LEXLIKE SEQUENCES

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Abstract: We study sets of monomials that have the same Hilbert function growth as initial lexicographic segments.

1. Introduction

Macaulay [Ma] showed in 1927 that for every graded ideal of $k[x_1, \cdots, x_n]$ there exists a lexicographic ideal with the same Hilbert function. Lexicographic ideals are highly structured: they can be defined combinatorially and it is easy to describe their Hilbert functions. In particular, Macaulay's theorem shows that d-generated lex ideals have minimal Hilbert function growth. In this paper we consider the following:

Question 1.1: What other d-generated monomial ideals have minimal Hilbert function growth?

In Section 3, we introduce lexlike sequences and study their properties. Let X be a sequence of the degree d monomials of $k[x_1, \cdots, x_n]$. We say that X is *lexlike* if the ideal generated by any initial segment of X has the same Hilbert function as the ideal generated by the initial segment with the same length of

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the degree d lex sequence. Theorems 3.8 and 3.13 provide criteria for a given sequence of monomials to be lexlike. Theorem 3.13 describes the structure of a lexlike sequence. This leads to Algorithm 3.14 producing all lexlike sequences and to Corollary 3.15 which gives the number of all lexlike sequences (in a fixed degree and fixed number of variables). Furthermore, we show that lexlike sequences share some of the nice properties of the lex sequence. In [MP2] we use lexlike sequences to build lexlike ideals and prove an analog of Macaulay's Theorem [Ma].

In Section 5, we introduce and study squarefree lexlike sequences in an exterior algebra. We classify such squarefree lexlike sequences and produce analogs to the results in Section 3. Theorem 5.26 shows that the Alexander dual of a squarefree lexlike sequence is squarefree lexlike. We introduce squarefree lexlike ideals in Definition 5.32 and show that in an exterior algebra E:

(1) If J is a graded ideal in E, then there exist (usually many) squarefree lexlike ideals with the same Hilbert function as J. This is an analog of the Kruskal-Katona theorem [Kr,Ka] which says that every Hilbert function is attained by a lex ideal.

(2) Every squarefree lexlike ideal has maximal graded Betti numbers over E among all ideals with the same Hilbert function. This is an analog of the Aramova-Herzog-Hibi theorem [AHH] which says that the lex ideals have maximal graded Betti numbers over E.

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2. Preliminaries

Let k be a field and $B = k[x_1, \dots, x_n]$ or $k[x_1, \cdots, x_n]/(x_1^2, \cdots, x_n^2)$ be graded by deg $(x_i) = 1$ for all i. We denote by B_d the k-vector space spanned by all monomials of degree d. Denote $\mathbf{m} = (x_1, \dots, x_n)_1$ the k-vector space spanned by the variables. We order the variables lexicographically by $x_1 > \dots > x_n$, and we denote by $\succ_{lex}$ the homogeneous lexicographic order on the monomials.

A monomial $x_1^{a_1}\dots x_n^{a_n}$ has *exponent vector* $\mathbf{a}=(a_1,\dots,a_n)$, and is sometimes denoted by $\mathbf{x^a}$. An ideal is called *monomial* if it can be generated by monomials; such an ideal has a unique minimal system of monomial generators.

Let I be a graded ideal in B. It decomposes as a direct sum of its components $I=\oplus_{d\geq 0} I_d$. Its *Hilbert function* $\mathrm{Hilb}_I^B:\ \mathbf{N}\cup 0\ \to \mathbf{N}\cup 0$ is defined by

$$\mathrm{Hilb}_I^B(d)=\dim_k(I_d) \quad \text{ for all } d\geq 0\,.$$

We use the following notation

$$|I_d|^B\ =\ \mathrm{Hilb}_I^B(d)\,;$$

and for simplicity, we write $|I_d|$ if it is clear over which ring we work.

An ideal is called *d-generated* if it has a system of generators of degree d.

We say that a vector space V is a *monomial space* if it is spanned by monomials, and a *degree d monomial space* if it is spanned by monomials of degree d, in which case we sometimes write it V_d.

Let L be a monomial ideal minimally generated by monomials $l_1,\dots,l_r$. The ideal L is called *lex*, (*lexicographic*), if the following property is satisfied:

$$\left.\begin{array}{c} m \text{ is a monomial} \\ m\succ_{lex} l_i \ \text{ and } \deg\,(m)=\deg\,(l_i),\ \text{ for some } 1\leq i\leq r \end{array}\right\} \quad\Longrightarrow\quad m\in L\,.$$

A sequence of monomials $X: m_1,\cdots,m_r$ will frequently be written as X for simplicity, or as $X: m_1,\cdots$ if its length is unknown or unimportant. X_i will refer not to the ith term of X but to the vector space spanned by the first i terms of X.

If we multiply a sequence by a monomial a via termwise multiplication, then we denote it by $aX: am_1,\cdots,am_r$. If $Y: t_1,\cdots,t_s$ is another sequence, we denote concatenation with a semicolon, so $X;Y: m_1,\cdots,m_r,t_1,\cdots,t_s$.

3. Lexlike sequences and redundancy

In this section we introduce lexlike sequences.

Definition 3.1. A *monomial sequence* (of degree d) is a sequence $X : \mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_s}$ of all the monomials of $B = k[x_1, \cdots, x_n]$ of degree d. We say that X is *lexlike* if, for every i, and for every vector space V generated by i degree d monomials, we have $|\mathbf{m}(\mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_i})| \leq |\mathbf{m}V|$.

Example 3.2. The sequence $X : ab, a^2, ac, bc, c^2, b^2$ is lexlike in $k[a, b, c]$, but not in $k[a, b, c, d]$. The sequence $Y : a^2, b^2, ab$ is not lexlike in $k[a, b]$.

Example 3.3. The lex sequence in degree d consists of all the degree d monomials ordered lexicographically. By Macaulay's Theorem [Ma], this sequence is lexlike. It is denoted by $\mathrm{Lex}(d)$ or simply Lex throughout the paper.

The Lex sequence begins with every monomial divisible by x_1, and ends with every monomial not divisible by x_1. It is thus partitioned into two lex sequences: Let A be the lex sequence in degree $d-1$ in n variables, and let C be the lex sequence in degree d in the variables $x_2, \cdots, x_n$. Then the lex sequence in degree d in n variables is $\mathrm{Lex}{:}x_1A; C$. We will see that all lexlike sequences can be partitioned similarly; this idea is at the heart of most of our proofs.

Definition 3.4. Let M_d be a monomial subspace of B_d and m a monomial of degree d. The *redundancy* $\mathrm{Red}(M, m)$ is the number of variables x_i such that $x_i m \in M$: $\mathrm{Red}(M, m) = |\mathbf{m}M \cap \mathbf{m}(m)|$. Alternatively, $\mathrm{Red}(M, m)$ is the k-dimension of the degree one part of the colon ideal $(M : m)$.

Corollary 3.5. *Let M and m be as above. Then*

$$|\mathbf{m}(M + (m))| = |\mathbf{m}M| + n - \mathrm{Red}(M, m)\,.$$

Definition 3.6. For a sequence $X : m_1, \cdots, m_s$ of monomials of the same degree, set

$$\mathrm{Red}_i(X) := \mathrm{Red}\left((m_i, \cdots, m_{i-1}), m_i\right),$$

Example 3.7. Let Lex:$m_1, \cdots, m_s$ be the lex sequence in degree d, and let $\max(m_i) = j$, where x_j is the lex-last variable dividing m_i. Then $\mathrm{Red}_i(\mathrm{Lex}) = j - 1$.

This leads to our first criterion for a monomial sequence to be lexlike.

Theorem 3.8. *A monomial sequence X is lexlike if and only if*

$$\mathrm{Red}_i(X) = \mathrm{Red}_i(\mathrm{Lex})$$

for all i (here Lex *stands for the lex sequence in the same degree as X).*

Proof: Inductively, we have $|\mathbf{m}X_{i+1}| = |\mathbf{m}X_i| + n - \mathrm{Red}_{i+1}(X)$ and $|\mathbf{m}\mathrm{Lex}_{i+1}| = |\mathbf{m}\mathrm{Lex}_i| + n - \mathrm{Red}_{i+1}(\mathrm{Lex})$. $\square$

Lemma 3.9. *Let $X : \mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_r}$ be a sequence of some of the monomials of B in degree d having the same redundancies as* Lex *(in degree d), and set $s = |(x_1)_d|$. Then:*
(1) If $r \leq s$ there exists a variable x_i such that x_i divides $\mathbf{x}^{t_j}$ for all j.
(2) If $r \geq s$ there exists a variable x_i such that $\{\mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_s}\} = \{\mathbf{m}^{d-1}x_i\}$.

Proof: We induct on r and d. The base case, where either is zero, is clear. If $r-1 \geq s$, there is nothing to prove, so we assume that $r - 1 < s$. Thus without loss of generality the first $r-1$ terms of X are divisible by x_1. Let Y be the sequence obtained by dividing every element of X by x_1. Set $t = |(x_1^2)(d)|$. Suppose $r - 1 \leq t$, so by induction the first $r-1$ elements of X are divisible by, without loss of generality, x_1x_2. If $\mathbf{x}^{t_r}$ is not divisible by x_1 or x_2 we have $0 = \mathrm{Red}_r(X) = \mathrm{Red}_r(\mathrm{Lex}) \geq 1$. If on the other hand $r - 1 > t$ and x_1 does not divide $\mathbf{x}^{t_r}$, we have $1 \geq \mathrm{Red}_r(X) = \mathrm{Red}_r(\mathrm{Lex}) \geq 2$. $\square$

Proposition 3.10. *Let $X : \mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_r}$ be a sequence of some of the degree d monomials of B having the same redundancies as Lex (in degree d). Then X can be extended to a lexlike sequence.*

Proof: It suffices to show that X may be extended to a sequence of length $r+1$. Let $s = \mathrm{Hilb}_B(x_1)(d)$. If $r < s$, we may without loss of generality divide every term of X by x_1, and the resulting sequence may be extended by induction on d; multiplying back by x_1 gives the desired extension. If $r \geq s$, we write $X = Y; Z$ where Y contains precisely the monomials divisible by x_1 and Z is a sequence of length $r - s$ of degree d monomials in the variables $x_2, \cdots, x_n$, having the same redundancies as Lex. By induction on n, Z may be extended to length $r - s + 1$; appending this to Y gives the desired result. $\square$

Corollary 3.11. *A monomial sequence $X : m_1, \cdots, m_s$ is lexlike if and only if, for all $i < j$, we have $\mathrm{Red}_{i+1}(X) \geq \mathrm{Red}\left((m_1, \cdots, m_i), m_j\right)$.*

Proof: Let X_i be the vector space generated by the first i terms of X, and L_i be the vector space generated by the first i terms of Lex. Suppose the inequalities hold. Then by induction we have $|\mathbf{m}X_i| = |\mathbf{m}L_i|$ and $\mathrm{Red}_j(X) = \mathrm{Red}_j(\mathrm{Lex})$ for $j \leq i$. By Proposition 3.10, $m_1, \cdots, m_i$ may be extended to a lexlike sequence Y. Let n be the $(i+1)$-th term of Y. We have $\mathrm{Red}_{i+1}(X) \geq \mathrm{Red}\left((m_1, \cdots, m_i), n\right)$; by Macaulay's theorem this must be an equality, so $\mathrm{Red}_{i+1}(X) = \mathrm{Red}_{i+1}(\mathrm{Lex})$ and $|\mathbf{m}X_{i+1}| = |\mathbf{m}L_{i+1}|$. On the other hand, if X is lexlike we immediately have $\mathrm{Red}_{i+1}(X) \geq \mathrm{Red}\left((m_1, \cdots, m_i), m_j\right)$.

$\square$

Example 3.12. Consider the sequence $X : abc, a^2b, ab^2, b^2c$, which has the same redundancies as Lex in $k[a, b, c]$ in degree 3. To extend X to length 5, we observe that every term of X is divisible by b; dividing by b yields the degree 2 sequence $Y : ac, a^2, ab, bc$. To extend Y, we observe that Y contains everything divisible by a. Truncating, we obtain the sequence $Z : bc$ in the variables b, c. To extend Z, we note that every term is divisible by b and by c; we arbitrarily choose c. Dividing by c yields the degree 1 sequence $W : b$ which may be extended to $W' : b, c$. The sequence Z is then extended to $Z' = cW' : bc, c^2$. The sequence Y is extended to $Y' : ac, a^2, ab, bc, c^2$ by appending Z', and X is extended to $X' = bY' : abc, a^2b, ab^2, b^2c, bc^2$.

Next, we obtain a complete structural description of the lexlike sequences. In particular, it can be used as a criterion for a monomial sequence to be lexlike.

Theorem 3.13. *Let $X : m_1, \cdots$ be a monomial sequence in degree d, and let $s = |(x_1)_d|$. Set $Y : m_1, \cdots, m_s$, and set $Z : m_{s+1}, \cdots$. The sequence X is lexlike if and only if the following hold:*

(1) There is a variable x_i such that $(x_i)_d = (m_1, \cdots, m_s)$.

(2) There is a lexlike sequence W in degree $d-1$ such that $x_i W = Y$.

(3) Z is lexlike in the variables $x_1, \cdots, \hat{x_i}, \cdots, x_n$, (here $\hat{x_i}$ means x_i is omitted.)

Proof: Suppose that X is lexlike. Then we obtain x_i from Lemma 3.8; writing $X = x_i W; Z$, it remains to show that W and Z are lexlike. We write $\mathrm{Lex} = x_1 A; C$ with A and C lexlike,as in Example 3.3 and compute:

$$\mathrm{Red}_i(W) = \mathrm{Red}_i(X) = \mathrm{Red}_i(\mathrm{Lex}) = \mathrm{Red}_i(A)$$

and

$$\mathrm{Red}_i(Z) = \mathrm{Red}_{i+s}(X) - 1 = \mathrm{Red}_{i+s}(\mathrm{Lex}) - 1 = \mathrm{Red}_i(C).$$

Conversely, if Z and W are lexlike, a similar computation shows that X is lexlike. □

This structure theorem yields the following algorithm enumerating all the lexlike sequences in n variables in degree d.

Algorithm 3.14. Inductively enumerate all lexlike sequences in n variables in degree $d-1$ and all lexlike sequences in $n-1$ variables in degree d. For each variable x_i and each sequence Z in $n-1$ variables, let Z_i be Z on the variables $\{x_1, \cdots, \hat{x_i}, \cdots, x_n\}$. For each variable x_i, lexlike sequence Y in degree $d-1$, and lexlike sequence Z in $n-1$ variables, write down the lexlike sequence $X_{i,Y,Z} : x_i Y; Z_i$.

Corollary 3.15. *Let $f(n,d)$ be the number of lexlike sequences in n variables in degree d. Then we have $f(n,d) = nf(n-1,d)f(n,d-1)$ if $n > 1$ and $f(1,d) = 1$. Thus,*

$$f(n,d) = \prod_{i=0}^{n-1} (n-i)^{\binom{d+i}{i+1}}.$$

Proof: Given a variable x_i, a lexlike sequence Y in n variables in degree $d-1$, and a lexlike sequence Z in the variables $x_1, \cdots, \hat{x}_i, \cdots, x_n$ in degree d, we may construct the lexlike sequence $X : x_iY; Z$. Every lexlike sequence in n variables in degree d may be attained in this way, and every choice of x_i, Y, and Z produces a different X. $\square$

Example 3.16. It is natural to ask if every monomial ideal with minimal Hilbert function growth occurs as an initial segment of some lexlike sequence. It does not. For example, $I := (a^2b, a^2c, ab^2, abc, ac^2, b^2c, bc^2)$ has minimal Hilbert function growth, but does not contain $(\mathbf{m}^2 x_i)$, for $x_i = a, b$, or c.

Definition 3.17. Let $Y : \mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_s}$ be a sequence of all the monomials of B of degree d, and let $X : \mathbf{x}^{v_1}, \cdots, \mathbf{x}^{v_r}$ be a sequence of all the monomials of B of degree $d-1$. We say that Y is *above* Y if, for all i, there is a j such that $\mathbf{m}X_i = Y_j$.

Example 3.18. The sequence $X : b^3, b^2c, ab^2, abc, bc^2, a^2b, a^2c, ac^2, a^3, c^3$ is above $Y : b^2, bc, ab, ac, a^2, c^2$.

Example 3.19 The Lex sequence in degree $d+1$ is above the Lex sequence in degree d.

Proposition 3.20. *Let X be a lexlike sequence in degree d.*

(1) There exists a monomial sequence Y above X.

(2) If Y is above X then Y is lexlike.

In particular, there exists a lexlike sequence Y in degree $d+1$ such that Y is above X.

Proof: In order to prove (1), we construct a sequence Y above X inductively: Y starts with the monomials in $\mathbf{m}X_1$ in any order; then we add the monomials of $\mathbf{m}X_2 \setminus \mathbf{m}X_1$ in any order; and so on.

Now, we will prove (2). Write $X : a_1, \cdots$ and $Y : b_1, \cdots$. It suffices to show that, for all i, $\mathrm{Red}_i(Y) \geq \mathrm{Red}_i(\mathrm{Lex})$. Let j be such that $|\mathbf{m}(a_1, \cdots, a_{j-1})| < i \leq$

$|\mathbf{m}(a_1,\cdots,a_j)|$, and set $s=|\mathbf{m}(a_1,\cdots,a_{j-1})|$. Then we claim that

$$\mathrm{Red}_i(Y)=\mathrm{Red}_j(X)+(i-s-1)=\mathrm{Red}_i(\mathrm{Lex})\,.$$

Since $b_i\in\{\mathbf{m}(a_1,\cdots,a_j)\}$ but $b_i\notin\{\mathbf{m}(a_1,\cdots,a_{j-1})\}$, we may write $b_i=ya_j$. We see that $x_pb_i\in(b_1,\cdots,b_{i-1})$ whenever $x_pa_j\in(a_1\cdots,a_{j-1})$ or $x_pa_j\in\{b_{s+1},\cdots,b_{i-1}\}$, i.e., $\mathrm{Red}_i(Y)=\mathrm{Red}_j(X)+(i-s-1)$.

Now suppose that X and Y are the Lex sequences in degrees d and $d+1$, respectively. Then $\mathrm{Red}_i(Y)=\max(a_i)-1=\max(b_j)-1+(\max(a_i)-\max(b_j))=\mathrm{Red}_j(X)+(i-s-1)$ as desired. ☐

Corollary 3.21. *Let V_d be spanned by an initial segment of a lexlike sequence X. Then $\mathbf{m}V_d$ is spanned by an initial segment of some lexlike sequence Y.*

Theorem 3.22. *Let Y be a lexlike sequence in degree d. There is a unique lexlike sequence X in degree $d-1$ such that Y is above X.*

Proof: By Lemma 2.6 we may write $Y=Y_1;Y_2$, where without loss of generality every monomial of Y_1 is divisible by x_1 and every monomial of Y_2 is not. Let Y_3 be the sequence obtained by dividing everything in Y_1 by x_1. Y_3 is a lexlike sequence in n variables and degree $d-1$, so by induction on d there is a unique lexlike sequence X_3 below it. Multiplying everything in X_3 by x_1 yields X_1. Y_2 is a lexlike sequence in $n-1$ variables and degree d, so by induction on n there is a unique X_2 below it. $X:=X_1;X_2$ is the unique lexlike sequence below Y.☐

Example 3.23. Consider the lexlike sequence $Y: abc, bc^2, b^2c, b^3, ab^2, a^2b, ac^2, a^2c, a^3, c^3$. The unique lexlike sequence below Y in degree 2 is the sequence $X: bc, b^2, ab, ac, a^2, c^2$.

4. Lex-favoring ideals

In this section we relate lexlike sequences to our work in [MP1]. Throughout the section, M stands for a monomial d-generated ideal.

Definition 4.1. Let Σ be the set of all monomial sequences in degree $d-1$. For a monomial sequence A of all the monomials in a fixed degree, set $A_i:=$

$(a_1, \cdots, a_i)$. Then Σ is partially ordered by

$$A \leq_M C \quad \text{if} \quad |\mathbf{m}A_i|^{B/M} \leq |\mathbf{m}C_i|^{B/M} \text{ for all } i.$$

Definition 4.2. We say that M is *lex-favoring* if for every lexlike sequence A in degree $d-1$ we have Lex $\leq_M A$.

The above two definitions are motivated by our work in [MP1]. We recall a definition from [MP1]:

Definition 4.3. We say that M is *p-pro-lex* if, for every p-generated monomial vector space N_p, there exists a p-generated lex vector space L_p such that $|L_p|^{B/M} = |N_p|^{B/M}$, and $|\mathbf{m}L_p|^{B/M} \leq |\mathbf{m}N_p|^{B/M}$.

Corollary 4.4. *The ideal M is $(d-1)$-pro-lex if and only if Lex is minimal among all monomial sequences with respect to $\leq_M$.*

Thus pro-lex ideals are lex-favoring, but lex-favoring ideals are not necessarily pro-lex. Example 4.6 shows that this can be the case even for Borel ideals.

Example 4.6. Set $M = (a^3, a^2b, a^2c, ab^2, abc, b^3, b^2c)$ in $B = k[a, b, c]$ is lex-favoring but not pro-lex: There is no lex ideal of B/M with the same Hilbert function as (a^2, ab, b^2).

Strongly stable ideals occur as generic initial ideals and have been extensively studied. We recall the definition: A monomial ideal W is called a *strongly stable ideal* or *Borel* if, for every monomial $m \in W$, and for every x_i dividing M and $j < i$, we have $(x_j/x_i)m \in W$.

Proposition 4.5. *A d-generated strongly stable ideal M is lex-favoring.*

Proof: Let $A : m_1, m_2, ...$ be any lexlike sequence in degree $d-1$, and let B be the sequence below A in degree 1. If x_1 is the first term of B, we have Lex $\leq_M A$ by induction on n and d. If not, assume without loss of generality that the first two terms of B are x_2 and x_1. Let $s = |(x_2)_{d-1}|^B$ and $t = |(x_2, x_1)_{d-1}|^B$. Let

$C : v_1, v_2 \cdots$ be the sequence obtained from A by setting $v_i = \frac{x_1}{x_2} m_i$ when $i \leq s$, $v_i = x_2^{e_i} n_i$ for $s < i \leq t$, $m_i = x_1^{e_i} n_i$, and $v_i = m_i$ for $t < i$.

Then for $j \leq s$, we have $ym_j \in M \Rightarrow yv_j \in M$, so $|\mathbf{m}C_k|^{B/M} \leq |\mathbf{m}A_k|^{B/M}$ whenever $k \leq s$. And for $s < j \leq t$ we have $yv_j \in M \Rightarrow ym_j \in M$, so for $s < k \leq t$ we have

$$|\mathbf{m}C_k|^{B/M} = |\mathbf{m}C_t|^{B/M} - \sum_{j=k+1}^{t} (|\{yv_j \notin M : yv_j \notin \mathbf{m}C_{j-1}\}|)$$

$$\leq |\mathbf{m}A_t|^{B/M} - \sum_{j=k+1}^{t} (|\{yv_j \notin M : ym_j \notin \mathbf{m}A_{j-1}\}|)$$

$$= |\mathbf{m}A_k|^{B/M}.$$

Thus we have $C \leq_M A$. □

Proposition 4.7. *Let M and N be d-generated d-lex-favoring ideals, and $(M \cap N)_d = 0$. Then $M + N$ is d-lex-favoring.*

Proof: Let A be any lexlike sequence in degree $d-1$, and let I and J be initial segments of Lex and A, respectively, of the same length. Then

$$|\mathbf{m}I \cap M| \geq |\mathbf{m}J \cap M| \quad \text{and} \quad |\mathbf{m}I \cap N| \geq |\mathbf{m}J \cap N|,$$

so

$$\begin{aligned} |\mathbf{m}I \cap (M+N)| &= |\mathbf{m}I \cap M| + |\mathbf{m}I \cap N| \\ &\geq |\mathbf{m}J \cap M| + |\mathbf{m}J \cap N| \\ &= |\mathbf{m}J \cap (M+N)| \end{aligned}$$

as desired. □

5. Lexlike sequences in an exterior algebra.

In this section we characterize the squarefree lexlike sequences. Our approach is almost identical to that used in section 3. Throughout this section, we will work over the ring $B = k[x_1, \cdots, x_n]/(x_1^2, \cdots, x_n^2)$. (Since we consider only Hilbert functions of monomial ideals, we can replace an exterior algebra by B.) We have numbered the results in this section not sequentially but in a way that emphasizes the analogy with the corresponding results in section 3. We omit proofs where they are essentially identical to those given in section 3.

Definition 5.1. A *squarefree monomial sequence* (of degree d) is a sequence $X : \mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_s}$ of all the degree d monomials of B. We say that X is *squarefree lexlike* if, for every i, and for every vector space V generated by i squarefree degree d monomials, we have $|\mathbf{m}(\mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_i})| \leq |\mathbf{m}V|$.

Example 5.2. The sequence $X : abc, ace, acd, abe, abd, ade, bce, bcd, bde, cde$ is squarefree lexlike in $B = k[a, b, c, d, e]/(a^2, b^2, c^2, d^2, e^2)$.

Example 5.3. The squarefree lex sequence in degree d consists of all the degree d monomials ordered lexicographically. By Kruskal-Katona's Theorem [Kr,Ka], this sequence is squarefree lexlike. It is denoted by $\text{Lex}(d)$ or simply Lex throughout this section.

As in the polynomial case, we may partition Lex into two smaller squarefree lex sequences. Let A be the squarefree lex sequence in degree $d-1$ and variables $x_2, \cdots, x_n$, and let C be the squarefree lex sequence in degree d and variables $x_2, \cdots, x_n$. Then we have $\text{Lex} = x_1 A; C$.

Definitions and results 3.4 through 3.12, and their proofs, hold verbatim in the squarefree case.

Theorem 5.13. *Let $X : m_1, \cdots$ be a squarefree monomial sequence in degree d, and let $s = |(x_1)_d|$. Set $Y : m_1, \cdots, m_s$, and set $Z : m_{s+1}, \cdots$. The sequence X is squarefree lexlike if and only if the following hold:*

(1) There is a variable x_i such that $(x_i)_d = (m_1, \cdots, m_s)$.

(2) There is a squarefree lexlike sequence W in degree $d-1$ and variables $x_1, \cdots, \hat{x_i}, \cdots, x_n$ such that $x_i W = Y$.

(3) Z is squarefree lexlike in the variables $x_1, \cdots, \hat{x_i}, \cdots, x_n$, (here $\hat{x_i}$ means x_i is omitted.)

Remark. The fundamental difference between the squarefree and polynomial case appears in (2). In the squarefree case, the variable x_i cannot appear in W, so W occurs in $n-1$ variables. In the polynomial case, there was no such restriction and W occured in n variables.

This structure theorem yields the following algorithm enumerating all the squarefree lexlike sequences in n variables in degree d.

Algorithm 5.14. Inductively enumerate all squarefree lexlike sequences in $n-1$ variables in degree $d-1$ and all squarefree lexlike sequences in $n-1$ variables in degree d. For each variable x_i and each sequence Z in $n-1$ variables, let Z_i be Z on the variables $\{x_1, \cdots, \hat{x_i}, \cdots, x_n\}$. For each variable x_i, lexlike sequence Y in degree $d-1$, and lexlike sequence Z in $n-1$ variables, write down the lexlike sequence $X_{i,Y,Z} : x_i Y_i; Z_i$.

Corollary 5.15. *Let $f(n,d)$ be the number of lexlike sequences in n variables in degree d. Then we have $f(n,d) = nf(n-1,d)f(n-1,d-1)$ if $n > d > 0$ and $f(n,d) = f(n,0) = 1$. Thus,*

$$f(n,d) = \prod_{i=0}^{n-1} (n-i)^{\sum_{j=i-d+1}^{n-d-1} \binom{i}{j}}.$$

(Here we use the convention that $\binom{i}{j} = 0$ if $i < j$ or $j < 0$.)

Remark. In particular, we have $f(n,d) = f(n,n-d)$. A nice bijection is given by the Alexander Dual in Theorem 5.26.

Example 5.16. It is natural to ask if every squarefree monomial ideal with minimal Hilbert function growth occurs as an initial segment of some squarefree lexlike sequence. It does not. For example, $I = (ac, ad, bc, bd)$ has minimal Hilbert function growth in $B = k[a,b,c,d]/(a^2, b^2, c^2, d^2)$ but does not contain $(\mathbf{m}x_i)$ for $x_i = a, b, c$, or d.

Definition 5.17. Let $Y : \mathbf{x}^{t_1}, \cdots, \mathbf{x}^{t_s}$ be a sequence of all the monomials of B of degree d, and let $X : \mathbf{x}^{v_1}, \cdots, \mathbf{x}^{v_r}$ be a sequence of all the monomials of B

of degree $d-1$. We say that Y is *above* Y if, for all i, there is a j such that $\mathbf{m}X_i = Y_j$.

Example 5.18. The sequence $X : bcd, abc, acd, abd$ is above $Y : bc, ac, cd, ab, ad, bd$.

Example 5.19 The squarefree Lex sequence in degree $d+1$ is above the squarefree Lex sequence in degree d.

Proposition 5.20. *Let X be a squarefree lexlike sequence in degree d.*
(1) There exists a squarefree monomial sequence Y above X.
(2) If Y is above X then Y is lexlike.

In particular, there exists a squarefree lexlike sequence Y in degree $d+1$ such that Y is above X.
Proof: We will prove (2). Write $X : a_1, \cdots$ and $Y : b_1, \cdots$. It suffices to show that, for all i, $\mathrm{Red}_i(Y) \geq \mathrm{Red}_i(\mathrm{Lex})$. Let j be such that $|\mathbf{m}(a_1, \cdots, a_{j-1})| < i \leq |\mathbf{m}(a_1, \cdots, a_j)|$, and set $s = |\mathbf{m}(a_1, \cdots, a_{j-1})|$. Then we claim that

$$\mathrm{Red}_i(Y) = \mathrm{Red}_j(X) + (i - s) = \mathrm{Red}_i(\mathrm{Lex})\,.$$

Since $b_i \in \{\mathbf{m}(a_1, \cdots, a_j)\}$ but $b_i \notin \{\mathbf{m}(a_1, \cdots, a_{j-1})\}$, we may write $b_i = ya_j$. We see that $x_p b_i \in (b_1, \cdots, b_{i-1})$ whenever $x_p a_j \in (a_1 \cdots, a_{j-1})$ or $x_p a_j \in \{b_{s+1}, \cdots, b_i\}$, i.e., $\mathrm{Red}_i(Y) = \mathrm{Red}_j(X) + (i - s - 1)$.

Now suppose that X and Y are the Lex sequences in degrees d and $d+1$, respectively. Then $\mathrm{Red}_i(Y) = \max(a_i) - 1 = \max(b_j) - 1 + (\max(a_i) - \max(b_j)) = \mathrm{Red}_j(X) + (i - s - 1)$ as desired. $\square$

Corollary 5.21. *Let V_d be spanned by an initial segment of a squarefree lexlike sequence X. Then $\mathbf{m}V_d$ is spanned by an initial segment of some squarefree lexlike sequence Y.*

Proposition 5.22. *Let Y be a lexlike sequence in degree d. There is a lexlike sequence X in degree $d-1$ such that Y is above X.*

Remark. In the polynomial case, X was unique. That uniqueness is lost here. For example, there are $n!$ squarefree lexlike sequences in degree $n-1$, and a unique squarefree lexlike sequence in degree n above them all.

Definition 5.23. Let $X : m_1, \cdots, m_s$ be a squarefree monomial sequence in degree d and variables $x_1, \cdots, x_s$. Let $u = \prod_{i=1}^{n} x_i$. Then the monomial sequence $X^\vee : \frac{u}{m_s}, \cdots, \frac{u}{m_1}$ in degree $n-d$ is called the *Alexander Dual* of X.

Remark This definition is motivated by the topological Alexander dual. The Stanley-Reisner simplicial complex of an initial segment $(\frac{u}{m_s}, \cdots, \frac{u}{m_{s-r+1}})$ of $X^\vee$ is the Alexander dual of the Stanley-Reisner complex of the initial segment $(m_1, \cdots, m_r)$ of X.

Example 5.24. The Alexander dual of X : $abd, acd, ade, bde, cde, bcd, bce, abe, abc, ace$ is $X^\vee$: $bd, de, cd, ad, ae, ab, ac, bc, be, cd$.

Proposition 5.25. *The Alexander dual of the squarefree Lex sequence in degree d is the squarefree lex sequence in degree $n-d$.*

Theorem 5.26. *The Alexander dual of a squarefree lexlike sequence is squarefree lexlike.*

Proof: Let X be a squarefree lexlike sequence in variables $x_1, \cdots, x_n$. Without loss of generality we may write $X = x_1Y; Z$ with Y and Z squarefree lexlike in the variables $x_2, \cdots, x_n$. Then we have $X^\vee = x_1Z^\vee; Y^\vee$, the duals on the right being taken with respect to the variables $x_2, \cdots, x_n$. $Z^\vee$ and $Y^\vee$ are lexlike by induction on n, so $X^\vee$ is lexlike by Theorem 5.13.

Proposition 5.27. *Let X and Y be squarefree lexlike sequences such that Y is above X. Then $X^\vee$ is above $Y^\vee$.*

Proof: Without loss of generality we may write $X = x_1A; B$ and $Y = x_1C; D$, with C above A and D above B. Then $A^\vee$ is above $C^\vee$ and $B^\vee$ is above $D^\vee$ by induction on n, so $X^\vee = x_1B^\vee; A^\vee$ is above $Y^\vee = x_1D^\vee; C^\vee$.

In [MP2] we construct lexlike ideals in the polynomial ring, and show that they have maximal Betti numbers. Here, we do the same for squarefree lexlike ideals.

Definition 5.28. A *squarefree lexlike tower* is a collection $\mathbf{X}$ of lexlike sequences $\mathbf{X}_d$, one for each degree $d < n$, such that $\mathbf{X}_{d+1}$ is above $\mathbf{X}_d$ for all d.

Example 5.29. **Lex**, defined by letting Lex_d equal the squarefree Lex sequence in degree d, is a lexlike tower.

Example 5.30. Let

$$\begin{aligned}
\mathbf{X}_1 &: b, a, c, d \\
\mathbf{X}_2 &: bc, ab, bd, ad, ac, cd \\
\mathbf{X}_3 &: bcd, abc, abd, acd \\
\mathbf{X}_4 &: abcd.
\end{aligned}$$

Then $\mathbf{X}$ is a squarefree lexlike tower.

Proposition 5.31. *Let X be any squarefree lexlike sequence in degree d. Then there exists a squarefree lexlike tower $\mathbf{X}$ with $\mathbf{X}_d = X$.*

Proof: Repeatedly apply Propositions 5.20 and 5.22. □

Definition 5.32. Fix a squarefree lexlike tower $\mathbf{X}$. Then a squarefree monomial ideal M is called an $\mathbf{X}$-*ideal* if for every degree d, M_d is an initial segment of $\mathbf{X}_d$. M is a *squarefree lexlike ideal* if M is an $\mathbf{X}$-ideal for some squarefree lexlike tower $\mathbf{X}$.

Example 5.33 The ideal $M = (bc, ab, acd)$ is an $\mathbf{X}$-ideal, for $\mathbf{X}$ as in Example 5.30.

Theorem 5.34. *For any squarefree monomial ideal I, there exists an $\mathbf{X}-$ideal J with $\mathrm{Hilb}^B(I) = \mathrm{Hilb}^B(J)$.*

Remark. If $\mathbf{X} = \mathbf{Lex}$, this is Kruskal-Katona's theorem [Kr,Ka].

Proof: For each d, let J_d be the initial segment of $\mathbf{X}_d$ with length $|J_d| = |I_d|$. It suffices to show that J is an ideal, that is, that $|J_{d+1}| \geq |\mathbf{m}J_d|$ for all d. We have $|J_{d+1}| = |I_{d+1}| \geq |\mathbf{m}I_d| \geq |\mathbf{m}J_d|$. $\square$

The proof of [MP2, Theorem 4.14] also yields the following result:

Theorem 5.35. *Let J be any squarefree monomial ideal, and let L be the squarefree lex ideal with the same Hilbert function as J. Let $\tilde{J}$ and $\tilde{L}$ be the ideals of $\tilde{A} = k[x_1, \cdots, x_n]$ generated by the monomials of J and L, respectively. Then $\tilde{J}$ and $\tilde{L}$ have the same graded Betti numbers over $\tilde{A}$.*

Theorem 5.36. *If J is a squarefree lexlike ideal, and L is the squarefree ideal with the same Hilbert function as J, then J and L have the same graded Betti numbers over B. In particular, the squarefree lexlike ideals have maximal graded Betti numbers over B among all monomial ideals with a fixed Hilbert function.*

Proof: Let I be any squarefree ideal, and define $\tilde{I}$ as in Theorem 5.35. Then the Betti numbers of I over B are related to those of $\tilde{I}$ over $\tilde{A}$ by the following formula [GHP]:

$$\sum_{i,j} \beta^B_{i,j}(B/I)t^i v^j = \sum_{i,j} \beta^{\tilde{A}}_{i,i+j}(\tilde{A}/\tilde{I})t^i v^j \frac{1}{(1-tv)^j}.$$

Combining this with Theorem 5.35 yields the desired result. $\square$

Theorem 5.37. *If J is a squarefree lexlike ideal, and L is the squarefree lex ideal with the same Hilbert function, then J and L have the same graded Betti numbers over the exterior algebra. In particular, lexlike ideals have maximal graded Betti numbers over the exterior algebra among all graded ideals with a fixed Hilbert function.*

Proof: The Betti numbers of any squarefree ideal I over the exterior algebra

E are related to those over $\tilde{A}$ by the following formula [AAH]:

$$\sum_{i,j} \beta^{E}_{i,j}(E/I)t^i v^j = \sum_{i,j} \beta^{\tilde{A}}_{i,j}(\tilde{A}/\tilde{I})t^i v^j \frac{1}{(1-tv)^j}.$$

Combining this with Theorem 5.35 yields the desired result. The fact that the Betti numbers are maximal follows from a result in [AHH] which states that the lex ideal has maximal graded Betti numbers over the exterior algebra among all graded ideals with a fixed Hilbert function. □

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