THREE SIMPLICIAL RESOLUTIONS

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ABSTRACT. We describe the Taylor and Lyubeznik resolutions as simplicial resolutions, and use them to show that the Scarf complex of a monomial ideal is the intersection of all its minimal free resolutions.

1. INTRODUCTION

Let $S = k[x_1, \ldots, x_n]$, and let $I \subset S$ be a monomial ideal. An important object in the study of I is its *minimal free resolution*, which encodes essentially all information about I. For example, the Betti numbers of I can be read off as the ranks of the modules in its minimal resolution.

There are computationally intensive algorithms to compute the minimal resolution of an arbitrary ideal (for example, in Macaulay 2 [GS], the command `res I` returns the minimal resolution of S/I), but no general description is known, even for monomial ideals. Thus, it is an ongoing problem of considerable interest to find classes of ideals whose minimal resolutions can be described easily. A related problem is to describe non-minimal resolutions which apply to large classes of monomial ideals.

The most general answer to the latter question is Taylor's resolution, a (usually highly non-minimal) resolution which resolves an arbitrary monomial ideal; it is discussed in Section 3.

A very successful approach to both problems in the last decade has been to find combinatorial or topological objects whose structures encode resolutions in some way. This approach began with *simplicial resolutions* [BPS], and has expanded to involve polytopal complexes [NR, Si], cellular complexes [BS], CW complexes [BW, Ve], lattices [Ma, Ph, PV], posets [Cl], matroids [Tc], and discrete Morse theory [JW].

Resolutions associated to combinatorial objects have distinguished bases, and relationships between the objects lead to relationships between these bases in a very natural way. It thus becomes possible to compare and combine resolutions in all the ways that we can compare or combine combinatorial structures. For example, most of these resolutions turn out to be subcomplexes of the Taylor resolution in a very natural way. The only new result in the paper is Theorem 7.1, which describes the intersection of all the simplicial resolutions of an ideal.

In section 2, we describe some background material and introduce notation used throughout the paper.

Section 3 introduces the Taylor resolution in a way intended to motivate simplicial resolutions, which are introduced in section 4.

Section 5 describes the Scarf complex, a simplicial complex which often supports the minimal resolution of a monomial ideal, and otherwise does not support any resolution.

Section 6 defines the family of Lyubeznik resolutions. This section is essentially a special case of an excellent paper of Novik [No], which describes a more general class of resolutions based on so-called "rooting maps".

Section 7 uses the Lyubeznik resolutions to prove Theorem 7.1, that the Scarf complex of an ideal is equal to the intersection of all its simplicial resolutions.

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2. Background and Notation

Throughout the paper $S = k[x_1, \ldots, x_n]$ is a polynomial ring over an arbitrary field k. In the examples, we use the variables $a, b, c, \ldots$ instead of $x_1, x_2, x_3, \ldots$.

We depart from the standard notation in two ways, each designed to privilege monomials. First, we write the standard or "fine" multigrading multiplicatively, indexed by monomials, rather than additively, indexed by n-tuples. Second, we index our simplices by monomials rather than natural numbers. Details of both departures, as well as some background on resolutions, are below.

2.1. **Algebra.** If $I \subset S$ is an ideal, then a *free resolution* of S/I is an exact sequence

$$\mathbb{F} : \ldots \xrightarrow{\phi_n} F_n \xrightarrow{\phi_{n-1}} F_{n-1} \ldots \xrightarrow{\phi_0} F_0 \to S/I \to 0$$

where each of the F_i is a free S-module.

We say that $\mathbb{F}$ is *minimal* if each of the modules F_i has minimum possible rank; in this case the ranks are the *Betti numbers* of S/I.

It is not at all obvious *a priori* that minimal resolutions should exist. For this reason, when I is homogeneous, most standard treatments take the following theorem as the definition instead:

Theorem 2.1. *Let I be a homogeneous ideal, and let $\mathbb{F}$ be a resolution of S/I. Write $\mathbf{m} = (x_1, \ldots, x_n)$. Then $\mathbb{F}$ is minimal if and only if $\phi_i(F_{i+1}) \subset \mathbf{m}F_i$ for all i.*

The proof of Theorem 2.1 is technical; see, for example, [Pe, Section 9].

All the ideals we consider are homogeneous; in fact, they are *monomial* ideals, which is a considerably stronger property.

Definition 2.2. An ideal I is a *monomial ideal* if it has a generating set consisting of monomials. There exists a unique minimal such generating set; we write gens(I) and call its elements the *generators* of I.

Monomial ideals respect a "multigrading" which refines the usual grading.

Notation 2.3. We write the multigrading multiplicatively. That is, for each monomial m of S, set S_m equal to the k-vector space spanned by m. Then $S = \oplus S_m$, and $S_m \cdot S_n = S_{mn}$, so this decomposition is a grading. We say that the monomial m has *multidegree* m. We allow multidegrees to have negative exponents, so, for example, the twisted module $S(m^{-1})$ is a free module with generator in multidegree m, and $S(m^{-1})_n \cong S_{m^{-1}n}$ as a vector space; this is one-dimensional if no exponent of $m^{-1}n$ is negative, and trivial otherwise. Note that $S = S(1)$.

If N and P are multigraded modules, we say that a map $\phi : N \to P$ is *homogeneous of degree m* if $\phi(N_n) \subset P_{mn}$ for all n, and that ϕ is simply *homogeneous* if it is homogeneous of degree 1. We say that a resolution (or, more generally, an algebraic chain complex) is *homogeneous* if all its maps are homogeneous.

The minimal resolution of S/I can be made homogeneous in a unique way by assigning appropriate multidegrees to the generators of its free modules; counting these generators by multidegree yields the *multigraded Betti numbers* of S/I.

2.2. **Combinatorics.** Let M be a set of monomials (typically, M will be the generators of I). The *simplex on M* is the set of all subsets of M; we denote this by Δ_M. We will sometimes refer to the elements of M as *vertices* of Δ_M.

A *simplicial complex* on M is a subset of Δ_M which is closed under the taking of subsets. If Γ is a simplicial complex on M and $F \in \Gamma$, we say that F is a *face* of Γ. Observe that if F is a face of Γ and $G \subset F$, then G is also a face of Γ. We require that simplicial complexes be nonempty; that is, the empty set must always be a face. (In fact, for our purposes, we may as well assume that every vertex must be a face.)

If F is a face of Γ, we assign F the multidegree $\mathrm{lcm}(m : m \in F)$. Note that the vertex m has multidegree m, and that the empty set has multidegree 1. The *order* of a face F, written $|F|$, is the number of vertices in F; this is one larger than its dimension. If $G \subset F$ and $|G| = |F| - 1$, we say that G is a *facet* of F.

We adopt the convention that the unmodified word "complex" will always mean an algebraic chain complex; simplicial complexes will be referred to with the phrase "simplicial complex". However, recall that every simplicial complex is naturally associated to a chain complex by the following standard construction from algebraic topology:

Construction 2.4. Let Γ be a simplicial complex on M, and impose an order on the monomials of M by writing $M = \{m_1, \ldots, m_r\}$. Then we associate to Γ the chain complex $\mathbb{C}_\Gamma$ as follows:

For every face $F \in \Gamma$, we create a formal symbol $[F]$. If we write $F = \{m_{i_1}, \ldots, m_{i_s}\}$ with increasing indices i_j, then for each facet G of F we may write $G = F \smallsetminus \{m_{i_j}\}$ for some j; we define an orientation by setting ε_G^F equal to 1 if j is odd and to -1 if j is even. For each s, let C_s be the k-vector space spanned by the symbols $[F]$ such that $|F| = s$, and define the map

$$\phi_{s-1} : C_s \to C_{s-1}$$

$$[F] \mapsto \sum_{G \text{ is a facet of } F} \varepsilon_G^F [G].$$

Then we set $\mathbb{C}_\Gamma$ equal to the complex of vector spaces

$$\mathbb{C}_\Gamma : 0 \to C_r \xrightarrow{\phi_{r-1}} \ldots \xrightarrow{\phi_1} C_1 \xrightarrow{\phi_0} C_0 \to 0.$$

The proof that $\mathbb{C}_\Gamma$ is a chain complex involves a straightforward computation of $\phi^2([m_{i_1}, \ldots, m_{i_s}])$. The (reduced) homology of Γ is defined to be the homology of this complex.

In section 4, we will replace this complex with a homogeneous complex of free S-modules.

3. The Taylor resolution

Let $I = (m_1, \ldots, m_s)$ be a monomial ideal. The Taylor resolution of I is constructed as follows:

Construction 3.1. For a subset F of $\{m_1, \ldots, m_r\}$, set $\mathrm{lcm}(F) = \mathrm{lcm}\{m_i : m_i \in F\}$. For each such F, we define a formal symbol $[F]$, called a *Taylor symbol*, with multidegree equal to $\mathrm{lcm}(F)$. For each i, set T_i equal to the free S-module with basis $\{[F] : |F| = i\}$ given by the symbols corresponding to subsets of size i. Note that $T_0 = S[\varnothing]$ is a free module of rank one, and that all other T_i are multigraded modules with generators in multiple multidegrees depending on the symbols $[F]$.

Define $\phi_{-1} : T_0 \to S/I$ by $\phi_{-1}(f[\varnothing]) = f$. Otherwise, we construct $\phi_i : T_{i+1} \to T_i$ as follows.

Given $F = \{m_{j_1}, \ldots, m_{j_i}\}$, written with the indices in increasing order, and $G = F \smallsetminus \{m_{j_k}\}$, we set the *sign* ε_G^F equal to 1 if k is odd and to -1 if k is even. Finally, we set

$$\phi_F = \sum_{G \text{ is a facet of } F} \varepsilon_G^F \frac{\mathrm{lcm}(F)}{\mathrm{lcm}(G)}[G],$$

and define $\phi_i : T_{i+1} \to T_i$ by extending the various ϕ_F. Observe that all of the ϕ_i are homogeneous with multidegree 1.

The *Taylor resolution* of I is the complex

$$\mathbb{T}_I : 0 \to T_r \xrightarrow{\phi_{r-1}} \ldots \xrightarrow{\phi_1} T_1 \xrightarrow{\phi_0} T_0 \to S/I \to 0.$$

It is straightforward to show that the Taylor resolution is a homogeneous chain complex.

The construction of the Taylor resolution is very similar to Construction 2.4; in fact, if Γ is the complete simplex, the only difference is the presence of the lcms in the boundary maps. We will explore this connection in the next section.

Example 3.2. Let $I = (a, b^2, c^3)$. Then the Taylor resolution of I is

$$\mathbb{T}_I : 0 \to S[a, b^2, c^3] \xrightarrow{\begin{pmatrix} a \\ -b^2 \\ c^3 \end{pmatrix}} \begin{matrix} S[b^2, c^3] \\ \oplus \\ S[a, c^3] \\ \oplus \\ S[a, b^2] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -c^3 & -b^2 \\ -c^3 & 0 & a \\ b^2 & a & 0 \end{pmatrix}} \begin{matrix} S[a] \\ \oplus \\ S[b^2] \\ \oplus \\ S[c^3] \end{matrix} \xrightarrow{\begin{pmatrix} a & b^2 & c^3 \end{pmatrix}} S[\varnothing] \to S/I \to 0.$$

Observe that I is a complete intersection and $\mathbb{T}_I$ is its Koszul complex. In fact, these two complexes coincide for all monomial complete intersections.

Example 3.3. Let $I = (a^2, ab, b^3)$. Then the Taylor resolution of I is

$$\mathbb{T}_I : 0 \to S[a^2, ab, b^3] \xrightarrow{\begin{pmatrix} a \\ -1 \\ b^2 \end{pmatrix}} \begin{matrix} S[ab, b^3] \\ \oplus \\ S[a^2, b^3] \\ \oplus \\ S[a^2, ab] \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -b^3 & -b \\ -b^2 & 0 & a \\ a^2 & a & 0 \end{pmatrix}} \begin{matrix} S[a^2] \\ \oplus \\ S[ab] \\ \oplus \\ S[b^3] \end{matrix} \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} S[\varnothing] \to S/I \to 0.$$

This is not a minimal resolution; the Taylor resolution is very rarely minimal.

Theorem 3.4. *The Taylor resolution of I is a resolution of I.*

It is not too difficult to show that $\phi^2 = 0$ in the Taylor complex, but it is not at all clear from the construction that the complex is exact. This seems to be most easily established indirectly by showing that the Taylor resolution is a special case of some more general phenomenon. We will prove Theorem 3.4 in the next section, using the language of simplicial resolutions. Traditionally, one builds the Taylor resolution as an iterated mapping cone; we sketch that argument below.

Sketch of Theorem 3.4. Write $I = (m_1, \ldots, m_r)$, and let $J = (m_1, \ldots, m_{r-1})$. Consider the short exact sequence

$$0 \to \frac{S}{(J : m_r)} \xrightarrow{m_r} \frac{S}{J} \to \frac{S}{I} \to 0.$$

If $(\mathbb{A}, \alpha)$ and $(\mathbb{B}, \beta)$ are free resolutions of $S/(J : m_r)$ and S/J, respectively, then multiplication by m_r induces a map of complexes $(m_r)_* : \mathbb{A} \to \mathbb{B}$. The *mapping cone complex* $(\mathbb{T}, \gamma)$ is defined by setting $T_i = B_i \oplus A_{i-1}$ and $\gamma|_{\mathbb{B}} = \beta$, $\gamma|_{\mathbb{A}} = (m_r)_* - \alpha$; it is a free resolution of S/I (see, for example, [Pe, Section 27]).

Inducting on r, S/J is resolved by the Taylor resolution on its generators $\{m_1, \ldots, m_{r-1}\}$, and $S/(J : m_r)$ is resolved by the Taylor resolution on its (possibly redundant) generating set $\{\frac{\text{lcm}(m_1, m_r)}{m_r}, \ldots, \frac{\text{lcm}(m_{r-1}, m_r)}{m_r}\}$. The resulting mapping cone is the Taylor resolution of I. □

4. Simplicial resolutions

If $\Gamma = \Delta$, the construction of the Taylor resolution differs from the classical topological construction of the chain complex associated to Γ only by the presence of the monomials $\frac{\text{lcm}(F)}{\text{lcm}(G)}$ in its differential maps. This observation leads naturally to the question of what other simplicial complexes give rise to resolutions in the same way. The resulting resolutions are called *simplicial*. Simplicial resolutions and, more generally, resolutions arising from other topological structures (it seems that the main results can be tweaked to work for anything defined in terms of skeletons and boundaries) have proved to be an instrumental tool in the understanding of monomial ideals. We describe only the foundations of the theory here; for a more detailed treatment, the original paper of Bayer, Peeva, and Sturmfels [BPS] is a very readable introduction.

Construction 4.1. Let M be a set of monomials, and let Γ be a simplicial complex on M (recall that this means that the vertices of Γ are the monomials in M). Fix an ordering on the elements of M; this induces an orientation ε on Γ. Recall that ε_G^F is either 1 or -1 if G is a facet of F (see Construction 2.4 for the details); it is often convenient to formally set ε_G^F equal to zero when G is not a facet of F.

We assign a multidegree to each face $F \in \Gamma$ by the rule $\text{mdeg}(F) = \text{lcm}(m : m \in F)$ (recall that F is a subset of M, so its elements are monomials).

Now for each face F we create a formal symbol $[F]$ with multidegree $\text{mdeg}(F)$. Let H_s be the free module with basis $\{[F] : |F| = s\}$, and define the differential

$$\phi_{s-1} : H_s \to H_{s-1}$$

$$[F] \mapsto \sum_{G \text{ is a facet of } F} \varepsilon_G^F \frac{\text{mdeg}(F)}{\text{mdeg}(G)} [G].$$

The *complex associated to* Γ is then the algebraic chain complex

$$\mathbb{H}_\Gamma : 0 \to H_r \xrightarrow{\phi_{r-1}} \ldots \xrightarrow{\phi_1} H_1 \xrightarrow{\phi_0} H_0 \to S/I \to 0.$$

Construction 4.1 differs from Construction 2.4 in that it is a complex of free S-modules rather than vector spaces. The boundary maps are identical except for the monomial coefficients, which are necessary to make the complex homogeneous.

Example 4.2. Let I be generated by M, and let Δ be the simplex with vertices M. Then the Taylor resolution of I is the complex associated to Δ.

Example 4.3. Let I be generated by $M = \{a^2, ab, b^3\}$, and let Δ be the full simplex on M, Γ the simplicial complex with facets $\{a^2, ab\}$ and $\{ab, b^3\}$, and Θ the zero-skeleton of Δ. These simplicial complexes, with their faces labeled by multidegree, are pictured in figure 1.
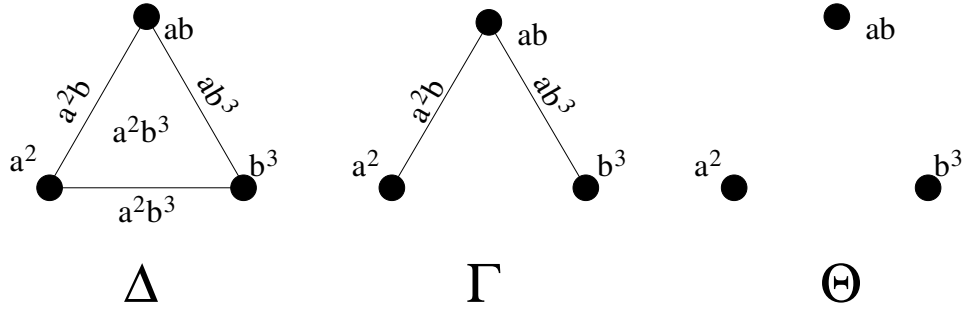

FIGURE 1. The simplicial complexes Δ, Γ, and Θ of example 4.3.

The algebraic complex associated to Δ is the Taylor resolution of Example 3.3. The other two associated complexes are

$$\mathbb{H}_\Gamma : 0 \to \begin{matrix} S[a^2, ab] \\ \oplus \\ S[ab, b^3] \end{matrix} \xrightarrow{\begin{pmatrix} -b & 0 \\ a & -b^2 \\ 0 & a \end{pmatrix}} \begin{matrix} S[a^2] \\ \oplus \\ S[ab] \\ \oplus \\ S[b^3] \end{matrix} \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} S[\varnothing] \to S/I \to 0$$

and

$$\mathbb{H}_\Theta : 0 \to \begin{matrix} S[a^2] \\ \oplus \\ S[ab] \\ \oplus \\ S[b^3] \end{matrix} \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} S[\varnothing] \to S/I \to 0.$$

$\mathbb{H}_\Gamma$ is a resolution (in fact, the minimal resolution) of S/I, and $\mathbb{H}_\Theta$ is not a resolution of I.

The algebraic complex associated to Γ is not always exact; that is, it does not always give rise to a resolution of I. When this complex is exact, we call it a *simplicial resolution*, or *the (simplicial) resolution supported on* Γ. It turns out that there is a topological condition describing whether Γ supports a resolution.

Definition 4.4. Let Γ be a simplicial complex on M, and let μ be a multidegree. We set $\Gamma_{\leq\mu}$ equal to the simplicial subcomplex of Γ consisting of the faces with multidegree divisible by μ,

$$\Gamma_{\leq\mu} = \{F \in \Gamma : \deg(F) \text{ divides } \mu\}.$$

Observe that $\Gamma_{\leq\mu}$ is precisely the faces of Γ whose vertices all divide μ.

Theorem 4.5 (Bayer-Peeva-Sturmfels). *Let Γ be a simplicial complex supported on M, and set $I = (M)$. Then Γ supports a resolution of S/I if and only if, for all μ, the simplicial complex $\Gamma_{\leq\mu}$ has no homology over k.*

Proof. Since $\mathbb{H}_\Gamma$ is homogeneous, it is exact if and only if it is exact (as a complex of vector spaces) in every multidegree. Thus, it suffices to examine the restriction of $\mathbb{H}_\Gamma$ to each multidegree μ.

Observe that $(S[F])_\mu \cong S(\frac{1}{\mathrm{mdeg}(F)})_\mu \cong S_{\frac{\mu}{\mathrm{mdeg}(F)}}$ is a one-dimensional vector space with basis $\frac{\mu}{\mathrm{mdeg}(F)}$ if $\mathrm{mdeg}(F)$ divides μ, and is zero otherwise. Furthermore, since the differential maps ϕ are homogeneous, the monomials appearing in their definition are precisely those which map these basis elements to one another. Thus $(\mathbb{H}_\Gamma)_\mu$ is, with minor abuse of notation, precisely the complex of vector spaces which arises when computing (via Construction 2.4) the homology of the simplicial complex $\{F \in \Gamma : \mathrm{mdeg}(F) \text{ divides } \mu\}$, and this complex is $\Gamma_{\leq\mu}$.

We conclude that Γ supports a resolution of I if and only if $(\mathbb{H}_\Gamma)_\mu$ is exact for every μ, if and only if $(\mathbb{H}_\Gamma)_\mu$ has no homology for every μ, if and only if $\Gamma_{\leq\mu}$ has no homology for every μ. $\square$

Example 4.6. The simplicial complexes $\Gamma_{\leq\mu}$ depend on the underlying monomials M, so that it is possible for a simplicial complex to support a resolution of some monomial ideals but not others. For example, the simplicial complex Γ in example 4.3 supports a resolution of $I = (a^2, ab, b^3)$ because no monomial is divisible by a^2 and b^3 without also being divisible by ab. However, if we were to relabel the vertices with the monomials a, b, and c, the resulting simplicial complex Γ' would not support a resolution of (a, b, c) because $\Gamma'_{\leq ac}$ would consist of two points; this simplicial complex has nontrivial zeroeth homology.

Remark 4.7. Note that the homology of a simplicial complex can depend on the choice of field, so some simplicial complexes support resolutions over some fields but not others. For example, if Γ is a triangulation of a torus, it may support a resolution if the field has characteristic zero, but will not support a resolution in characteristic two. In particular, resolutions of monomial ideals can be characteristic-dependent.

Theorem 4.5 allows us to give a short proof that the Taylor resolution is in fact a resolution.

Proof of Theorem 3.4. Let μ be given. Then $\Delta_{\leq\mu}$ is the simplex with vertices $\{m \in M : m \text{ divides } \mu\}$, which is either empty or contractible. $\square$

5. The Scarf complex

Unfortunately, the Taylor resolution is usually not minimal. The nonminimality is visible in the nonzero scalars in the differential maps, which occur whenever there exist faces F and G with the same multidegree such that G is in the boundary of

F. It is tempting to try to simply remove the nonminimality by removing all such faces; the result is the Scarf complex.

Construction 5.1. Let I be a monomial ideal with generating set M. Let Δ_I be the full simplex on M, and let Σ_I be the simplicial subcomplex of Δ_I consisting of the faces with unique multidegree,

$$\Sigma_I = \{F \in \Delta_I : \mathrm{mdeg}(G) = \mathrm{mdeg}(F) \implies G = F\}.$$

We say that Σ_I is the *Scarf simplicial complex* of I; the associated algebraic chain complex $\mathbb{S}_I$ is called the *Scarf complex* of I. The multidegrees of the faces of Σ_I are called the *Scarf multidegrees* of I.

Remark 5.2. It is not obvious that Σ_I is a simplicial complex. Let $F \in \Sigma_I$; we will show that every subset of F is also in Σ_I. Suppose not; then there exists a minimal $G \subset F$ which shares a multidegree with some other $H \in \Delta_I$. Let E be the symmetric difference of G and H. Then the symmetric difference of E and F has the same multidegree as F.

Example 5.3. Let $I = (a^2, ab, b^3)$. Then the Scarf simplicial complex of I is the complex Γ in figure 1. The Scarf complex of I is the minimal resolution

$$\mathbb{S}_I : 0 \to \begin{array}{c} S[a^2, ab] \\ \oplus \\ S[ab, b^3] \end{array} \xrightarrow{\begin{pmatrix} -b & 0 \\ a & -b^2 \\ 0 & a \end{pmatrix}} \begin{array}{c} S[a^2] \\ \oplus \\ S[ab] \\ \oplus \\ S[b^3] \end{array} \xrightarrow{\begin{pmatrix} a^2 & ab & b^3 \end{pmatrix}} S[\varnothing] \to S/I \to 0.$$

Example 5.4. Let $I = (ab, ac, bc)$. The Scarf simplicial complex of I consists of three disjoint vertices. The Scarf complex of I is the complex

$$\mathbb{S}_I : 0 \to \begin{array}{c} S[ab] \\ \oplus \\ S[ac] \\ \oplus \\ S[bc] \end{array} \xrightarrow{\begin{pmatrix} ab & ac & bc \end{pmatrix}} S[\varnothing] \to S/I \to 0.$$

It is not a resolution.

Example 5.4 shows that not every monomial ideal is resolved by its Scarf complex. We say that a monomial ideal is *Scarf* if its Scarf complex is a resolution.

Theorem 5.5. *If the Scarf complex of I is a resolution, then it is minimal.*

Proof. By construction, no nonzero scalars can occur in the differential matrices. $\square$

Bayer, Peeva and Sturmfels [BPS] call an ideal *generic* if no variable appears with the same nonzero exponent in more than one generator. They show that these "generic" ideals are Scarf.

Unfortunately, most interesting monomial ideals are not Scarf. However, Scarf complexes have proved an important tool in constructing ideals whose resolutions misbehave in various ways [Ve].

Theorem 5.6. *Let $\mathbb{F}$ be a minimal resolution of I. Then the Scarf complex of I is a subcomplex of $\mathbb{F}$.*

Proof. This is [Pe, Proposition 59.2]. The proof requires a couple of standard facts about resolutions, but is otherwise sufficiently reliant on the underlying simplicial complexes that we reproduce it anyway.

We know (see, for example, [Pe, Section 9]) that there is a homogeneous inclusion of complexes from $\mathbb{F}$ to the Taylor complex $\mathbb{T}$. We also know that the multigraded Betti numbers of I, which count the generators of $\mathbb{F}$, can be computed from the homology of the simplicial complexes $\Delta_{\lneqq m}$ ([Pe, Section 57]). If $m = \operatorname{mdeg}(G)$ is a Scarf multidegree, then $b_{|G|,m}(S/I) = 1$ and $b_{i,m}(S/I) = 0$ for all other I. If m divides a Scarf multidegree but is not itself a Scarf multidegree, then $b_{i,m}(S/I) = 0$ for all i. In particular, when m is a Scarf multidegree, the Betti numbers of multidegree m also count the number of faces of multidegree M in both Δ_I and Σ_I; these numbers are never greater than one.

By induction on multidegrees, each generator of $\mathbb{F}$ with a Scarf multidegree must (up to a scalar) be mapped under the inclusion to the unique generator of the Taylor resolution with the same multidegree. However, these are exactly the generators of the Scarf complex. Thus, the inclusion from $\mathbb{F}$ to $\mathbb{T}$ induces an inclusion from $\mathbb{S}$ to $\mathbb{F}$. $\qquad\square$

6. The Lyubeznik resolutions

If the Taylor resolution is too large, and the Scarf complex is too small, we might still hope to construct simplicial resolutions somewhere in between. Velasco [Ve] shows that it is impossible to get the minimal resolution of every ideal in this way, even if we replace simplicial complexes with much more general topological objects. However, there are still classes of simplicial resolutions which are in general much smaller than the Taylor resolution, yet still manage to always be resolutions. One such class is the class of Lyubeznik resolutions, introduced below.

Our construction follows the treatment in an excellent paper of Novik [No], which presents the Lyubeznik resolutions as special cases of resolutions arising from "rooting maps". The only difference between the following construction and Novik's paper is that the extra generality has been removed, and the notation is correspondingly simplified.

Construction 6.1. Let I be a monomial ideal with generating set M, and fix an ordering $\prec$ on the monomials appearing in M. (We do not require that $\prec$ have any special property, such as a term order; any total ordering will do.) Write $M = \{m_1, \ldots, m_s\}$ with $m_i \prec m_j$ whenever $i < j$.

Let Δ_I be the full simplex on M; for a monomial $\mu \in I$, set $\min(\mu) = \min_\prec\{m_i : m_i \text{ divides } \mu\}$. For a face $F \in \Delta_I$, set $\min(F) = \min(\operatorname{mdeg}(F))$. Thus $\min(F)$ is a monomial. We expect that in fact $\min(F)$ is a vertex of F, but this need not be the case: for example, if $F = \{a^2, b^2\}$, we could have $\min(F) = ab$.

We say that a face F is *rooted* if every nonempty subface $G \subset F$ satisfies $\min(G) \in G$. (Note that in particular $\min(F) \in F$.) By construction, the set $\Lambda_{I,\prec} = \{F \in \Delta_I : F \text{ is rooted}\}$ is a simplicial complex; we call it the *Lyubeznik simplicial complex* associated to I and $\prec$. The associated algebraic chain complex $\mathbb{L}_{I,\prec}$ is called a *Lyubeznik resolution* of I.

Example 6.2. Let $I = (ab, ac, bc)$. Then there are three distinct Lyubeznik resolutions of I, corresponding to the simplicial complexes pictured in Figure 2: Λ_{ab} arises from the orders $ab \prec ac \prec bc$ and $ab \prec bc \prec ac$, Λ_{ac} arises from the orders
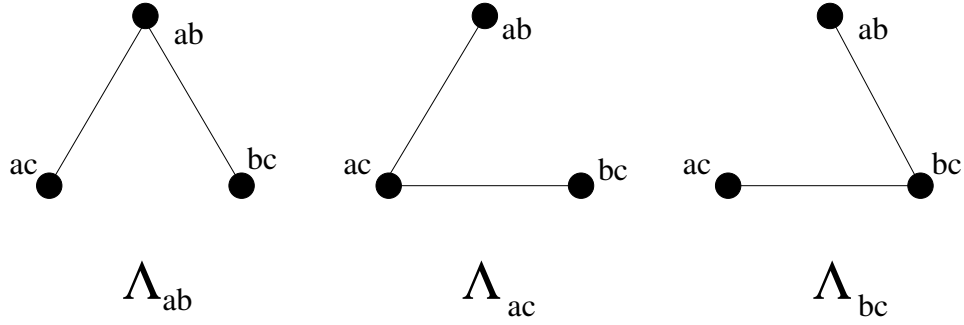
FIGURE 2. The Lyubeznik resolutions of $I = (ab, ac, bc)$.

with ac first, and Λ_{bc} arises from the orders with bc first. Each of these resolutions is minimal.

Example 6.3. Let $I = (a^2, ab, b^3)$. There are two Lyubeznik resolutions of I: the Scarf complex, arising from the two orders with ab first, and the Taylor resolution, arising from the other four orders. The corresponding simplicial complexes are pictured in figure 3.
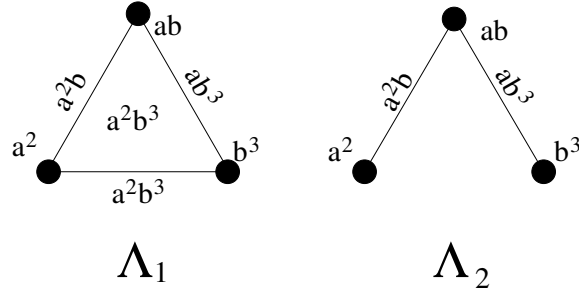

FIGURE 3. The Lyubeznik resolutions of $I = (a^2, ab, b^3)$.

Remark 6.4. It is unclear how to choose a total ordering on the generators of I which produces a smaller Lyubeznik resolution. Example 6.3 suggests that the obvious choice of a term order is a bad one: the lex and graded (reverse) lex orderings all yield the Taylor resolution, while the minimal resolution arises from orderings which cannot be term orders.

We still need to show that, unlike the Scarf complex, the Lyubeznik resolution is actually a resolution.

Theorem 6.5. *The Lyubeznik resolutions of I are resolutions.*

Proof. Let $M = (m_1, \ldots, m_s)$ be the generators of I and fix an order $\prec$ on M. For each multidegree μ, we need to show that the simplicial subcomplex $(\Lambda_{I,\prec})_{\leq \mu}$, consisting of the rooted faces with multidegree dividing μ, has no homology.

If $\mu \notin I$, this is the empty complex. If $\mu \in I$, we claim that $(\Lambda_{I,\prec})_{\leq \mu}$ is a cone.

Suppose without loss of generality that $m_1 = \min(\mu)$. We claim that, if F is a face of $(\Lambda_{I,\prec})_{\leq \mu}$, then $F \cup \{x_1\}$ is a face as well. First, note that $\mathrm{mdeg}(F \cup \{x_1\})$

divides μ because both x_1 and $\mathrm{mdeg}(F)$ do. Thus it suffices to show that $F \cup \{x_1\}$ is rooted. Observe that $\min(F \cup \{x_1\}) = x_1$ because $\mathrm{mdeg}(F \cup \{x_1\})$ divides μ and x_1 divides $\mathrm{mdeg}(F \cup \{x_1\})$. If $G \subset F$, then $\min(G) \in G$ because F is rooted, and $\min(G \cup \{x_1\}) = x_1$. Thus $F \cup \{x_1\}$ is rooted.

Hence $(\Lambda_{I,\prec})_{\leq \mu}$ is a simplicial cone on x_1 and is contractible. $\qquad\square$

7. Intersections

The only new result of this paper is that the Scarf complex of an ideal I is the intersection of all its minimal resolutions. To make this statement precise, we need to refer to some ambient space that contains all the minimal resolutions; the natural choice is the Taylor resolution.

Theorem 7.1. *Let I be a monomial ideal. Let $\mathbb{D}_I$ be the intersection of all isomorphic embeddings of the minimal resolution of I in its Taylor resolution. Then $\mathbb{D}_I = \mathbb{S}_I$ is the Scarf complex of I.*

Proof. We showed in Theorem 5.6 that the Scarf complex is contained in this intersection. It suffices to show that the intersection of all minimal resolutions lies inside the Scarf complex. We will show that in fact the intersection of all the Lyubeznik resolutions is the Scarf complex.

Suppose that F is a face of every Lyubeznik simplicial complex. This means that, regardless of the ordering of the monomial generators of I, the first generator dividing $\mathrm{mdeg}(F)$ appears in F. Equivalently, every generator which divides $\mathrm{mdeg}(F)$ appears as a vertex of F. Thus, F is the complete simplex on the vertices with multidegree dividing $\mathrm{mdeg}(F)$.

Now suppose that there exists another face G with the same multidegree as F. Every vertex of G divides $\mathrm{mdeg}(G) = \mathrm{mdeg}(F)$, so in particular $G \subset F$. But this means that G is also a face of every Lyubeznik simplicial complex, so every generator dividing $\mathrm{mdeg}(G)$ is a vertex of G by the above argument. In particular, $F = G$. This proves that F is the unique face with multidegree $\mathrm{mdeg}(F)$, i.e., F is in the Scarf complex. $\qquad\square$

8. Questions

The viewpoint that allows us to consider the statement of Theorem 7.1 requires that we consider a resolution together with its basis, so resolutions which are isomorphic as algebraic chain complexes can still be viewed as different objects. The common use of the phrase "*the* minimal resolution" (instead of "*a* minimal resolution") suggests that this this point of view is relatively new, or at any rate has not been deemed significant. In any event, there are some natural questions which would not make sense from a more traditional point of view.

Question 8.1. Let I be a monomial ideal. Are there (interesting) resolutions of I which are not subcomplexes of the Taylor resolution?

All the interesting resolutions I understand are subcomplexes of the Taylor complex in a very natural way: their basis elements can be expressed with relative ease as linear combinations of Taylor symbols. If a resolution is a subcomplex of the Taylor resolution, then it is simplicial if and only if all its basis elements are Taylor symbols. For a simplicial resolution to fail to be a subcomplex of the Taylor complex, the set of vertices of its underlying simplicial complex must not be a subset of the generators - in other words, the underlying presentation must not be minimal.

Question 8.2. Let I be a monomial ideal. Are there (interesting) resolutions of I with non-minimal first syzygies?

My suspicion is that such resolutions may exist, at least for special classes of ideals, and may be useful in the study of homological invariants such as regularity which are interested in the degree, rather than the number, of generators.

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