RESEARCH SUMMARY and PLANS

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1 Introduction

My research is in the field of Commutative Algebra, and has involved Hilbert functions, Betti numbers, and monomial ideals, especially lex ideals.

A well-studied and important numerical invariant of a homogeneous ideal over a graded polynomial ring is its Hilbert function. It measures the sizes of the graded components of the ideal. A case of particular importance is the Hilbert function of the vanishing ideal of a projective algebraic variety X; this function gives the dimensions of the spaces P_d of forms of degree d vanishing on X. Hilbert's motivation for studying Hilbert functions came from another source: Invariant Theory. For many years, Hilbert functions have been both central objects and fruitful tools in many fields, including Algebraic Geometry, Combinatorics, Commutative Algebra, and Computational Algebra.

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k graded by $\deg(x_i) = 1$ for all i. If $J \subset S$ is a homogeneous ideal, the *Hilbert function* of J is given by $\mathrm{Hilb}_J(d) = \dim_k(J_d)$, where J_d denotes the degree-d part of J. It measures the size of the ideal and encodes a lot of important information. Hilbert's insight was that it is determined by finitely many of its values. He proved that there exists a polynomial $h_J(t) \in \mathbb{Q}[t]$ such that $\mathrm{Hilb}_J(d) = h_J(d)$ for $d \gg 0$. Two major applications of Hilbert functions in Algebraic Geometry are the celebrated Riemann-Roch Formula (proved using a Hilbert polynomial) and Chern classes.

Algorithms for the computation of Hilbert functions are implemented in computer algebra systems such as MACAULAY, MACAULAY2, and COCOA. Hilbert functions are used in some algorithms to speed up computation or to compute other invariants.

Gröbner Basis Theory (from Computational Algebra) reduces many questions on properties of Hilbert functions to properties of Hilbert functions of ideals generated by monomials. This makes it possible to use combinatorial arguments.

What are the possible Hilbert functions of ideals in S? Macaulay showed [Ma] in 1927 that every Hilbert function is attained by a lex ideal (defined below).

Definition 1. Let L be an ideal in S minimally generated by monomials $l_1, \dots, l_r$. We say that L is *lex* if the following property is satisfied: if m is a monomial that is greater lexicographically than l_i and $\deg(m) = \deg(l_i)$ for some $1 \le i \le r$, then $m \in L$.

Lex ideals are highly structured: they are defined combinatorially and their Hilbert functions are easy to describe. Thus, Macaulay's theorem yields a characterization of all possible Hilbert functions of homogeneous ideals in S. The theorem also plays an important role in the study of homogeneous ideals; for example,

- Hartshorne's proof that the Hilbert scheme is connected [Ha] uses lex ideals in a fundamental way.
- The homological properties of lex ideals are combinatorially tractable [EK]. This leads to results by Bigatti [Bi], Hulett [Hu], Pardue [Pa], showing that lex ideals have extremal Betti numbers.

Other important numerical invariants of a homogeneous ideal J in S are its Betti numbers. A *free resolution* of S/J is an exact sequence

$$\mathbb{F} : \cdots \to F_2 \to F_1 \to F_0 \to S/J \to 0$$

with each F_i a free module. Free resolutions were introduced by Hilbert, and have been widely studied since they encode a lot of information about the ideal. $\mathbb{F}$ is *minimal* if each free module F_p has minimum possible rank (among all free resolutions); there exists a unique minimal free resolution up to an isomorphism. The *Betti numbers* of S/J are the ranks of the free modules in a minimal free resolution, $b_p(S/J) = \text{rk } F_p$.

We can grade each F_i so that all the maps in $\mathbb{F}$ are homogeneous of degree 0. If we do so, we can write $F_p = \oplus_s F_{p,s}$, where $F_{p,s}$ is generated in degree s. Then the *graded Betti numbers* of S/J are the ranks of these modules, $b_{p,s}(S/J) = \text{rk } F_{p,s}$.

2 Research Summary and Plans

2.1 Lexifying ideals

Macaulay's Theorem states that every Hilbert function in the ring S is attained by a lex ideal. One of the problems that I am interested in is:

Problem 2.1.1. *In what other rings does Macaulay's Theorem hold?*

Kruskal [Kr] and Katona [Ka] showed that Macaulay's theorem holds in the squarefree ring $R = S/(x_1^2, \cdots, x_n^2)$. This result is of great importance to the field of Combinatorics because it classifies the possible f-vectors of simplicial complexes. (Every simplicial complex Δ on n vertices is associated, under the Stanley-Reisner correspondence), to a monomial ideal I_Δ of R. The Hilbert function of R/I_Δ is the f-vector of Δ.)

Clements and Lindstrom [CL] extended Kruskal and Katona's result to the ring $S/(x_1^{e_1}, \cdots, x_r^{e_r})$ for any sequence of positive integers $e_1 \leq \cdots \leq e_r$.

Shakin [Sh] characterized the Borel-fixed ideals B such that Macaulay's Theorem holds in the quotient S/B. (Borel-fixed ideals are much-studied monomial ideals because they arise as generic initial ideals. They may be defined entirely combinatorially and are one of the largest classes of ideals whose minimal free resolutions are known [EK].)

In [MeP1,MeP2], [Me1], and [MM2], we have studied rings of the form S/M, with M a monomial ideal, and shown the following:

Theorem 2.1.2. [MeP1]

- *If Macaulay's Theorem holds in S/M, and L is a lex ideal of S/M, then Macaulay's Theorem holds in $S/M + L$.*
- *If Macaulay's Theorem holds in S/M, then it also holds in $S[y]/M$.*

Theorem 2.1.3. [Me1] *If M is generated by a regular sequence, then Macaualay's Theorem holds in S/M if and only if M has the form $(x_1^{e_1}, \cdots, x_{r-1}^{e_{r-1}}, x_r^{e_r - 1}y)$, where $e_1 \leq \cdots \leq e_r$ is an increasing sequence and $y \in \{x_r, \cdots, x_n\}$.*

Theorem 2.1.4. [MeP1] *Let $P = (x_1^{e_1}, \cdots, x_r^{e_r})$ with $e_1 \leq \cdots \leq e_r$, and K a compressed ideal of S/P,* *generated in degree d. If J is any homogeneous ideal of $S/(K+P)$, there exists a lex ideal $L \subset S/(K+P)$ such that the Hilbert functions of L and J agree in all degrees greater than or equal to d.*

Theorem 2.1.5. **[MM2]** *Let $\{x_1,\dots,x_n\} = V_1 \coprod \cdots \coprod V_r$, such that, whenever $a < b < d$ with $x_a \in V_i$ and $x_b, x_d \in V_j$, there exists $x_c \in V_i$ with $b < c < d$. Put $Q_i = (x_j : x_j \in Q_i)$ and $P = \sum Q_i^2$. Then Macaulay's theorem holds in S/P.*

The monomial ideals of the ring S/P in Theorem 2.1.5 correspond under the Stanley-Reisner correspondence to r-colored simplicial complexes, i.e., complexes on $\{x_1,\dots,x_n\}$ such that no face contains more than one vertex from any of the V_i. Thus, Theorem 2.1.5 characterizes the f-vectors of r-colored complexes with a fixed coloring. It generalizes a theorem of Frankl, Furedi, and Kalai [FFK], which characterized the f-vectors of r-colorable complexes.

I plan to continue my work on Problem 2.1.1.; for example, I will consider toric varieties. These varieties, which come equipped with a torus action, are of considerable importance in Algebraic Geometry, Commutative Algebra, and Combinatorics. They correspond to quotients of S by certain binomial ideals; these are highly structured (for example, they come with a natural multigrading that refines the grading by degree) and seem likely candidates to have an analog of Macaulay's theorem.

2.2 The lex-plus-powers conjecture

Bigatti [Bi], Hulett [Hu], and Pardue [Pa] showed that the lex ideals have maximal Betti numbers in S; that is, if L is the lex ideal having the same Hilbert function as J, $b_{p,s}(S/L) \geq b_{p,s}(S/J)$ for all p, s. Aramova, Herzog, and Hibi [AHH] proved the analogous result in the squarefree ring $S/(x_1^2,\cdots,x_n^2)$. In view of these results and of a geometrically motivated conjecture of Eisenbud, Green, and Harris [EGH1, EGH2], Graham Evans [FR] made the lex-plus-powers conjecture:

The Lex-plus-powers Conjecture 2.2.1. *Let J be a homogeneous ideal of S containing a regular sequence $f_1,\cdots,f_r$ with $e_i = \deg(f_i) \leq e_j = \deg(f_j)$ whenever $i \leq j$. Set $P = (x_1^{e_1},\cdots,x_r^{e_r})$. If L is a lex ideal such that $L+P$ has the same Hilbert function as J, then $b_{p,s}(S/L+P) \geq b_{p,s}(S/J)$ for all p, s.*

The Eisenbud-Green-Harris conjecture asserts the existence of a lex ideal L such that $L+P$ has the same Hilbert function as J.

Both conjectures are wide open. Some special cases are proved by G. Caviglia, S. Cooper, G. Evans, C. Francisco, D. Maclagan, B. Richert, and S. Sabourin [CM,Co1,Co2,ER,Fr,Fr2,Ri,RS]. An expository paper describing the the conjectures is [FR].

In a series of papers [MPS, Mu, MM1], Murai, Peeva, Stillman, and I prove the lex-plus-powers conjecture in the case that the regular sequence consists of powers of the variables.

Theorem 2.2.2. **[MPS]** *Set $P = (x_1^2,\cdots,x_n^2)$. Let N be any homogeneous ideal of S containing P. Let L be the lex ideal such that $N+P$ and $L+P$ have the same Hilbert function (L exists by Kruskal-Katona's Theorem). Then $b_{p,s}(S/L+P) \geq b_{p,s}(S/N+P)$ for all p, s.*

Theorem 2.2.3. **[MM1]** *Set $P = (x_1^{e_1},\cdots,x_n^{e_r})$ with $e_1 \leq \cdots \leq e_r$. Let N be any homogeneous ideal of S containing P. Let L be the lex ideal such that $N+P$ and $L+P$ have the same Hilbert function (L exists by Clements-Lindström's Theorem). Then $b_{p,s}(S/L+P) \geq b_{p,s}(S/N+P)$ for all p, s.*

In view of these, we made the following conjecture in [MeP2] (under some additional assumptions, for example char(k) = 0):

Conjecture 2.2.4. *Let S/M be a ring in which Macaulay's theorem holds. Let J be any homogeneous ideal of S/M, and let L be the lex ideal with the same Hilbert function as J. Then:*

(i) The Betti numbers of L over S/M are greater than or equal to those of J.

(ii) The Betti numbers of $L+M$ over S are greater than or equal to those of $J+M$.

Note that the first claim is usually about infinite resolutions, while the second deals exclusively with finite resolutions.

Theorem 2.2.3 proves Conjecture 2.2.4 (ii) in the Clements-Lindström case. Murai and Peeva [MuP] prove Conjecture 2.2.4 (i) in this case using a walk on the Hilbert scheme.

In [MM2], Murai and I produced a counterexample to Conjecture 2.2.4(i). Conjecture 2.2.4(ii) remains wide open, however.

I plan to continue my work on Conjectures 2.2.1 and 2.2.4. The proof of Theorems 2.2.2 and 2.2.3 make heavy use of compressed ideals. I plan to explore in what other rings one can use compressed ideals to study Betti numbers.

2.3 Compression

Compression is the main technique that I have used in various settings. This techinique was introduced by Macaulay [Ma]. Compression and compressed ideals have been used to study Hilbert functions in Macaulay [Ma], Clements-Lindstrom [CL], Mermin-Peeva [MeP1,MeP2], and Mermin [Me1,Me2]. Compression was used to study Betti numbers in [MPS] and [Me2].

Definition 2.3.1. Let N be a monomial ideal of S, and let $\mathcal{A}$ be a subset of $\{x_1, \cdots, x_n\}$. Let $\oplus_f$ denote a sum over all monomials of $k[\mathcal{A}^c]$. We may decompose N as a direct sum of $k[\mathcal{A}]$-modules, $N = \oplus_f f N_f$, with each N_f a monomial ideal of $k[\mathcal{A}]$. We say that N is *$\mathcal{A}$-compressed* if every N_f is a lex ideal of $k[\mathcal{A}]$. For each f, let T_f be the lex ideal of $k[\mathcal{A}]$ with the same Hilbert function as N_f. Put $T = \oplus_f f T_f$. We say that T is the *$\mathcal{A}$-compression* of N. If N is $\mathcal{A}$-compressed for all p-element sets $\mathcal{A}$, we say that N is *p-compressed*. If N is $\mathcal{A}$-compressed for all proper subsets $\mathcal{A}$ of $\{x_1, \cdots, x_n\}$, we say that N is *compressed*.

In [Me2] I have shown that 3-compressed ideals are lex. This leads to a very simple new proof of Macaulay's theorem, and gives hope that many questions about lex ideals can be solved by looking at compressed ideals instead. For example, Bigatti, Hulett, and Pardue's theorem [Bi,Hu,Pa] is an immediate corollary of the result in [Me2] that Betti numbers over S are nondecreasing under compression.

In [MPS], we used compression in the squarefree ring $R = S/(x_1^2, \cdots, x_n^2)$ in order to prove Theorem 2.2.2. It is not known how Betti numbers behave under compression, in any ring other than S, but it seems reasonable to expect they do not decrease. In fact, since any compression step may be viewed as replacing the ideal with a lex ideal in an associated multigrading, it seems reasonable to conjecture that these multigraded Betti numbers are nondecreasing under compression. I intend to conduct research exploring further the ideas in [Me2] and [MPS] on the following:

Problem 2.3.2. *How do Betti numbers behave under compression?*

2.4 Cellular resolutions

One of the most successful recent ideas in the study of resolutions is that of cellular resolutions [BS,BPS,JW, MS]. Let $M \subset S$ be a monomial ideal and

$$\mathbb{F} : \cdots \xrightarrow{\phi_2} F_1 \xrightarrow{\phi_1} F_0 \xrightarrow{\phi_0} M \to 0$$

be a free resolution; write each $F_p = \oplus S(f_{p,j})$. We say that $\mathbb{F}$ is *cellular* (respectively *simplicial*, *CW*) if there exists a regular cell complex (respectively simplicial complex, CW-complex) Δ and bijections ψ_p from the $f_{p,j}$ to the p-dimensional cells of Δ which commute with the boundary maps: $\partial_p\left(\psi_p(f_{p,j})\right) = \psi_{p-1}\left(\overline{\phi_p(f_{p,j})}\right)$, where ∂ is the boundary map in the topological chain complex of Δ and $\overline{(\cdot)}$ represents evaluating all x_i at 1.

The Taylor resolution, which (non-minimally) resolves every monomial ideal, is simplicial [BPS]. The situation for minimal resoutions is more complicated. Monomial ideals with "generic" exponents (such as complete intersections) are minimally resolved by the Scarf complex [BPS], which is simplicial. The largest other class of monomial ideals whose minimal resolutions are known is the stable ideals, which are resolved by the Eliahou-Kervaire resolution [EK]. The Eliahou-Kervaire resolution is not simplicial, but I show in [Me4] that it is cellular. In [Ve], Velasco constructs ideals whose minimal resolutions are not supported on any CW-complex.

There are general techniques [BS, BPS, PV] for using a cellular minimal resolution of an ideal M to obtain minimal resolutions of related ideals. For example, Sinefakopoulos [Si] uses a cellular structure on the minimal resolution of a power of the homogenous maximal ideal $(x_1, \cdots, x_n)$ to construct minimal resolutions of certain p-Borel-fixed ideals. (This is an important class of ideals which arise as generic initial ideals in characteristic p, whose resolutions were previously unknown.)

I am interested in the following problem:

Problem 2.4.1 *Identify (classes of) monomial ideals whose minimal resolutions are cellular, and construct those resolutions.*

One simplicial resolution which I find particularly interesting is the Lyubeznik resolution [Ly, No]. A monomial ideal usually has many Lyubeznik resolutions (corresponding to orderings on its generators), each of which sits canonically inside the Taylor resolution. Their intersection is the Scarf complex, which is in general not a resolution. I am interested in exploring two questions about the Lyubeznik resolution:

Problem 2.4.2

- *How can one choose a Lyubeznik resolution which is as close to minimal as possible?*
- *When can one use the various Lyubeznik resolutions to construct the minimal resolution?*

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