : & fab \in \text{Shift}_{a,b}(I) \end{array} \right),$$

for all squarefree monomials f not divisible by a or b.

Define the automorphism $\widetilde{\phi}$ of S^{po} by $\widetilde{\phi}(c_k) = b - c_k$ for all c_k and $\widetilde{\phi}(x_\ell) = x_\ell$ for all x_ℓ, and set $\widetilde{J} = \text{in}_{\text{rev}}(\widetilde{\phi}((I')^{\text{po}}))$.

A straightforward computation gives us $\widetilde{J} \supseteq JS^{\text{po}}$. Since $\text{Hilb}(I) = \text{Hilb}(I') = \text{Hilb}(J)$, it follows from Proposition 2.15(ii) that JS^{po} and $\widetilde{\phi}((I')^{\text{po}})$ have the same Hilbert function; hence JS^{po} and $\widetilde{J}$ have the same Hilbert function. Thus $\widetilde{J} = JS^{\text{po}}$. Hence, by Proposition 2.15(i), we have

$$b_{i,j}(S/J) = b_{i,j}(S^{\text{po}}/J^{\text{po}}) \geq b_{i,j}(S^{\text{po}}/(I')S^{\text{po}}) = b_{i,j}(S/I') \geq b_{i,j}(S/I). \qquad \square$$

5. Shifted ideals

Throughout the rest of the paper, we fix two variables a and b, with a before b in the lex order. Furthermore ℓ ("large") and s ("small") will always be integers with $\ell \gneqq s \geq 0$, and f will be a monomial not divisible by either a or b.

We begin by generalizing "shifting" to arbitrary monomial ideals.

Definition 5.1. Let I be a monomial ideal. We say that I is (a,b)-*shifted* if, whenever $fa^sb^\ell \in I$, we have $fa^\ell b^s \in I$ as well. For an integer t, we say that I is (a,b,t)-*shifted* if, whenever $fa^sb^{\ell+t} \in I$, we have $fa^\ell b^{s+t} \in I$ as well. Finally, we say that I is (a,b)-*strongly shifted* if I is (a,b,t)-shifted for all nonnegative t.

Remark. Suppose that I is a squarefree ideal. Then I is (a,b)-shifted if and only if I is $\{a,b\}$-squarefree compressed (as defined in [Me2, MPS]), and shifted if and only if it is (a,b)-shifted for all a and b.

Definition 5.2. Let I be a monomial ideal. We define the (a,b)-*shift of* I as the $\Bbbk$-vector space

$$J = \mathrm{Shift}_{a,b}(I) = \left\langle \begin{array}{ccc} fa^sb^s & : & fa^sb^s \in I \\ fa^\ell b^s & : & fa^\ell b^s \in I \text{ or } fa^sb^\ell \in I \\ fa^sb^\ell & : & fa^\ell b^s \in I \text{ and } fa^sb^\ell \in I \end{array} \right\rangle$$

this basis taken over all f and all pairs (s,ℓ) with $s \lneqq \ell$.

For nonnegative integers t, we would like to define the t^{th} (a,b)-shift of I as $\mathrm{Shift}_{a,b,t}(I) = a^{-t}\,\mathrm{Shift}_{a,b}(a^tI)$, but it is not obvious *a priori* that this even makes sense. Instead, we define the t^{th} (a,b)-*shift of* I as the $\Bbbk$-vector space

$$J = \mathrm{Shift}_{a,b,t}(I) = \left\langle \begin{array}{ccc} fa^sb^r & : & fa^sb^r \in I, r \lneqq t \\ fa^sb^{s+t} & : & fa^sb^{s+t} \in I \\ fa^\ell b^{s+t} & : & fa^\ell b^{s+t} \in I \text{ or } fa^sb^{\ell+t} \in I \\ fa^sb^{\ell+t} & : & fa^\ell b^{s+t} \in I \text{ and } fa^sb^{\ell+t} \in I \end{array} \right\rangle,$$

this basis taken over all f, all $r \lneqq t$, and all pairs $s \lneqq \ell$. In Proposition 5.4, we will show that this is equivalent to the desired definition.

The shifting operation modifies the ideal I by replacing, wherever possible, monomials of the form fa^sb^ℓ with the (lexicographically bigger) $fa^\ell b^s$. Where this is impossible (because $fa^\ell b^s$ is already present), it instead does nothing. Note that $\mathrm{Shift}_{a,b}(I) = \mathrm{Shift}_{a,b,0}(I)$.

Proposition 5.3. *Let* $J = \mathrm{Shift}_{a,b,t}(I)$. *Then:*

(i) J *is an ideal.*
(ii) J *is* (a,b,t)*-shifted.*
(iii) J *has the same Hilbert function as* I.
(iv) J *is reverse lexicographically greater than or equal to* I.

Proof. (ii), (iii), and (iv) are immediate; we prove (i).

It suffices to show that, for any monomial $m \in J$, we have $ma \in J$, $mb \in J$, and $mx_i \in J$ for any $x_i \neq a, b$. We consider four cases, depending on the form of m.

Suppose first that $m = fa^sb^r$ with $r \lneq t$. Then $m \in I$, so we have $ma \in I \Rightarrow ma \in J$ and $mx_i \in I \Rightarrow mx_i \in J$. Also, $mb = fa^sb^{r+1} \in I$. If $r+1 \lneq t$ this implies $mb \in J$ immediately; if $r+1 = t$ we have $mb = fa^sb^{t+0}$, and $s \geq 0$ gives us $mb \in J$.

Now suppose that $m = fa^sb^{s+t}$. Then $mx_i \in I \Rightarrow mx_i \in J$. Furthermore, $fa^{s+1}b^{s+t} \in I$ and $fa^sb^{s+1+t} \in I$, so these must both be in J as well.

Thirdly, suppose that $m = fa^\ell b^{s+t}$. Then we have $fa^\ell b^{s+t}$ or $fa^sb^{\ell+t}$ in I. It follows that $mx_i \in J$ because $fx_ia^\ell b^{s+t} \in I$ or $fx_ia^sb^{\ell+t} \in I$, that $ma \in J$ because $fa^{\ell+1}b^{s+t} \in I$ or $fa^sb^{\ell+1+t} \in I$, and that $mb \in J$ because $fa^\ell b^{s+1+t} \in I$ or $fa^{s+1}b^{\ell+t} \in I$.

Finally, suppose that $m = fa^sb^{\ell+t}$. Then we have $fa^\ell b^{s+t}$ and $fa^sb^{\ell+t}$ in I. It follows that $mx_i \in J$ because $fx_ia^\ell b^{s+t} \in I$ and $fx_ia^sb^{\ell+t} \in I$, that $ma \in J$ because $fa^{\ell+1}b^{s+t} \in I$ and $fa^sb^{\ell+1+t} \in I$, and that $mb \in J$ because $fa^\ell b^{s+1+t} \in I$ and $fa^{s+1}b^{\ell+t} \in I$. ☐

Remark. For simplicity, let $t = 0$ (or make the appropriate changes for arbitrary t). We could attempt to define a "pseudograding" on S by setting pdeg $m = m$ for a monomial not of the form fa^sb^ℓ, and pdeg $fa^sb^\ell = fa^\ell b^s$. (This is not an actual grading because $S_mS_n \nsubseteq S_{mn}$.) In this pseudograding, S_m has dimension 1 or 2 for every pseudodegree m, and the lex ideals are precisely the shifted ideals. Proposition 5.3 states that every pseudo-Hilbert function is attained by a pseudo-lex ideal, i.e., Macaulay's theorem [Ma] holds in this setting. The next natural question is whether the theorem of Bigatti, Hulett, and Pardue [Bi, Hu, Pa] on Betti numbers holds as well. Corollaries 5.9 and 5.11 will show that it does.

Proposition 5.4. *Let $J = \mathrm{Shift}_{a,b,t}(I)$. Then $a^tJ = \mathrm{Shift}_{a,b,0}(a^tI)$.*

Proof. As vector spaces, we have

$$\begin{aligned}
a^tJ &= \left\langle \begin{array}{ccc} fa^{s+t}b^r & : & r \lneq t, fa^sb^r \in I \\ fa^{s+t}b^{s+t} & : & fa^sb^{s+t} \in I \\ fa^{\ell+t}b^{s+t} & : & fa^\ell b^{s+t} \in I \text{ or } fa^sb^{\ell+t} \in I \\ fa^{s+t}b^{\ell+t} & : & fa^\ell b^{s+t} \in I \text{ and } fa^sb^{\ell+t} \in I \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} fa^{s+t}b^r & : & r \lneq t, fa^{s+t}b^r \in a^tI \\ fa^{s+t}b^{s+t} & : & fa^{s+t}b^{s+t} \in a^tI \\ fa^{\ell+t}b^{s+t} & : & fa^{\ell+t}b^{s+t} \in a^tI \text{ or } fa^{s+t}b^{\ell+t} \in a^tI \\ fa^{s+t}b^{\ell+t} & : & fa^{\ell+t}b^{s+t} \in a^tI \text{ and } fa^{s+t}b^{\ell+t} \in a^tI \end{array} \right\rangle \\
&= \left\langle \begin{array}{ccc} fa^{s+t}b^r & : & r \lneq t, fa^{s+t}b^r \in a^tI \text{ or } fa^rb^{s+t} \in a^tI \\ fa^{s+t}b^{s+t} & : & fa^{s+t}b^{s+t} \in a^tI \\ fa^{\ell+t}b^{s+t} & : & fa^{\ell+t}b^{s+t} \in a^tI \text{ or } fa^{s+t}b^{\ell+t} \in a^tI \\ fa^{s+t}b^{\ell+t} & : & fa^{\ell+t}b^{s+t} \in a^tI \text{ and } fa^{s+t}b^{\ell+t} \in a^tI \end{array} \right\rangle \\
&= \mathrm{Shift}_{a,b,0}(a^tI).
\end{aligned}$$

☐

We now study the effect of shifting on Betti numbers. Our main result is the following:

Theorem 5.5. *Let $J = \mathrm{Shift}_{a,b,t}(I)$. Then for all i, j one has $b_{i,j}(J) \geq b_{i,j}(I)$.*

The proof involves several lemmas and sub-propositions. We begin by considering the case $t = 0$. Our argument follows Murai and Hibi's original proof of Theorem 4.8 [MH] very closely. In the case that I is squarefree, the arguments are identical.

Definition 5.6. Let $\sigma : S \to S$ be the $\mathbb{k}$-algebra involution defined by $\sigma(a) = b$, $\sigma(b) = a$, and $\sigma(x_i) = x_i$ for all $x_i \neq a, b$.

Since σ is an automorphism, it extends to resolutions, and we have, for example, $b_{i,j}(I) = b_{i,j}(\sigma(I))$ for all graded ideals I. In fact, σ acts naturally on the multigrading, so we have $b_{i,m}(I) = b_{i,\sigma(m)}(\sigma(I))$ for all monomial ideals. Note that σ fixes monomials of the form fa^sb^s, and partitions the other monomials into orbits of cardinality two, $\sigma(fa^\ell b^s) = fa^s b^\ell$.

Proposition 5.7. *Let $J = \mathrm{Shift}_{a,b}(I)$. Then we have $I \cap \sigma(I) = J \cap \sigma(J)$ and $I + \sigma(I) = J + \sigma(J)$.*

Proof. Observe that, for any integers p and q, we have $fa^pb^q \in I$ and $fa^qb^p \in I$ if and only if $fa^pb^q \in J$ and $fa^qb^p \in J$. It follows that $I \cap \sigma(I) = J \cap \sigma(J)$. Similarly, $fa^pb^q \in I$ or $fa^qb^p \in I$ if and only if $fa^pb^q \in J$ or $fa^qb^p \in J$. It follows that $I + \sigma(I) = J + \sigma(J)$. □

Lemma 5.8. *Let $J = \mathrm{Shift}_{a,b}(I)$, and let m be a monomial fixed by σ. Then $\mathrm{Shadow}_m(J) = \mathrm{Shift}_{a,b}(\mathrm{Shadow}_m(I))$.*

Proof. Write $m = fa^sb^s$, and write $f = g\sqrt{f}$ for some monomial g. For a squarefree monomial μ dividing $\sqrt{m}$, we have $\mu \in \mathrm{Shadow}_m(J)$ if and only if $\mu\frac{m}{\sqrt{m}} = \mu g a^{s-1}b^{s-1} \in J$, and similarly for I. We write $\mu = f'$, $\mu = f'a$, $\mu = f'b$, or $\mu = f'ab$ with f' not divisible by a or b.

We consider the case $\mu = f'a$ (the other three cases are similar). In this case, we have $\mu = f'a \in \mathrm{Shadow}_m(J)$ if and only if $f'ga^sb^{s-1} \in J$, if and only if $f'ga^sb^{s-1} \in I$ or $f'ga^{s-1}b^s \in I$, if and only if $f'a \in \mathrm{Shadow}_m(I)$ or $f'b \in \mathrm{Shadow}_m(I)$, if and only if $\mu = f'a \in \mathrm{Shift}_{a,b}(\mathrm{Shadow}_m(I))$. □

Theorem 4.8 lets us compare the multigraded Betti numbers of I and J, in multidegrees fixed by σ:

Corollary 5.9. *Let $J = \mathrm{Shift}_{a,b}(I)$, and let m be a multidegree fixed by σ. Then for all i, one has $b_{i,m}(J) \geq b_{i,m}(I)$.*

Proof. Let $I' = \mathrm{Shadow}_m(I)$ and $J' = \mathrm{Shadow}_m(J)$. By Lemma 5.8, we have $J' = \mathrm{Shift}_{a,b}(I')$. Since I' and J' are squarefree ideals of $\mathbb{k}[\mathrm{supp}(m)]$, their Betti numbers are concentrated in squarefree multidegrees. Thus, in particular, $b_{i,|\mathrm{supp}(m)|}(I') = b_{i,\sqrt{m}}(I')$ (and likewise for J') since $\sqrt{m}$ is the only squarefree monomial of degree $|\mathrm{supp}(m)|$ in this ring. By Theorem 4.8 we have $b_{i,|\mathrm{supp}(m)|}(J') \geq b_{i,|\mathrm{supp}(m)|}(I')$, and by Theorem 4.5 we have $b_{i,m}(I) = b_{i,\sqrt{m}}(I')$ and $b_{i,m}(J) = b_{i,\sqrt{m}}(J')$. Putting all this together, we get $b_{i,m}(J) \geq b_{i,m}(I)$ as desired. □

Now we consider multidegrees not fixed by σ. The Mayer-Vietoris sequence,

$$0 \to I \cap \sigma(I) \to I \bigoplus \sigma(I) \to I + \sigma(I) \to 0,$$

gives rise to a long exact sequence in Tor:

$$\cdots \to \operatorname{Tor}_i(\Bbbk, I \cap \sigma(I)) \to \operatorname{Tor}_i(\Bbbk, I) \bigoplus \operatorname{Tor}_i(\Bbbk, \sigma(I)) \to \\ \operatorname{Tor}_i(\Bbbk, I + \sigma(I)) \to \operatorname{Tor}_{i-1}(\Bbbk, I \cap \sigma(I)) \to \cdots$$

We truncate and restrict to multidegree m, producing the exact sequence of vector spaces:

$$0 \to (\ker \Delta_{i,I})_m \to \operatorname{Tor}_i(\Bbbk, I \cap \sigma(I))_m \xrightarrow{\Delta_{i,I}} \operatorname{Tor}_i(\Bbbk, I)_m \bigoplus \operatorname{Tor}_i(\Bbbk, \sigma(I))_m \\ \to \operatorname{Tor}_i(\Bbbk, I + \sigma(I))_m \to (\ker \Delta_{i-1,I})_m \to 0$$

Proposition 5.10. *Suppose that I is (a,b)-shifted and that $m \neq \sigma(m)$. Then $(\ker \Delta_{i,I})_m = 0$ for all i.*

Proof. Suppose m has the form $fa^s b^\ell$. (The case $m = fa^\ell b^s$ is symmetric.) Let $g \in (\ker \Delta_{i,I})_m$ be given, and write $g = [\sum \alpha_j \gamma_j e_{\mu_j}]$ for some $\alpha_j \in \Bbbk$, monomials $\gamma_j \in I \cap \sigma(I)$, and squarefree monomials μ_j of degree i such that $\gamma_j \mu_j = m$ for all j. (The term $\sum \alpha_j \gamma_j e_{\mu_j}$ is an element of $K_i \otimes (I \cap \sigma(I))$; the brackets denote its class modulo the boundary in the Koszul complex.) We will show that $g = 0$ in $\operatorname{Tor}_i(\Bbbk, I \cap \sigma(I))$.

We have $\Delta_{i,I}(g) = ([g],[g]) = (0,0)$ by assumption, so, in particular, $\sum \alpha_j \gamma_j e_{\mu_j}$ is a boundary in $K_i \otimes I$. Thus, we may write $\sum \alpha_j \gamma_j e_{\mu_j} = D(\sum \beta_j h_j e_{\nu_j})$, for some coefficients $\beta_j \in \Bbbk$, monomials $h_j \in I$ and, ν_j squarefree of degree $i+1$ with $h_j \nu_j = m$ for all j.

We claim that $h_j \in I \cap \sigma(I)$. Indeed, h_j has the form $f' a^{s-\varepsilon_a} b^{\ell-\varepsilon_b}$, where $\varepsilon_a = 0$ if a does not divide ν_j and 1 if it does, and likewise for ε_b. Since $\ell \gneq s$, we have $\ell - \varepsilon_b \geq s - \varepsilon_a$, so, since I is shifted, $f' a^{s-\varepsilon_a} b^{\ell-\varepsilon_b} \in I \Rightarrow f' a^{\ell-\varepsilon_b} b^{s-\varepsilon_a} \in I$. Thus, $h_j = \sigma(f' a^{\ell-\varepsilon_b} b^{s-\varepsilon_a}) \in \sigma(I)$ as claimed.

Hence, $\sum \beta_j h_j e_{\nu_j} \in K_{i+1} \otimes (I \cap \sigma(I))$, so we have $[g] = [D(\sum \beta_j h_j e_{\nu_j})] = 0$ in $\operatorname{Tor}_i(\Bbbk, I \cap \sigma(I))$. $\square$

Corollary 5.11. *Let $J = \operatorname{Shift}_{a,b}(I)$, and let m be a multidegree not fixed by σ. Then for all i, one has $b_{i,m}(J) + b_{i,\sigma(m)}(J) \geq b_{i,m}(I) + b_{i,\sigma(m)}(I)$.*

Proof. From the Mayer-Vietoris sequence, we have

$$\begin{aligned}
b_{i,m}(I) + b_{i,\sigma(m)}(I) &= b_{i,m}(I+\sigma(I)) + b_{i,m}(I \cap \sigma(I)) \\
&\quad - \dim_{\Bbbk}(\ker \Delta_{i,I})_m - \dim_{\Bbbk}(\ker \Delta_{i-1,I})_m \\
&= b_{i,m}(J+\sigma(J)) + b_{i,m}(J \cap \sigma(J)) \\
&\quad - \dim_{\Bbbk}(\ker \Delta_{i,I})_m - \dim_{\Bbbk}(\ker \Delta_{i-1,I})_m \\
&\leq b_{i,m}(J+\sigma(J)) + b_{i,m}(J \cap \sigma(J)) \\
&= b_{i,m}(J+\sigma(J)) + b_{i,m}(J \cap \sigma(J)) \\
&\quad - \dim_{\Bbbk}(\ker \Delta_{i,J})_m - \dim_{\Bbbk}(\ker \Delta_{i-1,J})_m \\
&= b_{i,m}(J) + b_{i,\sigma(m)}(J),
\end{aligned}$$

the second equality by Proposition 5.7, and the fourth by Proposition 5.10. □

Corollaries 5.9 and 5.11 combine to prove Theorem 5.5 in the case that $t = 0$:

Theorem 5.12. *Let $J = \mathrm{Shift}_{a,b}(I)$. Then for all i, j one has $b_{i,j}(J) \geq b_{i,j}(I)$.*

Proof. We have

$$\begin{aligned}
b_{i,j}(I) &= \sum_{\deg(m)=j} b_{i,m}(I) \\
&= \sum_{\substack{\deg(m)=j \\ m=fa^s b^s}} b_{i,m}(I) + \sum_{\substack{\deg(m)=j \\ m=fa^\ell b^s}} \big(b_{i,m}(I) + b_{i,\sigma(m)}(I)\big),
\end{aligned}$$

and similarly for J. By Corollary 5.9, the inequality holds for the first sum, and by Corollary 5.11, it holds for the second. □

The proof of Theorem 5.5 is now immediate.

Proof of Theorem 5.5. Let $J = \mathrm{Shift}_{a,b,t}(I)$. Then, applying Proposition 5.4, we have $b_{i,j}(J) = b_{i,j+t}(a^t J) = b_{i,j+t}(\mathrm{Shift}_{a,b}(a^t I)) \geq b_{i,j+t}(a^t I) = b_{i,j}(I)$. □

In fact, this argument, combined with the proof of Theorem 5.12, proves the sharper result:

Proposition 5.13. *Let $J = \mathrm{Shift}_{a,b,t}(I)$. Then, for all f, all $r < t$, and all $s < \ell$, one has:*

- $b_{i,fa^s b^r}(J) \geq b_{i,fa^s b^r}(I)$.
- $b_{i,fa^s b^{s+t}}(J) \geq b_{i,fa^s b^{s+t}}(I)$.
- $b_{i,fa^s b^{\ell+t}}(J) + b_{i,fa^\ell b^{s+t}}(J) \geq b_{i,fa^s b^{\ell+t}}(I) + b_{i,fa^\ell b^{s+t}}(I)$.

6. Shifted-plus-powers ideals

The ideal $P = (x_1^{e_1}, \cdots, x_n^{e_n})$ is (a,b)-shifted, and, furthermore, if I is any monomial ideal containing P, then $\mathrm{Shift}_{a,b}(I)$ contains P as well. Unfortunately, this statement fails for (a,b,t)-shifted ideals. The goal of this section is to fix this problem.

Let I be a monomial ideal containing P, and write $I = I' + P$. We will show that, for appropriate choices of I' (namely, "deleting" the pure power of b from a minimal generating set for I) and t, the t-shifted-plus-P ideal $J = \mathrm{Shift}_{a,b,t}(I') + P$ has the same Hilbert function as I and satisfies $b_{i,j}(J) \geq b_{i,j}(I)$.

Notation 6.1. Throughout this section, fix integers $\beta > 1$ and $t \geq 0$. We denote by I an (a,b,t)-shifted ideal with no minimal generators divisible by b^β, and set $J = \mathrm{Shift}_{a,b,t+1}(I)$. By abuse of notation, we will often write $I + b^\beta$ in place of $I + (b^\beta)$.

Our goal is to show that $J + b^\beta$ has the same Hilbert function as $I + b^\beta$, and larger graded Betti numbers.

We break down the graded Betti numbers of $I + b^\beta$ and $J + b^\beta$ into a sum of multigraded Betti numbers according to the following formula. For a monomial m of the form $m = fa^s b^{\ell+t+1}$, set $n = fa^\ell b^{s+t+1}$. Then

$$\begin{aligned} b_{i,j}(J+b^\beta) \; &= \sum_{\substack{m \neq fa^s b^{\ell+t+1} \\ m \neq fa^\ell b^{s+t+1}}} b_{i,m}(J+b^\beta) + \sum_{\substack{m = fa^s b^{\ell+t+1} \\ \ell+t+1 \neq \beta}} \left(b_{i,m}(J+b^\beta) + b_{i,n}(J+b^\beta) \right) \\ &\quad + \sum_{\substack{m = fa^s b^{\ell+t+1} \\ \ell+t+1 = \beta}} \left(b_{i,m}(J+b^\beta) + b_{i,n}(J+b^\beta) \right), \end{aligned} \tag{6.2}$$

and likewise for $I + b^\beta$, all sums taken over monomials m with $\deg m = j$. We will show that each of the summands in formula (6.2) for J is larger than or equal to the corresponding summand for I.

We begin with a technical lemma.

Lemma 6.3. *Suppose that f is a monomial not divisible by a or b, and that $\ell+t+1 \geq \beta$. If $fa^s b^{\ell+t+1} \in I$, then $fa^\ell b^{s+t+1} \in I$ as well.*

Proof. Since I has no minimal generators divisible by b^β, we have $fa^s b^{\ell+t} \in I$. Since I is (a,b,t)-shifted, it follows that $fa^\ell b^{s+t} \in I$, so $fa^\ell b^{s+t+1} \in I$ as well. $\square$

Corollary 6.4. *$I \cap (b^\beta) = J \cap (b^\beta)$ and $(I : b^\beta) = (J : b^\beta)$.*

Proof. Let m be any monomial divisible by b^β, and write m as $fa^s b^{\ell+t+1}$, $fa^\ell b^{s+t+1}$, $fa^s b^{s+t+1}$, or $fa^s b^r$ with $r \lneqq t+1$, as appropriate.

First, if $m = fa^s b^{\ell+t+1}$, then $m \in J$ if and only if $m \in I$ and $fa^\ell b^{s+t+1} \in I$, if and only if (by Lemma 6.3) $m \in I$. Similarly, if $m = fa^\ell b^{s+t+1}$, then $m \in J$ if and only if $m \in I$ or $fa^s b^{\ell+t+1} \in I$, if and only if (by Lemma 6.3) $m \in I$. Finally, if $m = fa^s b^{s+t+1}$ or $fa^s b^r$, then $m \in J$ if and only if $m \in I$.

Thus, $I \cap (b^\beta) = J \cap (b^\beta)$, so $(I : b^\beta) = (J : b^\beta)$ as desired. $\square$

This corollary has several important corollaries of its own.

Corollary 6.5. $\mathrm{Hilb}(I + b^\beta) = \mathrm{Hilb}(J + b^\beta)$.

Corollary 6.6. *None of the minimal monomial generators of J is divisible by $b^{\beta+1}$.*

From the short exact sequence

$$0 \to \frac{S}{(I : b^\beta)}(b^{-\beta}) \xrightarrow{b^\beta} \frac{S}{I} \to \frac{S}{I + b^\beta} \to 0$$

there arises a long exact sequence in Tor, (the "mapping cone",)

$$0 \to \mathrm{Im}(b^\beta_{*,i,I}) \to \mathrm{Tor}_i\left(\Bbbk, \frac{S}{I}\right) \to \mathrm{Tor}_i\left(\Bbbk, \frac{S}{I + b^\beta}\right) \to \\ \mathrm{Tor}_{i-1}\left(\Bbbk, \frac{S}{(I : b^\beta)}\right)(b^{-\beta}) \to \mathrm{Im}(b^\beta_{*,i-1,I}) \to 0,$$

and similarly for J

$$0 \to \mathrm{Im}(b^\beta_{*,i,J}) \to \mathrm{Tor}_i\left(\Bbbk, \frac{S}{J}\right) \to \mathrm{Tor}_i\left(\Bbbk, \frac{S}{J + b^\beta}\right) \to \\ \mathrm{Tor}_{i-1}\left(\Bbbk, \frac{S}{(J : b^\beta)}\right)(b^{-\beta}) \to \mathrm{Im}(b^\beta_{*,i-1,J}) \to 0.$$

The following proposition is immediate from mapping cone theory.

Proposition 6.7. $\mathrm{Im}(b^\beta_{*,i,I}) = 0$ *for all i, and* $(\mathrm{Im}(b^\beta_{*,i,J}))_m = 0$ *for all i and all multidegrees m not equal to $fa^s b^\beta$.*

Proof. Observe that $(I : b^\beta)$ has no minimal generators divisible by b. Thus, by the Taylor resolution (see e.g. [Ei1, Exercise 17.11]), its Betti numbers are concentrated in multidegrees not divisible by b, and so $\mathrm{Tor}_i(\Bbbk, S/(I : b^\beta))(b^{-\beta})$ is nonzero only in multidegrees of the form $fa^s b^\beta$. Furthermore, again by the Taylor resolution, the Betti numbers of S/I (and so the $\mathrm{Tor}_i(\Bbbk, S/I)$) are concentrated in multidegrees not divisible by b^β, and those of S/J are concentrated in multidegrees not divisible by $b^{\beta+1}$. As the maps $b^\beta_{*,i,I}$ and $b^\beta_{*,i,J}$ are multihomogeneous, the proposition follows. □

Lemma 6.8. *If $m = fa^s b^\beta$, with $s \geq \beta - t - 1$, then* $\mathrm{Shadow}_m(J + b^\beta) = \mathrm{Shadow}_m(I + b^\beta)$.

Proof. Let n be a monomial dividing m, and such that $\frac{m}{n}$ is squarefree. We will show that $n \in I + b^\beta$ if and only if $n \in J + b^\beta$, from which the lemma follows. If b^β divides n, then $n \in I + b^\beta$ and $n \in J + b^\beta$. Otherwise, write $n = f'a^s b^{\beta-1}$ (or, mutatis mutandis, $f'a^{s-1}b^{\beta-1}$). Then $s \geq (\beta - 1) - t - 1$, so $n \in J$ if $n \in I$. Conversely, if $n \in J$, we have $n \in I$ or $f'a^{\beta-t-2}b^{s+t+1} \in I$. In the latter case, $s + t \geq \beta - 1$, so by construction $f'a^{\beta-t-2}b^{\beta-1} \in I$ and so $f'a^s b^{\beta-1} \in I$, i.e., $n \in I$. □

The following are immediate:

Lemma 6.9. *If m is not divisible by b^β, then* $\mathrm{Shadow}_m(I+b^\beta) = \mathrm{Shadow}_m(I)$ *and* $\mathrm{Shadow}_m(J+b^\beta) = \mathrm{Shadow}_m(J)$.

Lemma 6.10. *If m is divisible by $b^{\beta+1}$, then* $\mathrm{Shadow}_m(I+b^\beta) = \mathrm{Shadow}_m(J+b^\beta) = (1)$.

Using these shadows to compute Betti numbers via Theorem 4.5, we obtain the following:

Lemma 6.11.

(1) *Suppose $m = fa^sb^{s+t+1}$ or fa^sb^r with $r < t+1$. Then we have $b_{i,m}(J+b^\beta) \geq b_{i,m}(I+b^\beta)$.*
(2) *Suppose $s \gneqq \ell$, and $\ell+t+1 \neq \beta$. Put $m = fa^\ell b^{s+t+1}$ and $n = fa^sb^{\ell+t+1}$. Then $b_{i,m}(J+b^\beta) + b_{i,n}(J+b^\beta) \geq b_{i,m}(I+b^\beta) + b_{i,n}(I+b^\beta)$.*

Proof.

(1) If the exponent on b is less than β, apply Lemma 6.9 and Proposition 5.13. If it is greater than β, apply Lemma 6.10. If the exponent is equal to β, apply Lemma 6.8.
(2) If $s+t+1 = \beta$, we have $b_{i,n}(I+b^\beta) = b_{i,n}(J+b^\beta)$ by Lemma 6.10 and $b_{i,m}(I+b^\beta) = b_{i,m}(J+b^\beta)$ by Lemma 6.8. If $s+t+1 \neq \beta$, then, applying the mapping cone and Proposition 6.7, the left-hand side is equal to $b_{i,m}(J) + b_{i,n}(J) + b_{i-1,b^{-\beta}m}(J:b^\beta) + b_{i-1,b^{-\beta}n}(J:b^\beta)$, while the right-hand side is equal to $b_{i,m}(I) + b_{i,n}(I) + b_{i-1,b^{-\beta}m}(I:b^\beta) + b_{i-1,b^{-\beta}n}(I:b^\beta)$. Apply Proposition 5.13 and Corollary 6.4. □

Thus, the first two sums in formula (6.2) are larger for $J+b^\beta$ than for $I+b^\beta$. It remains to consider the case that $m = fa^sb^{\ell+t+1}$, with $\ell+t+1 = \beta$. We fix $m = fa^sb^\beta$ with $\beta = \ell+t+1 \gneqq s+t+1$, multiply I by a^{t+1}, and recall the Mayer-Vietoris sequence from the previous section:

$$\begin{aligned}0 \to (\ker \Delta_{i,a^{t+1}I})_{a^{t+1}m} \to \mathrm{Tor}_i\left(\Bbbk, a^{t+1}I \cap \sigma(a^{t+1}I)\right)_{a^{t+1}m} &\xrightarrow{\Delta_{i,a^{t+1}I}} \\ \mathrm{Tor}_i\left(\Bbbk, a^{t+1}I\right)_{a^{t+1}m} \bigoplus \mathrm{Tor}_i\left(\Bbbk, \sigma(a^{t+1}I)\right)_{a^{t+1}m} &\to \\ \mathrm{Tor}_i\left(\Bbbk, a^{t+1}I + \sigma(a^{t+1}I)\right)_{a^{t+1}m} &\to (\ker \Delta_{i-1,a^{t+1}I})_{a^{t+1}m} \to 0.\end{aligned}$$

Lemma 6.12. $\mathrm{Shadow}_{a^{t+1}m}(a^{t+1}J) = \mathrm{Shadow}_{a^{t+1}m}(a^{t+1}I \cap \sigma(a^{t+1}I))$.

Proof. Let n be a monomial dividing $a^{t+1}m$, and such that $\frac{a^{t+1}m}{n}$ is squarefree. We will show that $n \in a^{t+1}J$ if and only if $n \in a^{t+1}I \cap \sigma(a^{t+1}I)$, from which the lemma follows. We may write $n = f'a^{s+t+1-\varepsilon_a}b^{\ell+t+1-\varepsilon_b}$ with $\varepsilon_a, \varepsilon_b = 0$ or 1 (so $s-\varepsilon_a \leq \ell-\varepsilon_b$). By definition $n \in a^{t+1}J$ if and only if $n \in a^{t+1}I$ and $f'a^{\ell+t+1-\varepsilon_b}b^{s+t+1-\varepsilon_a} \in a^{t+1}I$, if and only if $n \in a^{t+1}I$ and $n \in \sigma(a^{t+1}I)$. □

Corollary 6.13. $\mathrm{Tor}_i\left(\Bbbk, J\right)_m \cong \mathrm{Tor}_i\left(\Bbbk, a^{t+1}I \cap \sigma(a^{t+1}I)\right)(a^{t+1})_m$.

Proof. From the proof of Theorem 4.5, the complex $(\mathbb{K}_\bullet \otimes M)_m$ depends only on $\mathrm{Shadow}_m(M)$ for any monomial ideal M and multidegree m.

We have $(\mathbb{K}_\bullet \otimes J)_m \cong (\mathbb{K}_\bullet \otimes a^{t+1}J)(a^{t+1})_m = (\mathbb{K}_\bullet \otimes (a^{t+1}I \cap \sigma(a^{t+1}I)))(a^{t+1})_m$, the first isomorphism given by multiplication by a^{t+1}, the second equality by applying Lemma 6.12. This isomorphism of complices induces an isomorphism on Tor,

$$\phi_{i,m} : \mathrm{Tor}_i(\Bbbk, J)_m \to \mathrm{Tor}_i(\Bbbk, a^{t+1}I \cap \sigma(a^{t+1}I))(a^{t+1})_m$$

given by $\phi_{i,m}([g]) = [a^{t+1}g]$ for any cycle $[g] \in K_i \otimes J$. □

We view $\mathrm{Im}(b^\beta_{*,i,J})_m$ and $(\ker \Delta_{i-1,a^{t+1}I})(a^{t+1})_m$ as submodules of $\mathrm{Tor}_{i-1}(\Bbbk, J)$ (via the natural isomorphism with $\mathrm{Tor}_i(\Bbbk, S/J)$) and of $\mathrm{Tor}_{i-1}(\Bbbk, a^{t+1}I \cap \sigma(a^{t+1}I))(a^{t+1})_m$, respectively. The isomorphism $\phi_{i,m}$ allows us to compare these two vector spaces.

Proposition 6.14. $\phi_{i,m}(\mathrm{Im}(b^\beta_{*,i,J})_m) \subset (\ker \Delta_{i-1,a^{t+1}I})(a^{t+1})_m$.

Proof. An element of $\mathrm{Im}(b^\beta_{*,i,J})_m$ has the form $[b^\beta g]$, where g is a cycle in $K_{i-1} \otimes (J : b^\beta)$. (Consider e.g. the connecting homomorphism arising from the short exact sequence $0 \to J \to S \to S/J \to 0$.) We have $\phi_{i,m}([b^\beta g]) = [a^{t+1}b^\beta g]$.

To show that $[a^{t+1}b^\beta g] \in (\ker \Delta_{i-1,a^{t+1}I})(a^{t+1})_m$, it suffices to show that $a^{t+1}b^\beta g$ is a boundary in both $a^{t+1}I \otimes K_{i-1}$ and $\sigma(a^{t+1}I) \otimes K_{i-1}$. From the Taylor resolution of I, we know that $a^{t+1}I \otimes \mathbb{K}_\bullet$ is exact in multidegree $a^{t+1}m$. Thus, since $a^{t+1}b^\beta g$ is a cycle in $a^{t+1}I \otimes K_{i-1}$, it is a boundary as well. Hence, we may write $a^{t+1}b^\beta g = D(h)$ for some $h \in a^{t+1}I \otimes K_i$.

Write $h = a^{s+t}b^{\beta-1}e_a \wedge e_b \wedge f_1 + a^{s+t+1}b^{\beta-1}e_b \wedge f_2 + a^{s+t}b^\beta e_a \wedge f_3 + a^{s+t+1}b^\beta f_4$, for $f_1, f_2, f_3, f_4 \in \mathbb{K}_\bullet$ not involving a, b, e_a, or e_b. Then, write $a^{s+t+1}b^\beta f_4$ (and, mutatis mutandis, $a^{s+t}b^\beta e_a \wedge f_3$) in the form $\sum \alpha_j a^{s+t+1}b^\beta \gamma_j e_{\mu_j}$ for coefficients $\alpha_j \in \Bbbk$ and monomials γ_j with $a^{s+t+1}b^\beta\gamma_j \in a^{t+1}I$, and hence $a^s b^\beta \gamma_j \in I \cap (b^\beta) = J \cap (b^\beta)$. Adjusting $b^\beta g$ in $\mathrm{Im}(b^\beta)_{*,i,J}$ if necessary, we may assume that $f_3 = f_4 = 0$.

Thus,

$$\begin{aligned} a^{t+1}b^\beta g &= D(h) \\ &= a^{s+t+1}b^{\beta-1}e_b \wedge f_1 - a^{s+t}b^\beta e_a \wedge f_1 + a^{s+t}b^{\beta-1}e_a \wedge e_b \wedge D(f_1) \\ &\quad + a^{s+t+1}b^\beta f_2 - a^{s+t+1}b^{\beta-1}e_b \wedge D(f_2). \end{aligned}$$

Since the left-hand side of this expression is divisible by b^β, it follows that both $a^{s+t+1}b^{\beta-1}e_b \wedge f_1 - a^{s+t+1}b^{\beta-1}e_b \wedge D(f_2)$ and $a^{s+t}b^{\beta-1}e_a \wedge e_b \wedge D(f_1)$ are equal to zero, and, in particular, $f_1 = D(f_2)$ (and $D(f_1) = 0$). Thus,

$$\begin{aligned} a^{t+1}b^\beta g &= a^{s+t+1}b^\beta f_2 - a^{s+t}b^\beta e_a \wedge D(f_2) \\ &= D(a^{s+t}b^\beta e_a \wedge f_2). \end{aligned}$$

We claim that this is a boundary in $\sigma(a^{t+1}I) \otimes K_{i-1}$. Indeed, we may write f_2 in the form $\sum \alpha_j \gamma_j e_{\mu_j}$ with $a^{s+t+1}b^{\beta-1}\gamma_j \in a^{t+1}I$, i.e., $a^s b^{\beta-1}\gamma_j = a^s b^{\ell+t}\gamma_j \in I$. Then, since I is (a,b,t)-shifted, we have $a^\ell b^{s+t}\gamma_j \in I$, so $a^{\ell+t+1}b^{s+t}\gamma_j \in a^{t+1}I$ and $a^{s+t}b^{\ell+t+1}\gamma_j = a^{s+t}b^\beta\gamma_j \in \sigma(a^{t+1}I)$. Thus, $a^{s+t}b^\beta e_a \wedge f_2 \in \sigma(a^{t+1}I) \otimes K_i$ as desired. □

Corollary 6.15. *For $m = fa^s b^\beta = fa^s b^{\ell+t+1}$, set $n = fa^\ell b^{s+t+1}$. Then, for all i, one has $b_{i,m}(S/(J+b^\beta)) + b_{i,n}(S/(J+b^\beta)) \geq b_{i,m}(S/(I+b^\beta)) + b_{i,n}(S/(I+b^\beta))$.*

Proof. The computation below appears daunting, but it is in fact merely long. The moral is that, by Proposition 6.14, the flexibility in the paired multidegrees m and n (given by $(\ker \Delta_{\bullet,a^{t+1}I})_{a^{t+1}m}$) is larger than the obstruction coming from the cancellation in the mapping cone (given by $\operatorname{Im} b^\beta_{*,\bullet,J}$).

Set $A = b_{i,m}(S/(J+b^\beta)) + b_{i,n}(S/(J+b_\beta)) - b_{i,m}(S/(I+b^\beta)) - b_{i,n}(S/(I+b^\beta))$. We will show that A is nonnegative.

Expanding each term of A with the mapping cone, we have

$$
\begin{aligned}
A = &\left(b_{i,m}\left(\frac{S}{J}\right) + b_{i-1,b^{-\beta}m}\left(\frac{S}{(J:b^\beta)}\right) - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i}\right)_m - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i-1}\right)_m\right)\\
&+\left(b_{i,n}\left(\frac{S}{J}\right) + b_{i-1,b^{-\beta}n}\left(\frac{S}{(J:b^\beta)}\right) - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i}\right)_n - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i-1}\right)_n\right)\\
&-\left(b_{i,m}\left(\frac{S}{I}\right) + b_{i-1,b^{-\beta}m}\left(\frac{S}{(I:b^\beta)}\right) - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,I,i}\right)_m - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,I,i-1}\right)_m\right)\\
&-\left(b_{i,n}\left(\frac{S}{I}\right) + b_{i-1,b^{-\beta}n}\left(\frac{S}{(I:b^\beta)}\right) - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,I,i}\right)_n - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,I,i-1}\right)_n\right).
\end{aligned}
$$

By Proposition 6.7, most of these images are empty, and by Corollary 6.4, the Betti numbers of the colon ideals all cancel. We are left with

$$
\begin{aligned}
A = &\left(b_{i,m}\left(\frac{S}{J}\right) + b_{i,n}\left(\frac{S}{J}\right)\right) - \left(b_{i,m}\left(\frac{S}{I}\right) + b_{i,n}\left(\frac{S}{I}\right)\right)\\
&- \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i}\right)_m - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i-1}\right)_m.
\end{aligned}
$$

We multiply the ideals by a^{t+1} (replacing $b_{i,m}\left(\frac{S}{J}\right)$ with $b_{i,a^{t+1}m}\left(\frac{S}{a^{t+1}J}\right)$, etc.), and then expand again with the Mayer-Vietoris sequence, yielding

$$
\begin{aligned}
A = &\Bigg[b_{i,a^{t+1}m}\left(\frac{S}{a^{t+1}J\cap\sigma(a^{t+1}J)}\right) + b_{i,a^{t+1}m}\left(\frac{S}{a^{t+1}J+\sigma(a^{t+1}J)}\right)\\
&\qquad - \dim_{\Bbbk}(\ker \Delta_{i-1,a^{t+1}J})_{a^{t+1}m} - \dim_{\Bbbk}(\ker \Delta_{i-2,a^{t+1}J})_{a^{t+1}m}\Bigg]\\
&-\Bigg[b_{i,a^{t+1}m}\left(\frac{S}{a^{t+1}I\cap\sigma(a^{t+1}I)}\right) + b_{i,a^{t+1}m}\left(\frac{S}{a^{t+1}I+\sigma(a^{t+1}I)}\right)\\
&\qquad - \dim_{\Bbbk}(\ker \Delta_{i-1,a^{t+1}I})_{a^{t+1}m} - \dim_{\Bbbk}(\ker \Delta_{i-2,a^{t+1}I})_{a^{t+1}m}\Bigg]\\
&- \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i}\right)_m - \dim_{\Bbbk}\left(\operatorname{Im} b^\beta_{*,J,i-1}\right)_m.
\end{aligned}
$$

The remaining Betti numbers cancel by Propositions 5.4 and 5.7, and the first two kernels are empty by Proposition 5.10. We are left with

$$A = \left(\dim_{\Bbbk} (\ker \Delta_{i-1,a^{t+1}I})_{a^{t+1}m} - \dim_{\Bbbk} \left(\operatorname{Im} b^{\beta}_{*,J,i}\right)_m\right) + \left(\dim_{\Bbbk} (\ker \Delta_{i-2,a^{t+1}I})_{a^{t+1}m} - \dim_{\Bbbk} \left(\operatorname{Im} b^{\beta}_{*,J,i-1}\right)_m\right).$$

By Proposition 6.14, each of these summands is nonnegative. □

Proposition 6.16. *For all i, j, one has $b_{i,j}(J + b^\beta) \geq b_{i,j}(I + b^\beta)$.*

Proof. For a monomial m of the form $m = fa^s b^{\ell+t+1}$, set $n = fa^\ell b^{s+t+1}$. We recall formula (6.2),

$$b_{i,j}(J+b^\beta) = \sum_{\substack{m \neq fa^s b^{\ell+t+1} \\ m \neq fa^\ell b^{s+t+1}}} b_{i,m}(J+b^\beta) + \sum_{\substack{m = fa^s b^{\ell+t+1} \\ \ell+t+1 \neq \beta}} \left(b_{i,m}(J+b^\beta) + b_{i,n}(J+b^\beta)\right) + \sum_{\substack{m = fa^s b^{\ell+t+1} \\ \ell+t+1 = \beta}} \left(b_{i,m}(J+b^\beta) + b_{i,n}(J+b^\beta)\right),$$

and similarly for I. By Lemma 6.11, the inequality holds for the first two sums, and by Corollary 6.15, it holds for the third. □

7. Strongly shifted ideals and compression

Let J be an (a, b)-strongly shifted ideal, none of whose generators is divisible by b^β, and suppose further that J contains a^α for some $\alpha \leq \beta$. Let T be the $\{a, b\}$-compression of J. We study the Betti numbers of J and T. We continue to denote by f a monomial not divisible by a or b.

The following observations are immediate:

Lemma 7.1.

(i) *If $T \neq J$, then T is reverse lexicographically greater than J.*
(ii) *$fa^r b^s \in T$ if and only if J contains at least $s + 1$ monomials of the form $fa^p b^q$ with $p + q = r + s$.*

Lemma 7.2. *The following are equivalent:*

(i) *$fa^r b^\beta \in J$.*
(ii) *$fa^r b^{\beta-1} \in J$.*
(iii) *$fa^p b^q \in J$ for all p, q such that $p + q = r + \beta - 1$ and $q \lneq \beta$.*
(iv) *$fa^r b^{\beta-1} \in T$.*
(v) *$fa^r b^\beta \in T$.*

Proof. (iii) $\Longrightarrow$ (ii) $\Longrightarrow$ (i) is obvious, as is (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v), and (i) $\Longrightarrow$ (ii) is immediate by construction. We will show that (ii) implies (iii) and (v) implies (i).

Suppose (ii) holds. If $p \geq \alpha$, then $fa^p b^q \in J$ because $a^\alpha \in J$. Otherwise, note that $p \geq r$, since $p - r = \beta - 1 - q \geq 0$ by assumption. Set $t = \beta - 1 - p$, which is nonnegative since $\beta - 1 \geq \alpha - 1 \geq p$. Since J is (a,b,t)-shifted and $fa^r b^{\beta-1} = fa^r b^{t+p} \in J$, we have $fa^p b^{t+r} = fa^p b^q \in J$. Thus (iii) holds.

Now suppose (v) holds. Then, by Lemma 7.1 (ii), J contains at least $\beta+1$ monomials of the form $fa^p b^q$, with $p+q = \beta + r$. By the pigeonhole principle, one of these, say $fa^P b^Q$, has $Q \geq \beta$ and $P \leq r$. By construction, then, J contains $fa^P b^\beta$ and so we have $fa^r b^\beta \in J$, and (i) is satisfied. ☐

Corollary 7.3. *No minimal monomial generator of T is divisible by b^β.*

Proof. Suppose that T contains a monomial m of the form $fa^r b^d$ with $d \geq \beta$. No minimal monomial generator of J is divisible by b^d, so Lemma 7.2 applies with d in place of β, and we have $fa^r b^{d-1} \in T$. Thus, m is not a minimal generator of T. ☐

Corollary 7.4. *We have $T \cap (b^\beta) = J \cap (b^\beta)$ and $(T : b^\beta) = (J : b^\beta)$.*

Proof. For any $q \geq \beta$, one has $fa^p b^q \in J$ if and only if $fa^p b^{\beta-1} \in J$, if and only if $fa^p b^{\beta-1} \in T$, if and only if $fa^p b^q \in T$. ☐

Proposition 7.5. *$b_{i,j}(T + b^\beta) \geq b_{i,j}(J + b^\beta)$ for all i,j.*

Proof. Both $T + b^\beta$ and $J + b^\beta$ are resolved by the mapping cone of b^β, via the short exact sequences

$$0 \to S/(J : b^\beta)(b^{-\beta}) \to S/J \to S/(J + b^\beta) \to 0$$

and

$$0 \to S/(T : b^\beta)(b^{-\beta}) \to S/T \to S/(T + b^\beta) \to 0.$$

By construction (for J) and Corollary 7.3 (for T), neither J nor T has any minimal generators divisible by b^β, so, by the Taylor resolution, their multigraded Betti numbers are concentrated in multidegrees not divisible by b^β. Thus, there is no cancellation in either mapping cone, and we have

$$\begin{aligned} b_{i,j}(S/(J + b^\beta)) &= b_{i,j}(S/J) + b_{i-1,j-\beta}(S/(J : b^\beta)) \\ b_{i,j}(S/(T + b^\beta)) &= b_{i,j}(S/T) + b_{i-1,j-\beta}(S/(T : b^\beta)). \end{aligned}$$

By Theorem 2.13, we have $b_{i,j}(S/T) \geq b_{i,j}(S/J)$, and by Corollary 7.4, $(J : b^\beta) = (T : b^\beta)$. ☐

8. The monomial case of the lex-plus-powers conjecture

In this section, we put everything together to prove the monomial case of the lex-plus-powers conjecture in arbitrary characteristic:

Theorem 8.1. *Suppose I is a homogeneous ideal containing a regular sequence of monomials $(f_1, \cdots, f_r)$ in degrees $e_1 \leq \cdots \leq e_r$. Put $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. Then there exists a lex-plus-P ideal L such that L has the same Hilbert function as I and $b_{i,j}(L) \geq b_{i,j}(I)$ for all i,j.*

Throughout the section, $e_1 \leq \cdots \leq e_n \leq \infty$. For variables $a = x_i, b = x_j$, set $e_a = e_i$ and $e_b = e_j$. For ideals containing $P = (x_1^{e_1}, \cdots, x_n^{e_n})$, we will frequently want to consider the ideal obtained by "deleting" b^{e_b} from a generating set:

Notation 8.2. Let a monomial ideal $I \supset P$ and variables a, b be given. We set I' equal to the ideal generated by all the minimal monomial generators of I except for b^{e_b}. (If b^{e_b} is not a minimal monomial generator of I, set $I' = I$.)

Lemma 8.3. *Let a and b be given. Then I is (a, b, t)-shifted-plus-P if and only if I' is (a, b, t)-shifted, and I is $\{a, b\}$-compressed-plus-P if and only if I' is $\{a, b\}$-compressed.*

Proof. Suppose that I is $\{a, b\}$-compressed-plus-P. (The proof for (a, b, t)-shifted is similar.) Then there exists an $\{a, b\}$-compressed ideal $\hat{I}$ such that $I = \hat{I} + P$. We will show that I' is $\{a, b\}$-compressed.

Fix a monomial f not divisible by a or b, and suppose that $u = fa^p b^q$ and $v = fa^r b^s$ are monomials of the same degree such that $v >_{\text{lex}} u$ (i.e., $r > p$) and $u \in I'$. We need to show that $v \in I'$ as well. If $p \geq e_a$, we have $v \in I'$ since $a^{e_a} \in I'$. Otherwise, u is divisible by some minimal generator w of I'; write $w = f' a^{p'} b^{q'}$ with $q' < e_b$. If $q' < s$ or $w = x_k^{e_k}$, we have $v \in I'$ immediately, otherwise $w \in \hat{I}$ since $w \in I \smallsetminus P$. Since $\hat{I}$ is $\{a, b\}$-compressed, it follows that $f' a^{p'+q'-s} b^s \in \hat{I}$. Since this is in I but is not divisible by b^{e_b}, it is in I', so we have $v \in I'$ as desired. $\square$

Proposition 8.4. *Let I be a monomial ideal which is (a, b, t)-shifted-plus-P. Then there exists an $(a, b, t+1)$-shifted-plus-P ideal J which has the same Hilbert function as I, is reverse lexicographically greater than or equal to I, and satisfies $b_{i,j}(J) \geq b_{i,j}(I)$.*

Proof. Set $J' = \text{Shift}_{a,b,t+1}(I')$ and $J = J' + P$. We have $I = I' + b^{e_b}$ and $J = J' + b^{e_b}$, so, by Corollary 6.5, I and J have the same Hilbert function, by Proposition 5.3, J is reverse lexicographically greater than or equal to I, and, by Proposition 6.16, $b_{i,j}(J) \geq b_{i,j}(I)$. $\square$

Proposition 8.5. *Let I be a monomial ideal containing P, and fix a and b. Then there exists an (a, b)-strongly shifted-plus-P ideal J which is reverse lexicographically greater than or equal to I, has the same Hilbert function as I, and satisfies $b_{i,j}(J) \geq b_{i,j}(I)$ for all i, j.*

Proof. Clearly, $\text{Shift}_{a,b}(I)$ contains P. Thus, replacing I with $\text{Shift}_{a,b}(I)$ if necessary (and applying Proposition 5.3 and Theorem 5.12,) we may assume that I is (a, b)-shifted. If I is not already (a, b)-strongly shifted-plus-P, there exist integers $t \gneqq 0$ such that I is not (a, b, t)-shifted-plus-P. Choose the smallest such t. Then by Proposition 8.4 there exists an (a, b, t)-shifted-plus-P ideal with the same Hilbert function as I and larger graded Betti numbers. Replace I with this new ideal and repeat. This process must terminate, since there are only finitely many monomial ideals with the same Hilbert function, and at each step we replace the ideal with a reverse lexicographically greater one. Let J be the resulting ideal. $\square$

Proposition 8.6. *Let I be (a, b)-strongly-shifted-plus-P. Then there exists an $\{a, b\}$-compressed-plus-P ideal T which is reverse lexicographically greater than or equal to I, has the same Hilbert function as I, and satisfies $b_{i,j}(T) \geq b_{i,j}(I)$.*

Proof. Let T' be the $\{a,b\}$-compression of I', and put $T = T' + P$. We have $I = I' + b^{e_b}$ and $T = T' + b^{e_b}$, so, by Corollary 7.4, I and T have the same Hilbert function, and, by Proposition 7.5, $b_{i,j}(T) \geq b_{i,j}(I)$. □

Proposition 8.7. *Let I be a monomial ideal containing P. Then there exists a Borel-plus-P ideal B such that B has the same Hilbert function as I, and $b_{i,j}(B) \geq b_{i,j}(I)$ for all i, j.*

Proof. If I is not already Borel-plus-P, there exist pairs of variables a, b such that I is not $\{a,b\}$-compressed-plus-P. Choose any such pair. By Propositions 8.5 and 8.6, there exists an $\{a,b\}$-compressed-plus-P ideal T with the same Hilbert function as I and larger Betti numbers. Replace I with T and repeat. This process must terminate because there are only finitely many monomial ideals with the same Hilbert function, and at each step we are replacing the ideal with a reverse lexicographically greater one. Let B be the resulting ideal. □

Theorem 1.4 completes the proof of Theorem 8.1.

Proof of Theorem 8.1. By Lemma 3.2, we may assume without loss of generality that $(f_1, \cdots, f_r) = P$. By Proposition 8.7, we may assume that I is Borel-plus-P. Thus, the desired inequality holds by Theorem 1.4. □

9. Consecutive Cancellation

A *consecutive cancellation* in the graded Betti numbers of a module M is the simultaneous subtraction of 1 from consecutive Betti numbers in the same internal degree, i.e., replacing $b_{i,j}(M)$ and $b_{i-1,j}(M)$ with $(b_{i,j}(M) - 1)$ and $(b_{i-1,j}(M) - 1)$.

We say that the graded Betti numbers of an ideal I are obtained from those of L by consecutive cancellations if we can perform a sequence of consecutive cancellations on the $b_{i,j}(L)$ to produce the Betti numbers of I. Heuristically, this happens because the minimal resolution of L "deforms" into a (non-minimal) resolution of I, which can be decomposed into a direct sum of the minimal resolution of I and some trivial complices $0 \to S \to S \to 0$; the cancellations are in the degrees of these trivial complices. We define this more formally as follows:

Definition 9.1. Let L and I be two homogeneous ideals. We say that the graded Betti numbers of I are *obtained from those of L by consecutive cancellations* if there exist nonnegative integers $c_{i,j}$ such that, for all i and j, we have $b_{i,j}(I) = b_{i,j}(L) - c_{i,j} - c_{i-1,j}$.

Peeva shows in [Pe] that, if L is the lex ideal with the same Hilbert function as I, the graded Betti numbers of I are obtained from those of L by consecutive cancellations; similar results are known (often with the same proof) in many settings where the lex ideals attain all Hilbert functions.

We will show:

Theorem 9.2. *Let I, P, and L be as in the statement of Theorem 8.1. Then the graded Betti numbers of I are obtained from those of L by consecutive cancellations.*

Proof. The proof of Theorem 8.1 consists of a series of compressions, shifts, and t-shifts-plus-P, followed by a jump from Borel-plus-P to lex-plus-P. We will show that at each step the graded Betti numbers are obtained by consecutive cancellations. Thus, what we must show is that the graded Betti numbers of I are obtained from those of J (or T) in Theorem 1.4, Lemma 3.2, and Propositions 8.5 and 8.6.

Murai shows in [Mu, Theorem 5.1] that the Betti numbers of a Borel-plus-P ideal are obtained from those of the lex-plus-P ideal by consecutive cancellations. Since Lemma 3.2 and Proposition 8.6 use only coordinate changes, initial ideals, and compressions, the statement follows from [Pe] and [Me2, Theorem 5.10] in these cases. Lemma 9.3 below completes the proof. □

Lemma 9.3. *Let I be a monomial ideal.*

(1) *Set $J = \mathrm{Shift}_{a,b}(I)$. Then the graded Betti numbers of I are obtained from those of J by consecutive cancellations.*
(2) *Suppose that I is (a,b,t)-shifted and has no generators divisible by b^β. Set $J = \mathrm{Shift}_{a,b,t}(I)$. Then the graded Betti numbers of $I + b^\beta$ are obtained from those of $J + b^\beta$ by consecutive cancellations.*

Proof.

(1) Let m be a multidegree. If m has the form $fa^s b^s$, we have $b_{i,m}(I) = b_{i,|\operatorname{supp}(m)|}(\mathrm{Shadow}_m(I))$, and likewise for J. By Lemma 5.8, we can compare these Betti numbers with Theorem 4.8. Applying Peeva's proof [Pe] to our proof of Theorem 4.8, the graded Betti numbers of $\mathrm{Shadow}_m(I)$ and $\mathrm{Shadow}_m(J)$ differ by consecutive cancellations. Thus, there exist integers $c_{i,m}$ such that $b_{i,m}(J) - b_{i,m}(I) = c_{i,m} + c_{i-1,m}$.

If m has the form $fa^\ell b^s$, put $n = fa^s b^\ell$. Then by the Mayer-Vietoris sequence and Proposition 5.10, we have

$$b_{i,m}(J) + b_{i,n}(J) = b_{i,m}(I) + b_{i,n}(I) + \dim_{\Bbbk}(\ker \Delta_{i,I})_m + \dim_{\Bbbk}(\ker \Delta_{i-1,I})_m.$$

Set $c_{i,m} = \dim_{\Bbbk}(\ker \Delta_{i,I})_m$.

Finally, we put

$$c_{i,j} = \sum_{\substack{m=\sigma(m)\\ \deg m = j}} c_{i,m} + \sum_{\substack{m=fa^\ell b^s\\ \deg m=j}} c_{i,m}.$$

The statement follows from the formula in the proof of Theorem 5.12.

(2) The statement follows from the proof of Proposition 6.16 in the same way that (1) follows from the proof of Theorem 5.12. Let m be a multidegree. If $m = fa^\ell b^{s+t+1}$ with $\ell + t + 1 = \beta$, put $c_{i,m} = \dim_{\Bbbk}(\ker \Delta_{i,a^{t+1}I'})_{a^{t+1}m} - \dim_{\Bbbk}(\operatorname{Im} b^\beta_{*,i-1,J})_m$. Otherwise, define $c_{i,m}$ as in the proof of (1) (making the obvious changes).

Again, we put

$$c_{i,j} = \sum_{\substack{m=fa^s b^{s+t+1}\\ \deg m=j}} c_{i,m} + \sum_{\substack{m=fa^\ell b^{s+t+1}\\ \deg m=j}} c_{i,m}.$$

□

10. Open problems

We recall some related problems, and make some brief remarks about them.

10.1. **The general Lex-Plus-Powers Conjecture.** In Evans' original conjecture, the regular sequence was not required to consist of monomials:

Conjecture 10.1 (Evans, The Lex-Plus-Powers Conjecture)**.** *Suppose that $F = (f_1, \cdots, f_r)$ is any regular sequence with $\deg f_i = e_i$, and define $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. Let I be any homogeneous ideal containing F. Then there exists a lex-plus-P ideal L such that I and L have the same Hilbert function. Furthermore, $b_{i,j}(L) \geq b_{i,j}(I)$ for all i, j.*

A few special cases and reductions are known, due to Francisco, Richert, and Sabourin [Fr, FR, Ri, RS], but the conjecture appears to be wide open. Indeed, the mere existence of the lex-plus-P ideal L is far from certain; this is the Eisenbud-Green-Harris conjecture [EGH1, EGH2]. Some special cases are due to Caviglia and Maclagan, Cooper, and Richert [CM, Co1, Co2, FR]. A good survey article on both conjectures is [FR].

The problem for both conjectures is that the usual first step in proving Macaulay-type theorems is to take an initial ideal, but doing so in this setting ruins the regular sequence. Without a monomial ideal, most of our other techniques are useless. Unfortunately, Theorem 8.1 does nothing to resolve this. It does, however, reduce both conjectures to the same obstacle. The following statement is equivalent to the Lex-Plus-Powers conjecture (and, without the last sentence, has been known for some time to imply the Eisenbud-Green-Harris conjecture):

Conjecture 10.2. *Let $(f_1, \cdots, f_r)$ be a regular sequence with $deg f_i = e_i$, and let $P = (x_1^{e_1}, \cdots, x_r^{e_r})$. Then there exists a monomial ideal M containing P such that I and M have the same Hilbert function. Furthermore, M may be chosen so that $b_{i,j}(M) \geq b_{i,j}(I)$ for all i, j.*

Conjecture 10.2 holds if all but one of the f_i are pure powers. (For example, if $(f_1, \cdots, f_{r-1}) = (x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}})$, then the monomial support of f_r must contain some term not divisible by any of $x_1, \cdots, x_{r-1}$. Take an initial ideal in some appropriate order, and if necessary apply Lemma 3.2.) Thus, the Lex-Plus-Powers Conjecture holds for these regular sequences as well:

Proposition 10.3. *Let F, I, and L be as in Conjecture 10.1, and suppose further that all but one of the f_i is a pure power. Then there exists a lex-plus-P ideal L with the same Hilbert function as I, and, for all i and j, we have $b_{i,j}(L) \geq b_{i,j}(I)$.*

In [CM], Caviglia and Maclagan prove that the Eisenbud-Green-Harris Conjecture holds whenever $e_k > \sum_{\ell=1}^{k-1} e_\ell$ for all k. In light of this result, we consider the following question potentially tractable:

Problem 10.4. *Suppose that, for all k, $e_k > \sum_{\ell=1}^{k-1} e_\ell$. Does the Lex-Plus-Powers Conjecture hold for ideals containing a regular sequence in degrees $(e_1, \cdots, e_r)$?*

10.2. **Betti numbers over S/P.** Gasharov, Hibi, and Peeva make the following conjecture [GHP]:

Conjecture 10.5. *Let I be a homogeneous ideal containing P, and let L be the lex-plus-P ideal with the same Hilbert function. Let $\bar{I}$ and $\bar{L}$ be their images in $R = S/P$. Then the graded Betti numbers of $\bar{I}$ and $\bar{L}$ satisfy $b^R_{i,j}(\bar{L}) \geq b^R_{i,j}(\bar{I})$ for all i and j.*

This conjecture deals with infinite resolutions. Nevertheless, our techniques may give some indication of how to proceed. For example, after replacing the Koszul complex with a resolution of $\Bbbk$ over R, an analog of Corollary 5.11 continues to hold. It is less clear how the rest of the argument might translate, however.

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